CONCURRENT PARITY GAMES

REDUCING CONCURRENT TO TURN-BASED BÜCHI AND CO-BÜCHI GAMES

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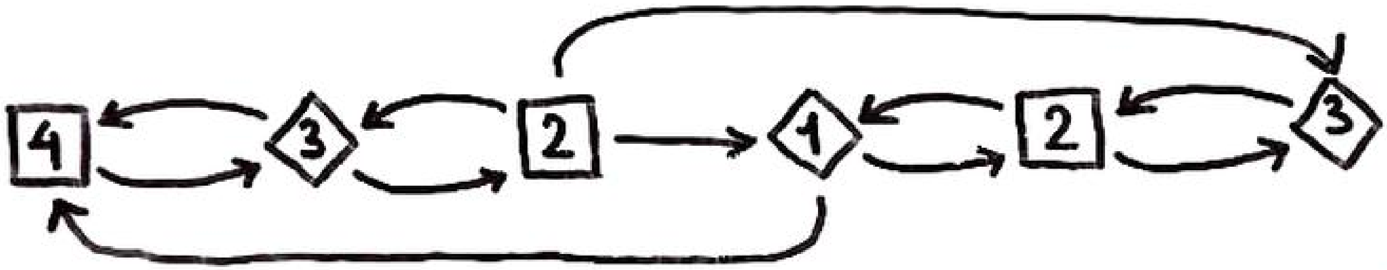
PARITY GAMES

FOCUS HERE ON:

- GENERALIZATION TO CONCURRENT GAMES (PROBABILISTIC STRATEGIES NEEDED)
- GENERALIZATION OF DETERMINACY
- STRUCTURE OF WINNING STRATEGIES
- ALGORITHMICS OF SOLVING GAMES

PARITY GAME EXAMPLE



□ PLAYER 0

◇ PLAYER 1

- PLAY : INFINITE PATH
- WINNING PLAY (FOR PL. 0):

  HIGHEST PRIORITY OCCURRING INFINITELY OFTEN IS EVEN

- STRATEGY
- WINNING STRATEGY

STRATEGIES

		HISTORY		CURRENT VERTEX		NEXT VERTEX
PURE	GENERAL	V^*	$\times$	V	$\longrightarrow$	V
	W/MEMORY	M	$\times$	V	$\longrightarrow$	$V \times M$
	MEMORYLESS			V	$\longrightarrow$	V

PURE: OUTCOME (v,σ,τ) : AN INFINITE PATH

MIXED (PROBABILISTIC)	GENERAL	V^*	$\times$	V	$\longrightarrow$	DISTR(V)
	W/MEMORY	M	$\times$	V	$\longrightarrow$	DISTR$(V) \times M$

MIXED (PROBABILISTIC): OUTCOMES (v,σ,τ) : PROBABILITY SPACE OVER INFINITE PATHS

DETERMINACY OF TURN-BASED PARITY GAMES

<u>THM</u> [EJ'91, Mos'91]

THERE IS A UNIQUE PARTITION

$$V = W_0 \cup W_1$$

S.T. PLAYER i WINS ON W_i

WINNING STRATEGIES FOR BOTH PLAYERS ARE

- PURE
- MEMORYLESS

CONCURRENT PARITY GAMES [dAHK'98'c

<u>FACT</u> FOR EACH PURE STRATEGY α FOR PLAYER 0 THERE IS A STRATEGY β FOR PLAYER 1, S.T. OUTCOME(v, α, β) IS LOSING FOR PL. 0

$\Rightarrow$ PURE STRATEGIES DETERMINACY FAILS FOR CONCURRENT GAMES

A PROBABILISTIC STRATEGY FOR PL. 0

MIXED STRATEGY α FOR PLAYER 0:

$$\mathbb{P}^{\alpha}[\text{THROW L}] = \varepsilon$$

$$\mathbb{P}^{\alpha}[\text{THROW R}] = 1-\varepsilon$$

<u>FACT</u> FOR EVERY MIXED STRATEGY τ OF PL. 1

$$\mathbb{P}_{v}^{\alpha,\tau}[\text{HIT}] \geq \varepsilon \qquad \text{IF } \varepsilon \leq \frac{1}{2}$$

<u>COROLLARY</u> $\boxed{\mathbb{P}_{v}^{\alpha,\tau}[\text{PL. 0 WINS}] = 1}$

ALMOST SURE / POSITIVE PROBABILITY WIN "DETERMINACY" FOR CONCURRENT PARITY GAMES

<u>THM</u> [dAH'00] THERE IS A UNIQUE PARTITION

$$V = W_0^{AS} \cup W_1^{P} \quad \text{S.T.}$$

- PLAYER 0 HAS A STRATEGY α ON W_0^{AS}, S.T.

  $$\mathbb{P}^{\alpha,\tau}[\text{PL. 0 WINS}] = 1$$

  FOR EVERY STRATEGY τ FOR PLAYER 1

- PLAYER 1 HAS A STRATEGY β ON W_1^{P}, S.T.

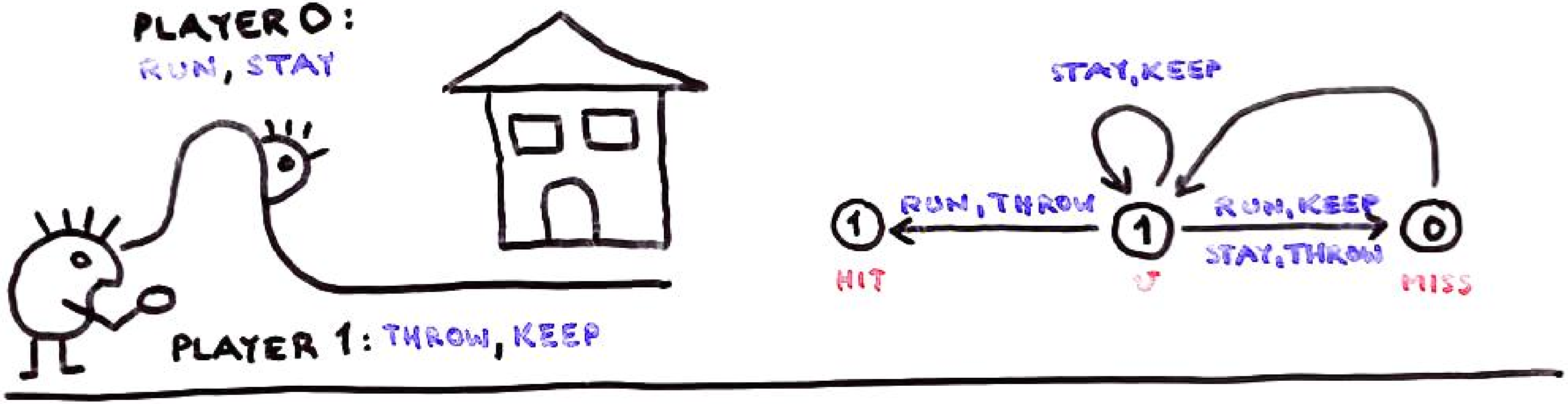
  $$\mathbb{P}^{\sigma,\beta}[\text{PL. 1 WINS}] > 0$$

  FOR EVERY STRATEGY σ FOR PLAYER 0

WINNING STRATEGIES α, β ARE:

- MIXED
- WITH "COUNTING" MEMORY (INFINITE!)

