DECIMAL REPRESENTATION OF REAL NUMBERS

Recall that if \( |r| < 1 \) then the geometric series (Note 17) \( \sum_{n \geq 0} r^n \) has sum \( 1/(1 - r) \), i.e.,

\[
1 + r + r^2 + r^3 + \cdots = \frac{1}{1 - r}.
\]

If \( a_1, a_2, \cdots \) is a sequence of decimal digits, so that each \( a_i \) belongs to \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \), then \( .a_1a_2a_3\cdots \) denotes the real number

\[
\frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots.
\]

To see why this sum exists, note that we have \( a_n/10^n \leq 9/10^n \) for each \( n \). Now the series \( \sum 1/10^n \) converges (\( r = 1/10 \) above), so \( \sum 9/10^n \) converges, and it follows from the Comparison Test (Note 17) that \( \sum a_n/10^n \) converges.

It is clear that a terminating decimal \( .a_1a_2\cdots a_n \) represents a rational number since

\[
.a_1a_2\cdots a_n = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}.
\]

The general form of a repeating decimal is

\[
.a_1a_2\cdots a_mb_1b_2\cdots b_nb_1b_2\cdots b_n\cdots,
\]

which is denoted by \( .a_1a_2\cdots a_m\dot{b}_1\dot{b}_2\cdots \dot{b}_n \).

Repeating decimal as rational number. A repeating decimal represents a rational number and can be expressed as a quotient of two integers by summing an appropriate geometric series as the following problem illustrates.

Problem. Express the repeating decimal 0.59\( \dot{1} 0 \dot{2} \) as the quotient of two integers.
Solution. We have
\[
0.59102102 \cdots = \frac{59}{100} + \frac{102}{10^5} + \frac{102}{10^8} + \frac{102}{10^{11}} + \cdots
\]
\[
= \frac{59}{100} + \frac{102}{10^5} \left( 1 + \frac{1}{10^3} + \frac{1}{10^6} + \cdots \right)
= \frac{59}{100} + \frac{102}{10^5} \left( \frac{1}{1-1/10^3} \right)
= \frac{59}{100} + \frac{102}{1000} \frac{1}{999} = \frac{59 \times 999 + 102}{99900}
= \frac{59043}{99900}.
\]

A non-zero number with a terminating decimal representation can also be written as a non-terminating decimal since we can introduce an infinity of repeating 9’s. For example \(0.5 = 0.4\dot{9}\) and \(1 = 0.\dot{9}\).

Problem. Show that \(0.4\dot{9} = 1/2\).

Solution. From the definition of a repeating decimal we have
\[
0.4\dot{9} = \frac{4}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots = \frac{4}{10} + \frac{9}{10^2} \left( \frac{1}{1-1/10} \right)
= \frac{4}{10} + \frac{9}{100-10} = \frac{4}{10} + \frac{1}{10} = \frac{1}{2}.
\]

Irrational numbers such as \(\sqrt{2}, \pi\) and \(e\) are represented by decimal expansions which do not terminate or repeat.

The decimal expansion of an arbitrary real number, pictured at \(x\) on the real line, can be computed to any desired accuracy in the following way. If \(x\) is not an integer then it lies between two consecutive integers \(a_0\) and \(a_0 + 1\). Now divide the interval between these two integers into 10 equal parts. If \(x\) is not on one of the subdivision points then it must lie between two consecutive subdivision points. Hence we have
\[
a_0 + \frac{a_1}{10} < x < a_0 + \frac{a_1 + 1}{10},
\]
where \(a_1 \in \{0, 1, \ldots, 9\}\). Dividing this new interval containing \(x\) into 10 equal subintervals (each of length \(1/10^2\)) gives an inequality of the form
\[
a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} < x < a_0 + \frac{a_1}{10} + \frac{a_2 + 1}{10^2}.
\]
Continuing in this way, we obtain two sequences of finite decimals each converging to \(x\), one increasing from below and the other decreasing from above. By repeating this procedure sufficiently many times, the decimal expansion of \(x\) can be obtained to any desired degree of accuracy.
Rational numbers as repeating decimals. We have seen that terminating and repeating decimal expansions represent rational numbers. Now we show that, conversely, every rational number has a terminating or repeating expansion.

The decimal expansion of a rational number $r/s \in (0, 1)$ can be computed using the division algorithm (generations of schoolchildren used to call this process long division).

Dividing $10r$ by $s$ we have

$$10r = sq_1 + r_1$$

where $0 \leq r_1 < s$.

Repeatedly multiplying remainders by 10 and dividing by $s$ produces sequences $q_1, q_2, \ldots$ and $r_1, r_2, \ldots$ with

$$10r_i = sq_{i+1} + r_{i+1}$$

where $0 \leq r_i < s$.

Hence

$$\frac{r_i}{s} = \frac{q_{i+1}}{10} + \frac{1}{10} \frac{r_{i+1}}{s}$$

for each $i$.

