

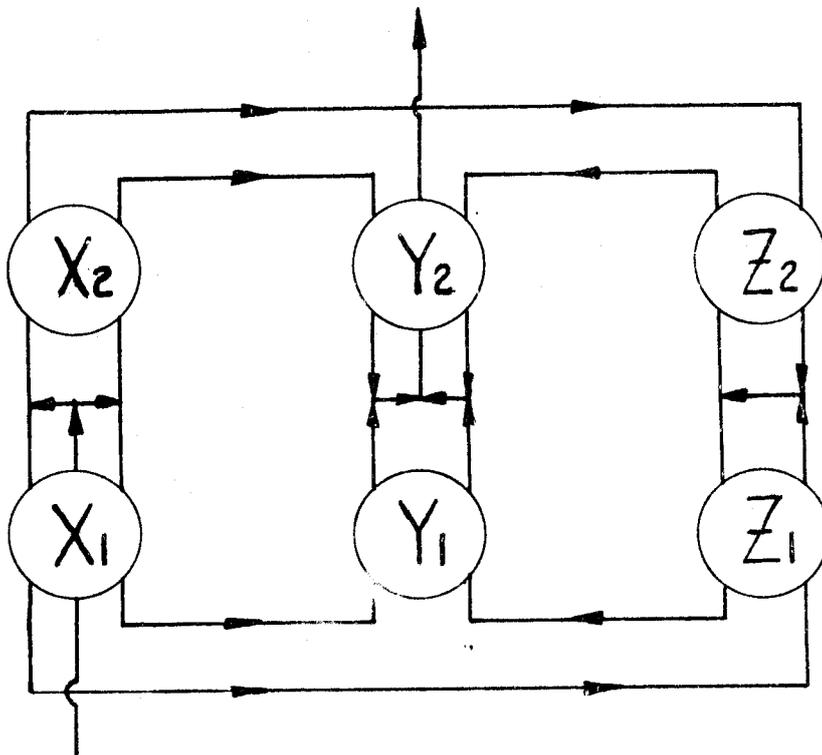
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THEORY OF COMPUTATION REPORT

NO. 5

A Complete Natural Deduction System
for the
Relational Calculus

WILLIAM W. WADGE



0. Introduction

A relational calculus is a formal system in which relation is the fundamental concept. The simplest relational calculus, that of ordinary binary relations, was introduced by Tarski in [4]. Tarski's system is essentially an algebra in which the operations are the usual Boolean operations (on sets of ordered pairs) together with the two special operations converse (denoted by \checkmark) and composition (denoted by ";" or by juxtaposition). For any binary relations P and Q

$$\checkmark P = \{ \langle x, y \rangle \mid \langle y, x \rangle \in P \}$$

$$P;Q = \{ \langle x, y \rangle \mid \langle x, v \rangle \in P \text{ and } \langle v, z \rangle \in Q \text{ for some } v \}.$$

The unusual feature of the relational calculi is the manner in which it is possible to make assertions about relations without referring to points. For example, the inclusion $R;R \subseteq R$ holds iff R is transitive, and $R = \checkmark R$ iff R is symmetric. In a sense the relational calculi are to the concept of relation as category theory is to the concept of function.

Interest in relational calculi has been spurred recently by the search for suitable mathematical systems in which to express the semantics of programming languages. Given a machine of the appropriate type we can associate with each program or part of a program the binary relation which characterizes its input-output behaviour: as de Roever says in [3,p.1]

"the collection of all pairs [consisting] of an initial state of the memory, for which the program terminates, and its corresponding final state of the memory."

For example, if relations R_1 and R_2 represent programs P_1 and P_2 respectively, then the relation $R_1;R_2$ represents the program "first P_1 then P_2 ", and the equation

$$(R_1;R_2);U = U$$

(where U is the universal relation) asserts that the resulting program halts for every initial state.