"ALMOST" ALMOST SURE WIN

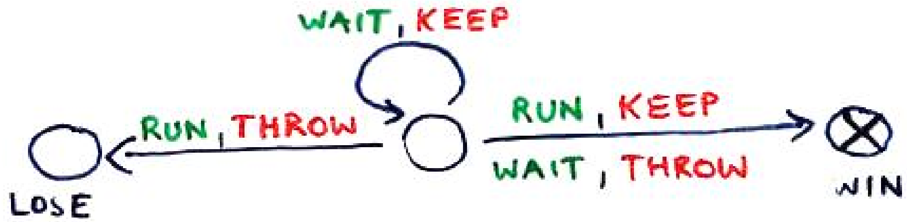


FACT 1 PLAYER 0 CANNOT WIN ALMOST SURELY

FACT 2 FOR EVERY $\varepsilon > 0$,
PLAYER 0 HAS A STRATEGY TO WIN
WITH PROBABILITY $\geqslant 1 - \varepsilon$

(LIMIT SURE WIN)

CONCURRENT REACHABILITY GAME : LIMIT-SURE WIN



$$\alpha : \begin{bmatrix} \text{RUN} \longmapsto \varepsilon \\ \text{WAIT} \longmapsto 1-\varepsilon \end{bmatrix} \qquad \beta : \begin{bmatrix} \text{THROW} \longmapsto 1-K \\ \text{KEEP} \longmapsto K \end{bmatrix}$$

FACT IF PLAYER 0 PLAYS α IN ALL STEPS THEN HE WINS WITH PROBABILITY $\geq 1-2\varepsilon$

CLAIM

$$\boxed{\begin{array}{ll} (1) & \mathbb{P}^{\alpha\beta}[\text{WIN}] \geq \varepsilon \\ (2) & \mathbb{P}^{\alpha\beta}[\text{LOSE}] \leq 2\varepsilon \cdot \mathbb{P}^{\alpha\beta}[\text{WIN}] \end{array}}$$

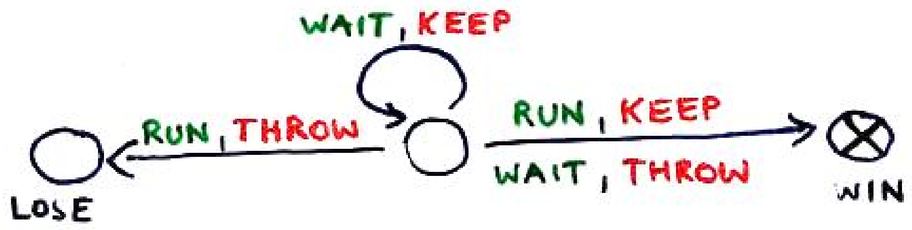
PROOF

ASS. $\varepsilon \leq \frac{1}{2}$

(1) $\mathbb{P}^{\alpha\beta}[\text{WIN}] = \varepsilon \cdot K + (1-\varepsilon)\cdot(1-K) \geq \varepsilon$

(2) $$\frac{\mathbb{P}^{\alpha\beta}[\text{LOSE}]}{\mathbb{P}^{\alpha\beta}[\text{WIN}]} \leq \frac{\varepsilon\cdot(1-K)}{(1-\varepsilon)\cdot(1-K)} \leq 2\varepsilon$$

CONCURRENT REACHABILITY GAME : LIMIT-SURE WIN



$$\alpha : \begin{bmatrix} \text{RUN} \longmapsto \varepsilon \\ \text{WAIT} \longmapsto 1-\varepsilon \end{bmatrix}$$

<u>FACT</u> IF PLAYER 0 PLAYS α IN ALL STEPS THEN HE WINS WITH PROBABILITY $\geq 1-2\varepsilon$

<u>CLAIM</u>

$$\boxed{\begin{array}{ll} (1) & \mathbb{P}^{\alpha\beta}[\text{WIN}] \geq \varepsilon \\ (2) & \mathbb{P}^{\alpha\beta}[\text{LOSE}] \leq 2\varepsilon \cdot \mathbb{P}^{\alpha\beta}[\text{WIN}] \end{array}}$$

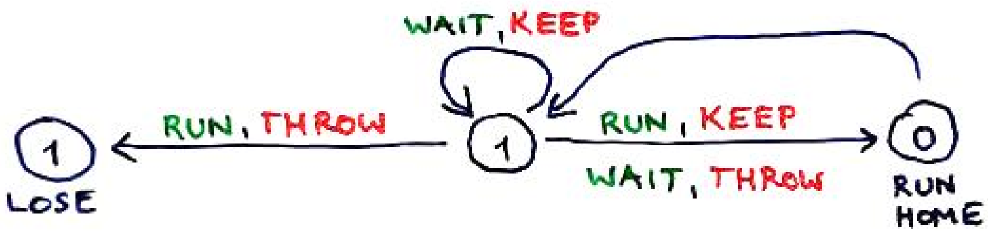
<u>PROOF</u> $L \overset{df}{=}$ PROBABILITY OF REACHING <u>LOSE</u> BEFORE <u>WIN</u>

$$\boxed{L \leq \mathbb{P}^{\alpha\beta}[\text{LOSE}] + (1-\mathbb{P}^{\alpha\beta}[\text{WIN}]) \cdot L}$$

$$\Longrightarrow \quad L \leq \frac{\mathbb{P}^{\alpha\beta}[\text{LOSE}]}{\mathbb{P}^{\alpha\beta}[\text{WIN}]} \leq 2\varepsilon$$