Now for any $n$ we have

$$\frac{r}{s} = \frac{q_1}{10} + \frac{1}{10} \frac{r_1}{s}$$

$$= \frac{q_1}{10} + \frac{1}{10} \left( \frac{q_2}{10} + \frac{1}{10} \frac{r_2}{s} \right) = \frac{q_1}{10} + \frac{q_2}{10^2} + \frac{1}{10^2} \frac{r_2}{s}$$

$$\vdots$$

$$= \frac{q_1}{10} + \frac{q_2}{10^2} + \cdots + \frac{q_n}{10^n} + \frac{1}{10^n} \frac{r_n}{s}.$$

Also each $q_i$ is one of the digits $0, 1, \ldots, 9$ since

$$0 \leq 10r_i < 10s \implies 0 \leq sq_{i+1} + r_{i+1} < 10s$$

$$\implies 0 \leq sq_{i+1} < 10s - r_{i+1} \leq 10s - s$$

$$\implies 0 \leq q_{i+1} \leq 9.$$

If at some stage in the division we obtain a remainder $r_n = 0$ then

$$\frac{r}{s} = \frac{q_1}{10} + \frac{q_2}{10^2} + \cdots + \frac{q_n}{10^n} = 0.q_1q_2 \ldots q_n.$$
Otherwise, since there are only finitely many values for the remainders (which must lie between 0 and \( s - 1 \)), the sequence \( r_1, r_2, \ldots \) must contain repetitions.

Suppose that \( r_n \) is the first repeated remainder, so that \( r_n = r_m \) for some \( m < n \). Dividing \( 10r_n \) by \( s \) is the same as dividing \( 10r_m \) by \( s \), so the quotient and remainder are the same as before.

Hence the process repeats and we have \( q_{n+1} = q_{m+1} \), \( q_{n+2} = q_{m+2} \ldots \), which gives

\[
\frac{r}{s} = 0.q_1q_2\ldots\dot{q}_m\ldots\dot{q}_n.
\]

**Problem.** Find the decimal expansion of \( \frac{5}{14} \).

**Solution.**

\[
\begin{array}{c|ccccccccc}
& 0 & . & 3 & 5 & 7 & 1 & 4 & 2 & 8 & 5 \\
\hline
14 & 5 & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
& 4 & 2 & & & & & & & & \\
& 8 & 0 & & & & & & & & \\
& 7 & 0 & & & & & & & & \\
& 1 & 0 & 0 & & & & & & & \\
& 9 & 8 & & & & & & & & \\
& 2 & 0 & & & & & & & & \\
& 1 & 4 & & & & & & & & \\
& 6 & 0 & & & & & & & & \\
& 5 & 6 & & & & & & & & \\
& 4 & 0 & & & & & & & & \\
& 2 & 8 & & & & & & & & \\
& 1 & 2 & 0 & & & & & & & \\
& 1 & 1 & 2 & & & & & & & \\
& 8 & 0 & & & & & & & & \\
& 7 & 0 & & & & & & & & \\
\end{array}
\]

Hence \( \frac{5}{14} = 0.3\dot{5}714\dot{2}\dot{8} \) since the remainder 8 is repeated.

**Non-denary bases.** We can also represent real numbers in bases other than ten. In base two, if \( b_1, b_2, b_3, \ldots \) are binary digits (0 or 1), then \( .b_1b_2b_3\ldots \) represents the number

\[
\frac{b_1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \cdots.
\]

For example we have

\[
1/2 = 0.5_{\text{ten}} = 0.1_{\text{two}} \quad \text{and} \quad 1/4 = 0.25_{\text{ten}} = 0.01_{\text{two}}.
\]
Problem. Find the binary representation of the base-ten fraction $1/3$.

Solution. We divide 3 into 1 in base two i.e. divide 11 into 1.

\[
\begin{array}{c|c c c}
& 0 & 1 & 0 \\
\hline
11 & 1 & 0 & 0 \\
\hline
& & 1 & 1 \\
\hline
& & 1 & 0 & 0 \\
\end{array}
\]

so $(1/3)_{\text{ten}} = 0.010101\cdots = 0.\dot{0}\dot{1}$ since the remainder 1 is repeated.

ABSTRACT

Content terminating decimals, repeating decimal as rational number, rational numbers as repeating decimals, non-denary bases

In this Note we consider decimal numbers in terms of an associated geometric series. We also study the relationship between repeating decimals and rational numbers and vice-versa. We finish by considering real numbers expressed in non-decimal bases such as binary.

History

The regular use of the decimal point appears to have been introduced about 1585, but the occasional use of decimal fractions can be traced back as far as the 12th century.

Muhammad ibn-Musa Khwarizmi, [c. 780-c. 850] was a Persian mathematician who wrote a book on algebra, from part of whose title (al-jabr) comes the word 'algebra', and a book in which he introduced to the West the Hindu-Arabic decimal number system. The word 'algorithm' is a corruption of his name. He compiled astronomical tables and was responsible for introducing the concept of zero into Arab mathematics.

Simon Stevinus [c. 1548-1620] was a Flemish scientist who, in physics, developed statics and hydrodynamics; he also introduced decimal notation into Western mathematics.

John Napier [1550-1617], 8th Laird of Merchiston, was a Scottish mathematician who invented logarithms in 1614 and 'Napier’s bones', an early mechanical calculating device for multiplication and division.

Napier arranged his logarithmic calculations in convenient tables which evolved into what generations of schoolchildren came to know as ‘log tables’. These have now been superseded by the use of hand calculators and digital computers.

It was Napier who first used and then popularised the decimal point to separate the whole part from the fractional part of a number.