For this approach to work it is necessary to consider not just simple binary relations (between points and points) but also typed binary relations (between sequences of points and sequences of points). For example, the system described by Hitchcock and Park in [1] has for each natural numbers n and m relations of type $n \rightarrow m$ which under a standard interpretation would denote subsets of $D^n \times D^m$.

Certain difficulties arise when it comes to constructing proof systems for relational calculi. The usual approach is to axiomatize the notion of a relational algebra. Tarski gave as his axioms the usual Boolean identities plus

$$T1: (X;Y);Z = X;(Y;Z)$$

$$T2: \check{\check{X}} = X$$

$$T3: (X;Y)\check{\check{}} = \check{\check{Y}};\check{\check{X}}$$

$$T4: X;E = X$$

$$T5: (X;Y) \cap Z = \Omega \rightarrow (Y;\check{\check{Z}}) \cap \check{\check{X}} = \Omega$$

where U is the universal relation, Ω is its complement and E is equality. De Roever's axioms are basically a typed version of Tarski's with some added rules for projections. Neither system has any non-logical rules of inference, i.e. any rules other than substitution, replacement, modus ponens etc. A model of these axioms is a relation algebra, and a relation algebra is standard iff it is isomorphic to an algebra obtained by taking a collection of relations on some domain and interpreting the relation constants and operations in the usual way (there are nonstandard relation algebras).

The trouble with at least Tarski's axioms is that they are incomplete: Lyndon in [2] showed that the inclusion

$$X_1 X_2 \cap Y_1 Y_2 \cap Z_1 Z_2 \subseteq X_1 (\check{\check{X}}_1 Y_1 \cap X_2 \check{\check{Y}}_2 \cap (\check{\check{X}}_1 Z_1 \cap X_2 \check{\check{Z}}_2)) (\check{\check{Z}}_1 Y_1 \cap Z_2 \check{\check{Y}}_2) Y_2,$$

though true in every standard relation algebra, does not follow from Tarski's axioms. The theory of standard relation algebras is nevertheless axiomatizable simply because it is enumerable: every formula φ of the relational calculus

can be translated into a formula φ' of the predicate calculus in such a way that models of φ' correspond to standard models of φ and vice versa. The difficulty lies in finding a sufficiently elegant axiomatization. It is possible that de Roever's axioms are enough, for he has proved Lyndon's formula by using projections, but the question has so far not been resolved.

The most serious drawback of the algebraic approach is the tendency of proofs of even simple assertions to be complex and unnatural. Consider, for example, the proof on [3, p37] of the important lemma

$$X;Y \cap Z = X;(\check{X};Z \cap Y) \cap Z .$$

the proof is nontrivial, uses the somewhat cryptic axiom T5 and has as a substep the verification of

$$\overline{(X;\check{X};Z)} \cap Z = \Omega .$$

In this work we attempt to remedy this defect by constructing a system of natural deduction for the relational calculus (typed or untyped) which is complete for standard relation algebras, i.e. in which a formula is provable iff it is true in every standard relation algebra. The system is natural in the sense that rules used in proving an assertion are determined by the structure of the assertion. The only disadvantage of the method is that it expands the formalism by adding point variables.

1. Informal Explanation

To understand how the system works, consider how one might prove the lemma given above in a nonalgebraic manner. The equality is actually two inclusions, the most difficult being

$$X;Y \cap Z \subseteq X;(\check{X};Z \cap Y) \cap Z .$$

In terms of points, we must show that for any a and b , if $aX;Y \cap Zb$ (i.e. if $\langle a, b \rangle \in X;Y \cap Z$) then $aX;(\check{X};Z \cap Y) \cap Zb$. The assertion that $aX;Y \cap Zb$ is equivalent to the assertion that $aX;Yb$ and that aZb , and the assertion that $aX;Yb$ is in turn equivalent to the assertion that aXc and that cYb for some c .