CONCURRENT BÜCHI GAME : LIMIT-SURE WIN

INFINITE MEMORY STRATEGY



α_δ : STR. FOR PL. 0 TO RUN HOME WITH PROB. $\geq \delta$

$\bar{\alpha}$: AFTER RUNNING HOME k TIMES PLAY α_{δ_k}

WHERE $\delta_k \stackrel{df}{=} (1-\varepsilon)^{1/2^{k+1}}$

FACT PLAYER 0 USING $\bar{\alpha}$ RUNS HOME ∞ MANY TIMES WITH PROBABILITY $\geq 1-\varepsilon$

PROOF R_k : EVENT THAT PL. 0 RUNS HOME $\geq k$ TIMES

$$\mathbb{P}^{\bar{\alpha}\beta}[R_\infty] = \prod_{i=0}^{\infty} \mathbb{P}^{\bar{\alpha}\beta}[R_{i+1} \mid R_i] \geq \prod_{i=0}^{\infty} \delta_i = 1-\varepsilon$$

LIMIT SURE / POSITIVE BOUNDED PROBABILITY WIN "DETERMINACY" FOR CONCURRENT PARITY GAMES

<u>THM</u> [dAH'00] THERE IS A UNIQUE PARTITION

$$V = W_0^{LS} \cup W_1^{B} \quad \text{s.t.}$$

- PLAYER 0 WINS LIMIT-SURELY ON W_0^{LS}, I.E.

FOR ALL $\varepsilon > 0$, HE HAS A STRATEGY α ON W_0^{LS}, S.T. $\inf_\tau \mathbb{P}^{\alpha,\tau}[\text{PL. 0 WINS}] \geq 1 - \varepsilon$

- PLAYER 1 WINS WITH PROB. BOUNDED AWAY FROM 0 ON W_1^B, I.E.

THERE IS $K > 0$, S.T. HE HAS A STRAT. β ON W_1^B, S.T. $\inf_\sigma \mathbb{P}^{\sigma,\beta}[\text{PL. 1 WINS}] \geq K$

"QUANTITATIVE" DETERMINACY
FOR CONCURRENT PROBABILISTIC PARITY GAMES

THM [dAM'01]

FOR EACH VERTEX v,
THERE IS A NUMBER π_v, S.T.

- $\sup_\sigma \inf_\tau \mathbb{P}_v^{\sigma,\tau}[\text{PL. 0 WINS}] = \pi_v$
- $\sup_\tau \inf_\sigma \mathbb{P}_v^{\sigma,\tau}[\text{PL. 1 WINS}] = 1 - \pi_v$

