Predicates and Propositions

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What Is A Predicate?
Strictly, the meaning of a certain kind of sentence, but often used (conveniently) to refer to the sentence itself.

Example: “Student S1 is enrolled on course C1.”
Note that the meaning is language-independent; the sentence is not!
Note also that we get a very similar sentence, with very similar meaning, if we just change either of the designators, S1 and C1 (e.g., replace C1 by C2).

Recommended Book
Highly recommended if you like this sort of thing (but definitely not a course requirement):

Wilfrid Hodges: “Logic”.

What Kind of Sentence?
A sentence having the grammatical form of a statement — something that can be believed, or not believed.

In English, if “Is it true that \(x\)” is a grammatical English question, then \(x\) is a statement (having the form of a declarative sentence).

Might need to paraphrase \(x\). E.g. (from Shakespeare):
“O for a muse of fire” ≡ “I wish for a muse of fire”.
“To be or not to be, that is the question” ≡ “The question is whether to be or not to be”

Some Counterexamples
Sentences that are not declarative:
• “Let’s all get drunk.”
• “Will you marry me?”
• “Please pass me the salt.”
• “If music be the food of love, play on.”

Some Examples
Sentences that are declarative (and so denote predicates):
• “Student S1 is enrolled on course C1.”
• “I will marry you.”
• “The king of France is bald”
• “\(2 + 2 = 5\)”
• “\(a + \bar{a} = c\)”
• “Student \(s\) is enrolled on course \(c\)”
• “\(P(x)\)” (notation for the general form)
Deriving Predicates from Predicates (1)

Substitution: of a designator for a parameter
Given an \( n \)-adic predicate, yields an \((n-1)\)-adic predicate.
E.g., in \( "a < b" \) substitute 10 for \( b \) to give \( "a < 10" \).
Now substitute 5 for \( a \), and we get \( "5 < 10" \), a proposition.

Instantiation: substitution of all the parameters, yielding a proposition.

Deriving Predicates from Predicates (2)

The familiar logical operators:

conjunction: “Student \( s \) is enrolled on course \( c \) and \( s \) is called name.”

disjunction: “\( a < b \) or \( c < d \)”

negation: “It is not the case that I will marry you.”

Deriving Predicates from Predicates (3)

Conditionals:

implication: “If you ask me nicely, then I will marry you.”

only if: “I will marry you only if you ask me nicely.”

biconditional: “I will marry you if and only if you ask me nicely.” (equivalence)

Deriving Predicates from Predicates (4)

Quantifiers:

existential: “There exists \( s \) such that \( s \) is a student and \( s \) is enrolled on course \( c \).” (≡ “At least one student is enrolled on course \( c \).”)

universal: “For all \( s \), if \( s \) is a student then \( s \) is enrolled on course \( c \).” (≡ “All the students are enrolled on course \( c \).”)

Quantification, like substitution, binds a parameter.

Intension and Extension

Of a predicate:

Intension: its meaning (loosely speaking).

Extension: all the instantiations that are (believed to be) true.

The concept of extension is crucially important in relational theory. Note that it is a set of propositions. Alternatively, it is a single proposition formed by connecting all the members of that set together using “and”.

Note in passing that the extension of a niladic predicate is either itself (if it is true) or the empty set (if it is false).

Sets

Let \( P(x) \) be a predicate. If object \( a \) is such that \( P(a) \) is true, then \( a \) is said to satisfy \( P \). And \( P(x) \) is called a membership predicate for the set consisting of all such objects \( a \).

Example: “\( x \) is an integer such that \( 1 < x < 4 \)”

“\( x \) is an integer such that \( 1 < x < 4 \)” is a membership predicate for the set consisting of the elements 2 and 3, denoted by the expression \{ 2, 3 \} (an enumeration).

This set is also denoted by \{ \( x : x \in \mathbb{Z} \) and \( 1 < x \) and \( x < 4 \) \}.
The Language of Sets (1)

Let \( A \) and \( B \) be sets with membership predicates \( P_A(x) \) and \( P_B(x) \), respectively. Let \( a \) be an element. Then we have the following comparisons:

- **membership:** \( a \in A \) (\( a \) is a member of \( A \))
- **containment:** \( B \subseteq A \) (\( B \) is a subset of \( A \))
  
  - \( A \subseteq B \) (\( A \) is a superset of \( B \))
  
  - \( B \subset A \) (\( B \) is a proper subset of \( A \))
  
  - \( A \supset B \) (\( A \) is a proper superset of \( B \))
- **equality:** \( A = B \) (\( A \subseteq B \) and \( B \subseteq A \))
- **disjointness:** \( A \) and \( B \) are disjoint (have no members in common)

The Language of Sets (2)

And the following operations on sets that yield sets:

- **union:** \( A \cup B = \{ x : x \in A \text{ or } x \in B \} \) (disjunction)
- **intersection:** \( A \cap B = \{ x : x \in A \text{ and } x \in B \} \) (conjunction)
- **complement:** \( (A) = \{ x : \neg x \in A \} \) (negation)
- **difference:** \( A - B = \{ x : x \in A \text{ and } \neg x \in B \} \)

EXERCISES

Assume that the membership predicate for the following relation is “Student StudentId, named Name, is enrolled on course CourseId.”

<table>
<thead>
<tr>
<th>StudentId</th>
<th>Name</th>
<th>CourseId</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>Anne</td>
<td>C1</td>
</tr>
<tr>
<td>S1</td>
<td>Anne</td>
<td>C2</td>
</tr>
<tr>
<td>S2</td>
<td>Boris</td>
<td>C1</td>
</tr>
<tr>
<td>S3</td>
<td>Cindy</td>
<td>C3</td>
</tr>
<tr>
<td>S4</td>
<td>Devinder</td>
<td>C1</td>
</tr>
</tbody>
</table>

(The exercises are in the Notes)