Thus we must derive $aX;(\check{X};ZNY)NZb$ from the assumption that aXc , cYb and aZb hold for some c . Now the former assertion is equivalent to the assertion that both $aX;(\check{X};ZNY)b$ and aZb and since aZb is an assumption, we need only derive $aX;(\check{X};ZNY)b$. This is in turn equivalent to aXv and $v\check{X};ZNYb$ for some v . Since we have aXc as an assumption it will be enough to prove $c\check{X};ZNYb$. Next we break $c\check{X};ZNYb$ down into cYb (an assumption) and $c\check{X};Zb$ which in turn follows from $c\check{X}w$ and wZb for any w . But both aXc and aZb are assumptions and so the proof is complete.

The important point is that in formalizing this type of proof it is not necessary to add the quantifiers and logical connectives of first order logic. For example, although in connection with ";" we used the phrases "for some" and "for any" in fact only the following two rules are needed: (1) to derive $aR;Sv$ derive aRv and vSb for some v , and (2) to derive a result from some assumptions plus $aR;Sb$ derive the result from the same assumptions plus aRv and vSb for any variable v not appearing in the result or in the other assumptions.

2. Formalism

Our formal system consists of (1) a language whose formulas are of the form xRy with R a relation expression and x and y point variables; (2) a collection of interpretations corresponding to standard relation algebras; and (3) a relation " \vdash " of deducibility generated by a set of natural deduction rules. The completeness result states that a formula F can be derived from a set Γ of formulas iff every interpretation which makes every formula in Γ true also makes F true.

2.1 Syntax

The alphabet of the formal language contains the symbols

- | | |
|--|-------------------------------|
| (i) a, b, c, \dots | a set V of point variables; |
| (ii) A, B, C, \dots | relation variables; |
| (iii) Ω, U, E | relation constants; |
| (iv) $\cup, \cap, ;, \check{}, \check{}$ | relation operations. |

A term, for example " $A;B\cap C$ ", is an expression formed in the usual way from the relation constants, variables and expressions. A formula, for example " $dA;B\cap Ca$ ", is a word of the form xRy with R a term and x and y point variables. The negation of the formula xRy is the formula $x\bar{R}y$. Strictly speaking the set of formulas depends on the set V of point variables and at times we will indicate this dependence by referring to " V -formulas". Note that the symbol " \sqsubseteq " is not in the alphabet; instead, we define " $R \subseteq S$ " to be the formula " $aU;(R\cap\bar{S});Ua$ ".

2.2 Semantics

For any nonvoid set D and any set V of point variables a (V,D) -interpretation \mathcal{A} is a map which assigns to each term a subset of $D \times D$ and to each point variable an element of D . The map \mathcal{A} must be consistent with the usual interpretation of the relation operations and constants, e.g. $\mathcal{A}(R\cap S)$ must be the intersection of $\mathcal{A}(R)$ and $\mathcal{A}(S)$, and $\mathcal{A}(\Omega)$ must be the empty set. Note that an interpretation is therefore determined by its restriction to the point and relation variables.

If \mathcal{A} is an interpretation, x and y are point variables and R is a term, then $\mathcal{A} \models xRy$ (\mathcal{A} satisfies xRy , or xRy is true in \mathcal{A}) iff $\langle \mathcal{A}(x), \mathcal{A}(y) \rangle$ is in $\mathcal{A}(R)$. If Γ is a set of formulas and F is a formula then $\mathcal{A} \models \Gamma$ iff $\mathcal{A} \models G$ for every formula G in Γ , and $\Gamma \models F$ iff $\mathcal{A} \models \Gamma$ implies $\mathcal{A} \models F$ for every interpretation \mathcal{A} .