OPEN PROBLEM (EFFICIENT) ALGORITHMS FOR SOLVING

QUANTITAVE CONCURRENT PARITY GAMES

THE MAIN RESULT

LINEAR TIME REDUCTIONS

- FROM CONCURRENT BÜCHI GAMES
  TO TURN-BASED BÜCHI GAMES

- FROM CONCURRENT CO-BÜCHI GAMES
  TO TURN-BASED PARITY (0,2) GAMES

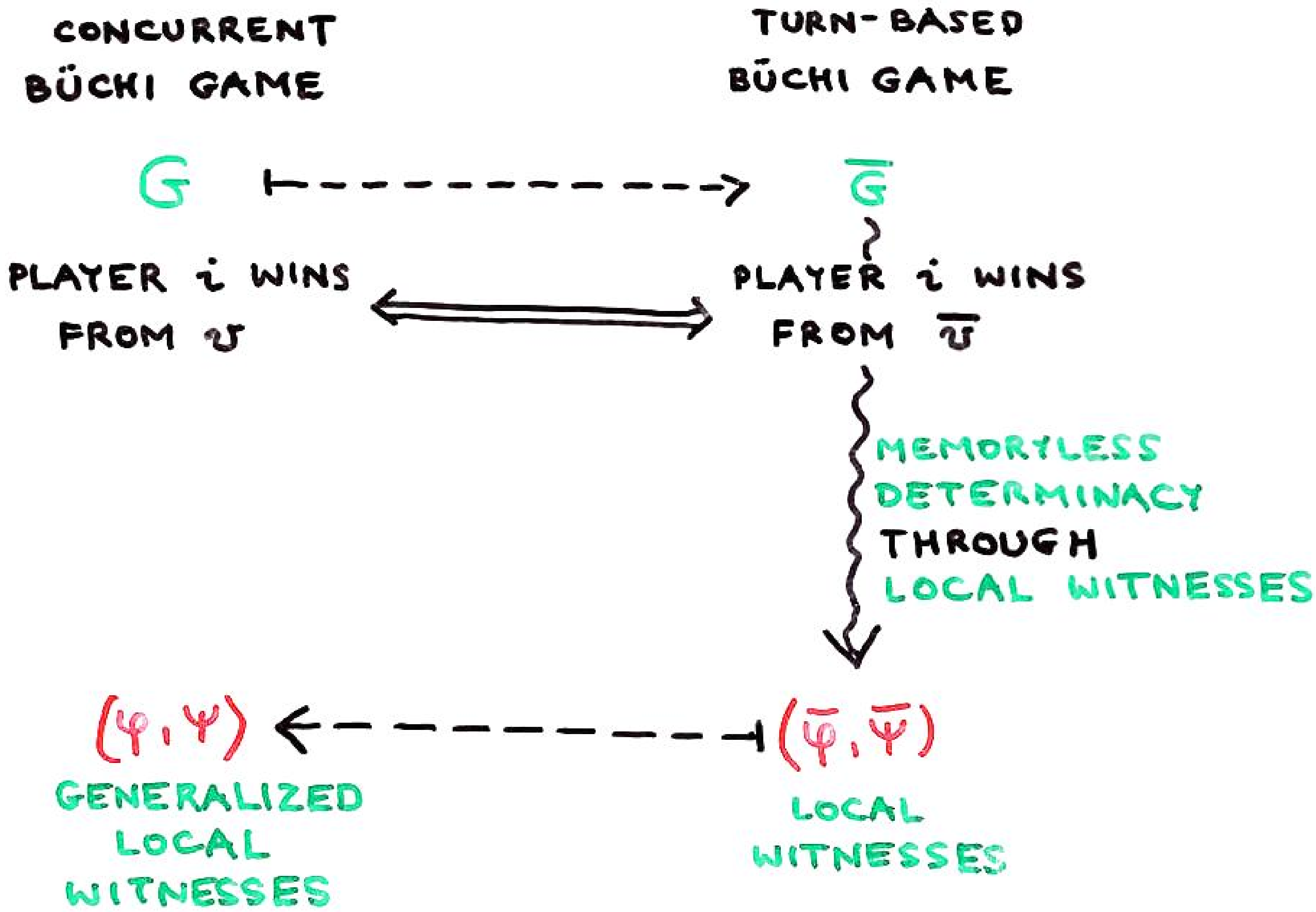
COROLLARY

CONCURRENT CO-BÜCHI GAMES CAN BE SOLVED IN QUADRATIC TIME

CONSEQUENCES AND BY-PRODUCTS

- THE REDUCTIONS TURN SOLVERS OF TURN-BASED PARITY GAMES INTO SOLVERS OF CONCURRENT BÜCHI AND CO-BÜCHI GAMES
- A GENERALIZATION OF LOCAL WITNESSES (A.K.A. PROGRESS MEASURES / SIGNATURES) TO CONCURRENT GAMES
- AN ALTERNATIVE PROOF OF ALMOST SURE / POSITIVE PROB. WIN DUALITY FROM MEMORYLESS DETERMINACY OF TURN-BASED PARITY GAMES

TOWARDS THE TRANSLATION



THE INGREDIENTS WE NEED

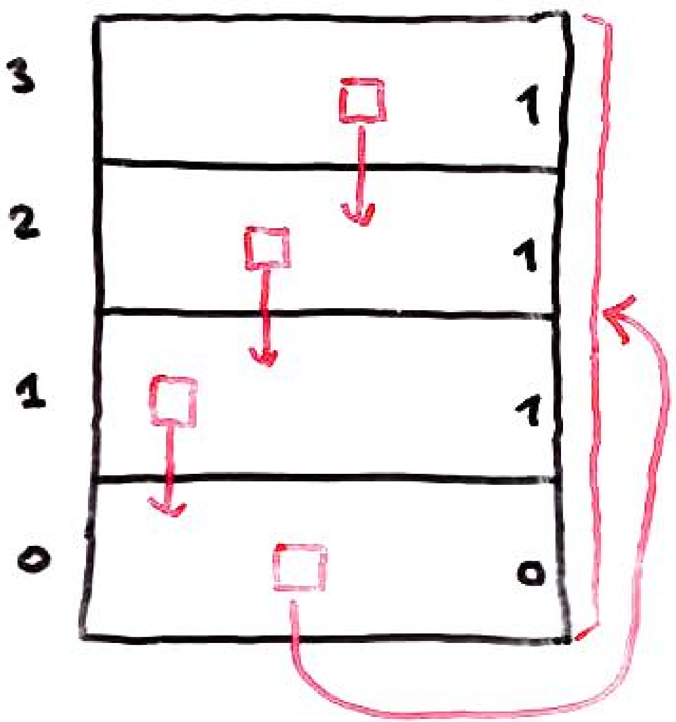
1. DEFINITION OF LOCAL WITNESSES FOR CONCURRENT GAMES
2. CORRECTNESS PROOF FOR LOCAL WITNESSES: THEY INDUCE WINNING STRATEGIES
3. PROOF THAT LOCAL WITNESSES IN $\overline{G}$ INDUCE LOCAL WITNESSES IN G

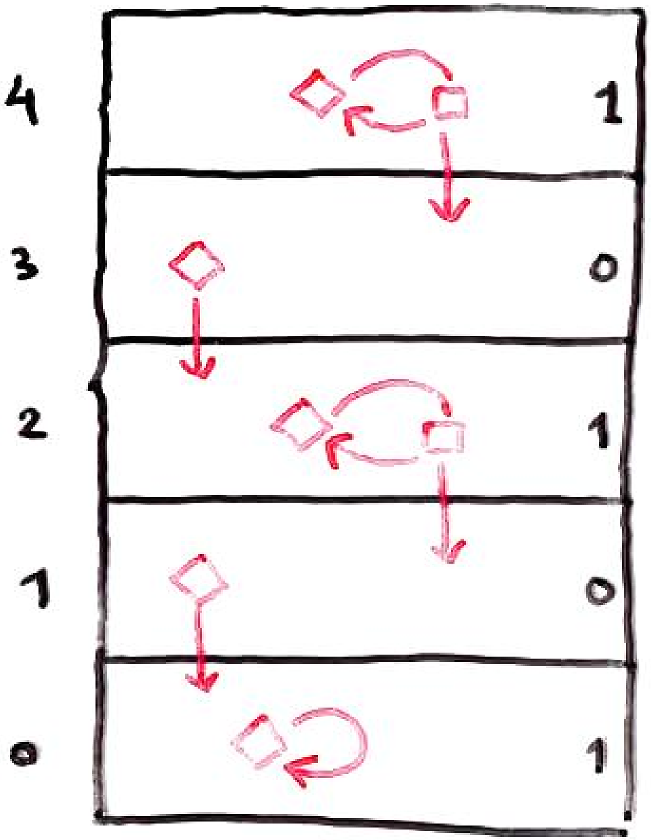
LOCAL WITNESSES (FOR BÜCHI GAMES)

PLAYER 0 | **PLAYER 1**

$\varphi : V \to \mathbb{N} \cup \{\infty\}$

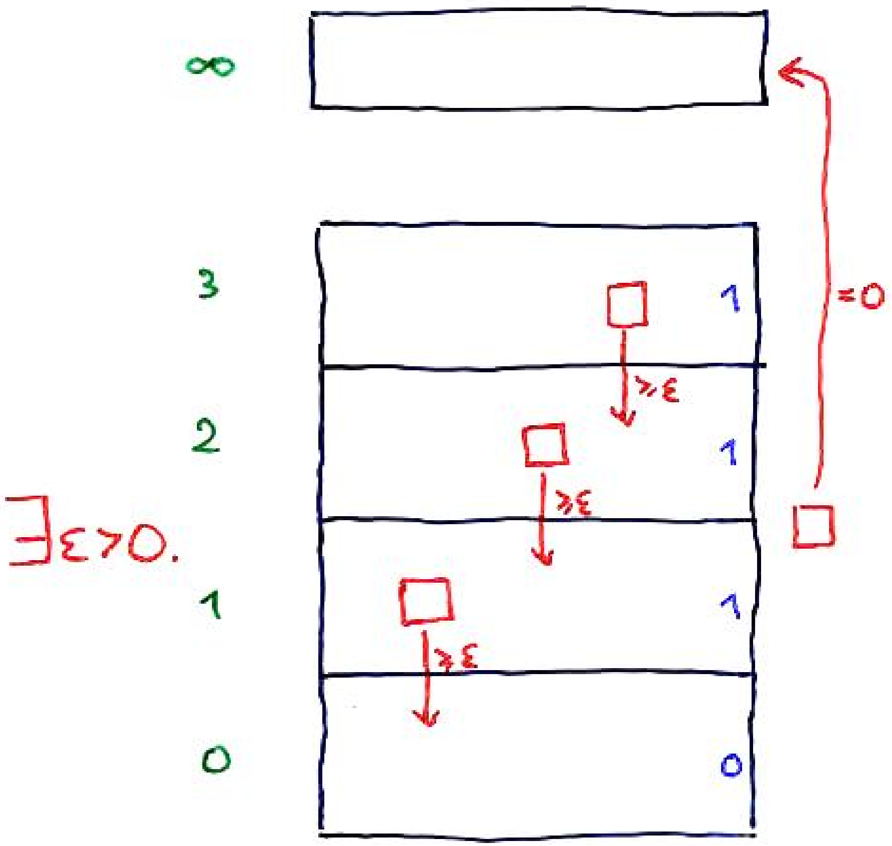
$\psi : V \to \mathbb{N} \cup \{\infty\}$





GENERALIZED LOCAL WITNESSES FOR PLAYER 0 IN $C_{as}(0,1)$ GAMES

$$\varphi : V \longrightarrow \mathbb{N} \cup \{\infty\}$$



<u>FACT</u> PLAYER 0 WINS WITH PROBABILITY 1

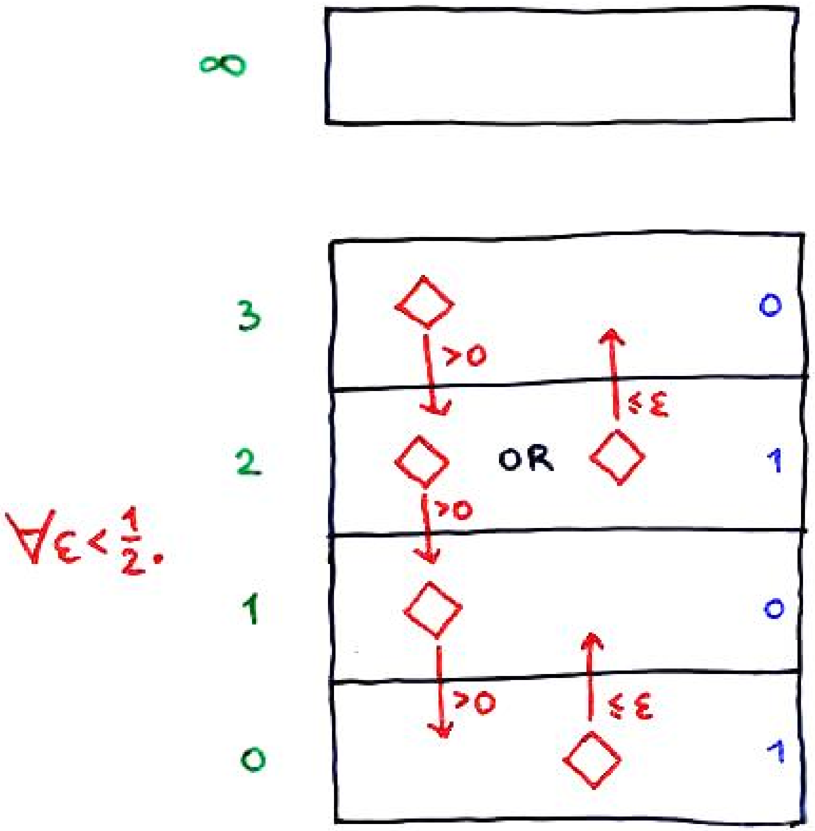
<u>PROOF</u>

R_k: EVENT THAT RANK 0 IS REACHED $\geq k$ TIMES

$$\mathbb{P}[R_\infty] = \prod_{i=0}^{\infty} \mathbb{P}[R_{i+1} \mid R_i] = 1$$

GENERALIZED LOCAL WITNESSES FOR PLAYER 1 IN $C_{as}(0,1)$ GAMES

$\Psi : V \longrightarrow \mathbb{N} \cup \{\infty\}$



FACT PLAYER 1 WINS WITH PROBABILITY > 0

PROOF (INDUCTION ON RANK)

- RANK $r = 0$

CLAIM PLAYER 1 CAN STAY IN Ψ_0 WITH PROB. $\geq 1-\varepsilon$

PROOF

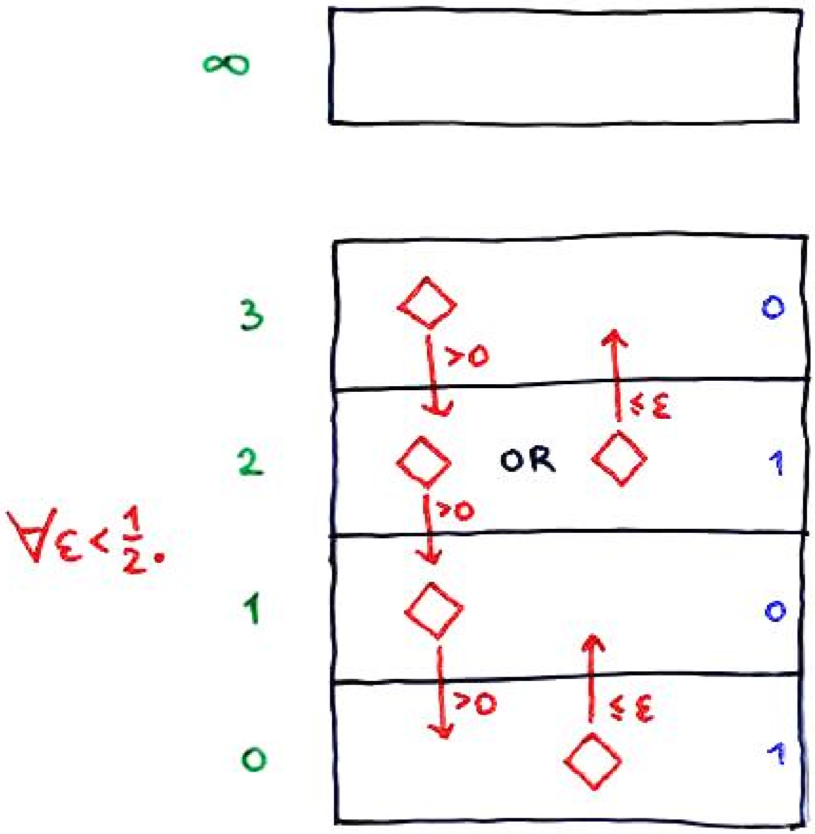
$\bar{\alpha}$: IN K-TH STEP CHOOSE α_{ε_k} WHERE $\varepsilon_k = \frac{\varepsilon}{2^k}$

PROBABILITY OF LEAVING Ψ_0:

$$\leq \sum_{i=1}^{\infty} \varepsilon_i = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon$$

GENERALIZED LOCAL WITNESSES FOR PLAYER 1 IN $C_{as}(0,1)$ GAMES

$\psi : V \longrightarrow \mathbb{N} \cup \{\infty\}$



FACT PLAYER 1 WINS WITH PROBABILITY > 0

PROOF (INDUCTION ON RANK)

- RANK r IS EVEN

$\bar{\alpha}$: IN k-TH STEP PLAY α_{ε_k}

S.T. $\mathbb{P}^{\alpha_{\varepsilon_k}\beta}[\psi_{<r}] > 0$ (*)

OR $\mathbb{P}^{\alpha_{\varepsilon_k}\beta}[\psi_{>r}] \leq \varepsilon_k$

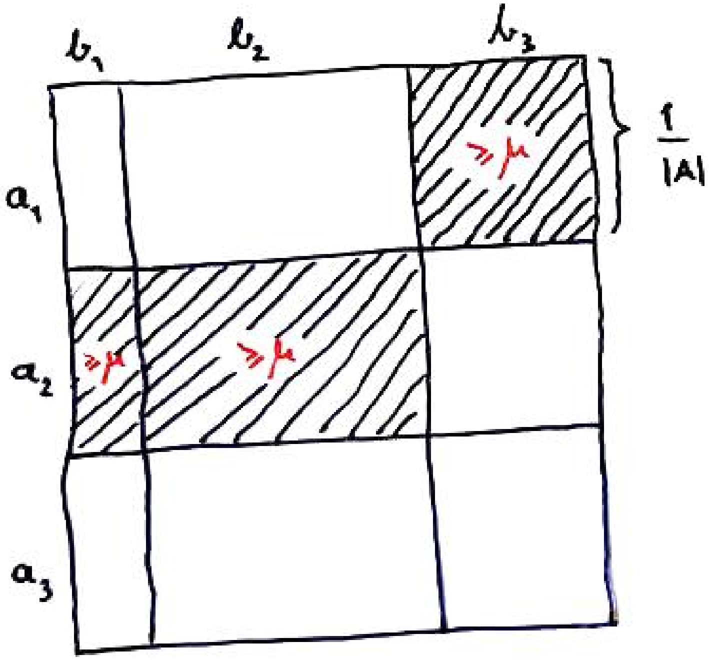
IF (*) NEVER HOLDS THEN PROBABILITY OF INCREASING RANK IS

$$\leq \sum_{i=1}^{\infty} \varepsilon_i = \varepsilon$$

TRANSLATING THE $\left[\exists \varepsilon>0.\ \overset{v}{\square}\downarrow_{\geqslant\varepsilon}\right]$ CONDITION

$$\boxed{\exists \varepsilon>0.\ \exists \alpha.\ \forall \beta.\ \mathbb{P}_v^{\alpha\beta}[\varphi_\infty]=0 \wedge \mathbb{P}_v^{\alpha\beta}[\varphi_{<v}] \geqslant \varepsilon}$$

$$\alpha : \begin{bmatrix} A \longmapsto \frac{1}{|A|} \\ \bar{A} \longmapsto 0 \end{bmatrix} \quad \text{S.T.} \quad \boxed{\forall b.\ \exists a \in A.\ \mathbb{P}_v^{ab}[\varphi_{<v}] > 0}$$



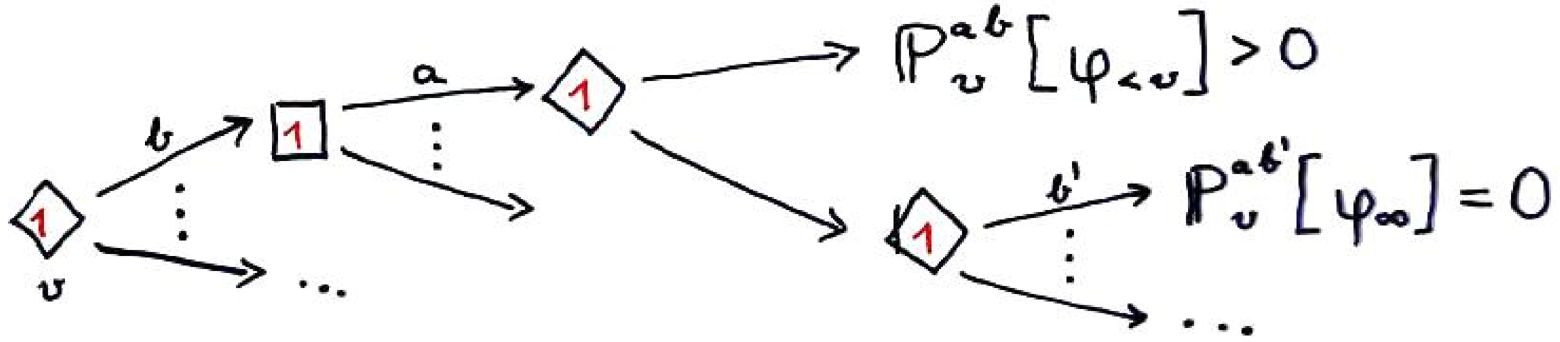
$$\Longrightarrow \mathbb{P}^{\alpha\beta}[\varphi_{<v}] \geqslant \frac{\mu}{|A|}$$

μ = THE SMALLEST NON-ZERO $\mathbb{P}_v^{ab}[w]$.

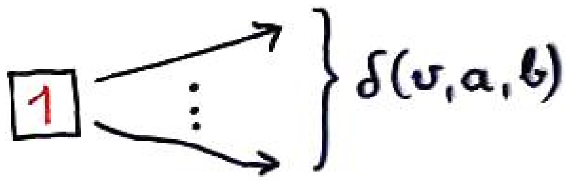
TRANSLATING THE $[\exists \varepsilon > 0. \overset{v}{\square}\downarrow_{\geq \varepsilon}]$ CONDITION

$$\boxed{\exists \varepsilon > 0.\ \exists \alpha.\ \forall \beta.\ \mathbb{P}_v^{\alpha\beta}[\varphi_\infty] = 0 \ \wedge\ \mathbb{P}_v^{\alpha\beta}[\varphi_{<v}] \geq \varepsilon}$$

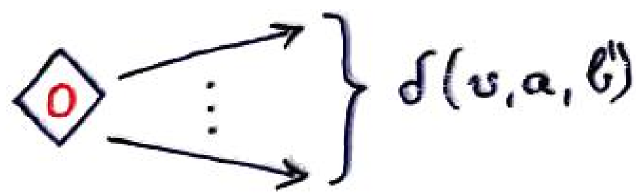
$$\forall b.\ \exists a.\ \left(\forall b'.\ \mathbb{P}_v^{ab'}[\varphi_\infty] = 0\right) \wedge\ \mathbb{P}_v^{ab}[\varphi_{<v}] > 0$$



$\mathbb{P}_v^{ab}[\varphi_{<v}] > 0$?

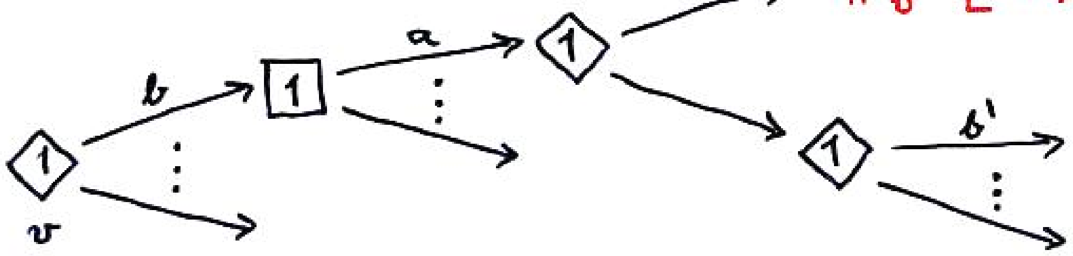


$\mathbb{P}_v^{ab}[\varphi_\infty] = 0$?



THE $[\forall \varepsilon < \frac{1}{2}.\ \Diamond\!\downarrow_{>0}$ OR $\Diamond\!\uparrow_{\le \varepsilon}]$ IS DUAL TO $[\exists \varepsilon > 0.\ \Box\!\downarrow_{\ge \varepsilon}]$

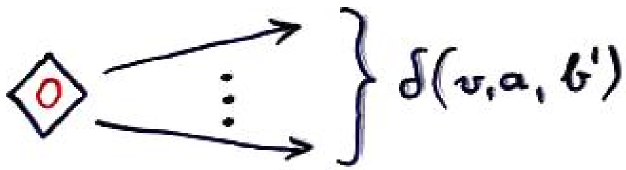
$$\boxed{\forall \varepsilon < \tfrac{1}{2}.\ \exists \beta.\ \forall \alpha.\quad \mathbb{P}_v^{\alpha\beta}[\Psi_{<v}] > 0 \ \vee\ \mathbb{P}_v^{\alpha\beta}[\Psi_{>v}] \le \varepsilon}$$



$\mathbb{P}_v^{ab}[\Psi_{>v}] = 0$

$\mathbb{P}_v^{ab'}[\Psi_{<v}] > 0$

$\mathbb{P}_v^{ab}[\Psi_{>v}] = 0$



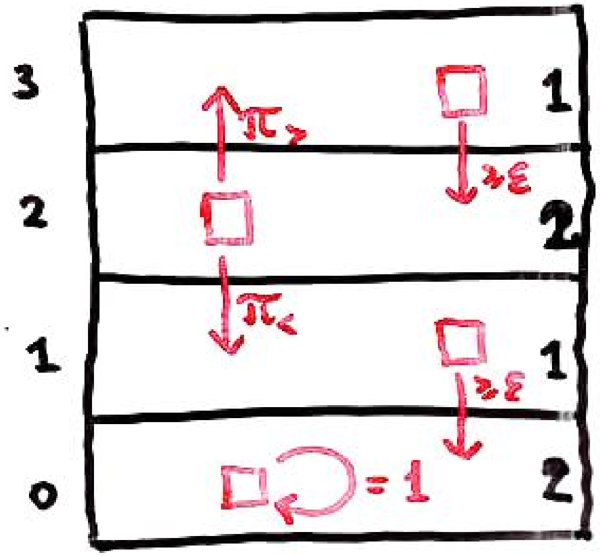
$\mathbb{P}_v^{ab'}[\Psi_{<v}] > 0$

GENERALIZED LOCAL WITNESSES FOR CONCURRENT CO-BÜCHI GAMES

PLAYER 0

$$\varphi : V \to \mathbb{N} \cup \{\infty\}$$

$\exists \varepsilon > 0.$



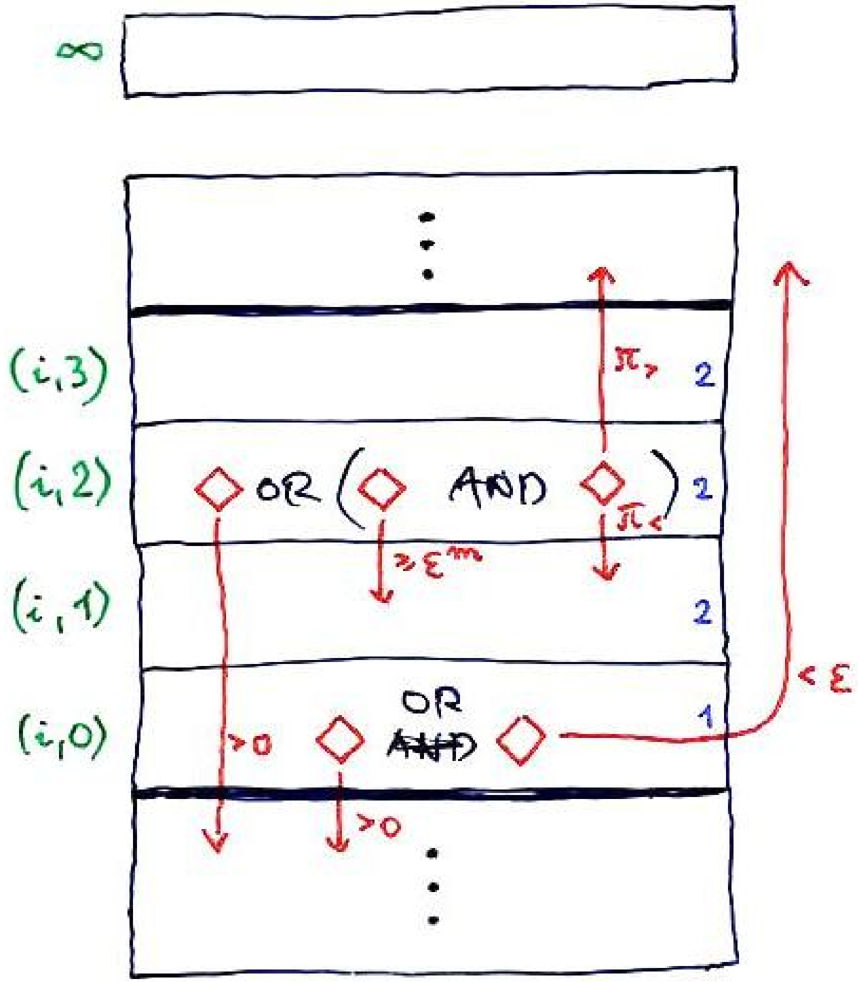
$\pi_< \geq \varepsilon \cdot \pi_>$

PLAYER 0 WINS W/PROBABILITY 1

GENERALIZED LOCAL WITNESSES FOR CONCURRENT CO-BÜCHI GAMES

PLAYER 1 $\quad \Psi : V \longrightarrow \mathbb{N}^2 \cup \{\infty\}$

$\boxed{\exists m \geq 1.\ \forall \varepsilon \leq \tfrac{1}{2}.}$

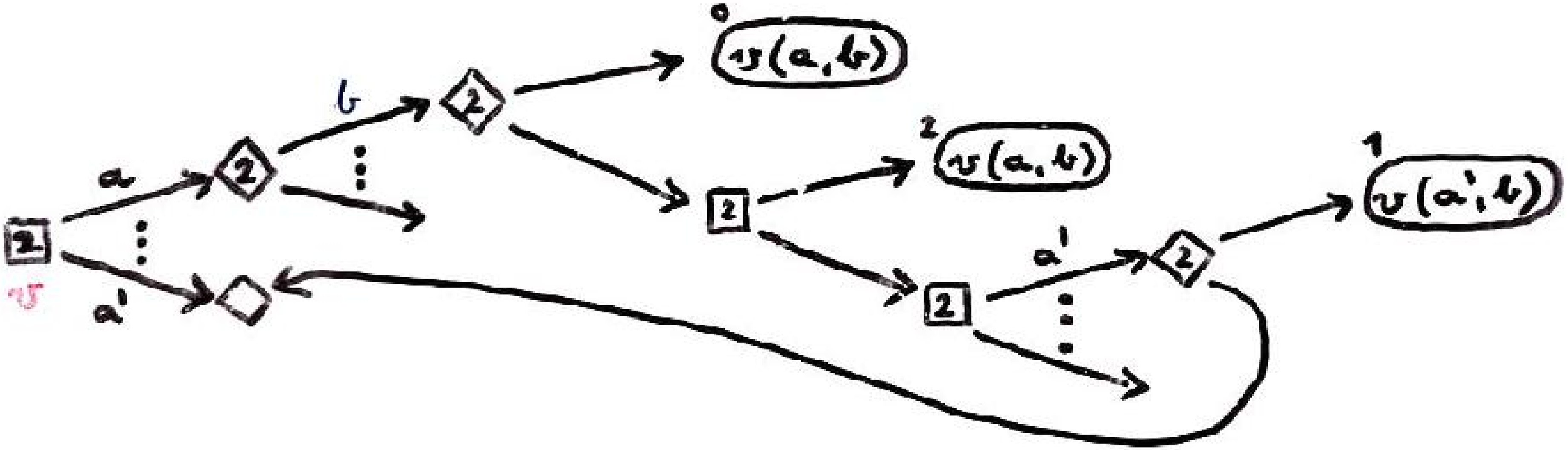


$\boxed{\pi_> \leq \varepsilon \cdot \pi_<}$

TRANSLATING THE $\left[\overset{\uparrow \pi_>}{\underset{\downarrow \pi_<}{\Box\, \varepsilon}} \,\Big/\, \underset{\downarrow >0}{\Diamond} \text{ OR } \left(\underset{\downarrow \geq \frac{1}{\delta^{m}}}{\Diamond} \text{ AND } \overset{\uparrow}{\underset{\downarrow}{\Diamond}}\, \delta \right) \right]$ CONDITION

PLAYER 0: $\exists \varepsilon > 0.\ \exists \alpha.\ \forall \beta.\ \mathbb{P}_v^{\alpha\beta}[\varphi_{<\infty}] = 1 \ \wedge$

$\wedge\ \mathbb{P}_v^{\alpha\beta}[\varphi_{<v}] \geq \varepsilon \cdot \mathbb{P}_v^{\alpha\beta}[\varphi_{>v}]$



PLAYER 1: $\exists m.\ \forall \varepsilon \leq \frac{1}{2}.\ \exists \beta.\ \forall \alpha.\ \mathbb{P}_v^{\alpha\beta}[\psi^{0}_{<v}] > 0 \ \vee$

$\vee \left(\mathbb{P}_v^{\alpha\beta}[\psi_{<v}] \geq \varepsilon^{m} \ \wedge\ \mathbb{P}_v^{\alpha\beta}[\psi^{0}_{>v}] \leq \varepsilon \cdot \mathbb{P}_v^{\alpha\beta}[\psi_{<v}] \right)$

TO BE DONE

- TRANSLATIONS OF $C(1,d)$ AND $C(0,d)$ GAMES TO TURN-BASED $(0,d)$ GAMES

  $\Rightarrow$ IMPROVING COMPLEXITY FROM $O(n^d)$ TO $O(n^{d/2})$

- SAME FOR LIMIT-SURE WIN

- GENERALIZATION TO QUANTITATIVE SOLUTION OF CONCURRENT PROBABILISTIC GAMES ?

  $\Rightarrow$ ALGORITHMS ?