2.3 Rules of Inference

The rules of inference of our system are the introduction and elimination rules given on the next page together with the rule $F \vdash F$. In these rules F and G are arbitrary formulas; x, y and v are arbitrary variables; Γ is an arbitrary set of formulas and R and S are arbitrary terms. In $(;E)$ " v " is assumed not to occur in any formula in Γ and in $(\sqsubseteq I)$ " x " and " y " are assumed not to occur in any formula in Γ . Formally speaking the relation " \vdash " between sets of formulas and formulas is the least one for which these rules are valid.

Rules of Inference

$$(AI) \frac{\Gamma \vdash F}{\Gamma, G \vdash F}$$

$$(AE) \frac{\Gamma \vdash G \quad \Gamma, G \vdash F}{\Gamma \vdash F}$$

$$(UI) \frac{xRy \quad xSy}{xRUSy}$$

$$(UE) \frac{\Gamma, xRy \vdash F \quad \Gamma, xSy \vdash F \quad \Gamma \vdash xRUSy}{\Gamma \vdash F}$$

$$(\cap I) \frac{xRy \quad xSy}{xR\cap Sy}$$

$$(\cap E) \frac{xR\cap Sy}{xRy \quad xSy}$$

$$(\neg I) \frac{\Gamma, xRy \vdash a\Omega a}{\Gamma \vdash xRy}$$

$$(\neg E) \frac{\Gamma, xRy \vdash F \quad \Gamma, x\bar{R}y \vdash F}{\Gamma \vdash F}$$

$$(\Omega I) \frac{xRy \quad x\bar{R}y}{a\Omega a}$$

$$(\Omega E) \frac{a\Omega a}{F}$$

$$(UI) \frac{}{xUy}$$

$$(UE) \frac{\Gamma, xUy \vdash F}{\Gamma \vdash F}$$

$$(\sim I) \frac{xRy}{y\check{R}x}$$

$$(E) \frac{y\check{R}x}{xRy}$$

$$(;I) \frac{xRv \quad vSy}{xR;Sy}$$

$$(;E) \frac{\Gamma, xRv, vSy \vdash F \quad \Gamma \vdash xR;Sy}{\Gamma \vdash F}$$

$$(\subseteq I) \frac{\Gamma, xRy \vdash xSy}{\Gamma \vdash R \subseteq S}$$

$$(\subseteq E) \frac{R \subseteq S \quad xRy}{xSy}$$

$$(EI) \frac{}{xEx}$$

$$(EE) \frac{xRv \quad yEv}{xRy}$$

3. Examples

For our first example we verify Lyndon's inclusion (see p2). The proof illustrates the manner in which a natural deduction system allows one to work backward from the statement of a problem. It is for this reason that natural deduction systems are easier to mechanize than axiomatic-algebraic ones.

For convenience let

$$\begin{aligned} \psi_1 &= \check{X}_1 Z_1 \cap X_2 \check{Z}_2 \\ \psi_2 &= \check{Z}_1 Y_1 \cap Z_2 \check{Y}_2 \\ \varphi &= \check{X}_1 Y_1 \cap X_2 \check{Y}_2 \cap \psi_1 \psi_2 \end{aligned}$$

so that we wish to show

$$X_1 X_2 \cap Y_1 Y_2 \cap Z_1 Z_2 \subseteq X_1 \varphi Y_2$$

(note we have omitted ";"'s).. Then by (\subseteq) it is enough to show that

$$aX_1X_2 \cap Y_1Y_2 \cap Z_1Z_2b \vdash aX_1\varphi Y_2b.$$

It follows from the (\cap E) and the (AE) rules that it is enough to establish

$$aX_1X_2b, aY_1Y_2b, aZ_1Z_2b \vdash aX_1\varphi Y_2b$$

and it follows from the ($;$ E) rule in turn that it is enough to show

$$aX_1p, pX_2b, aY_1q, qY_2b, aZ_1r, rZ_2b \vdash aX_1\varphi Y_2b.$$

Let us call the set of formulas on the left hand side Γ . Then we will have

$$\Gamma \vdash aX_1\varphi Y_2b$$

iff from Γ we can derive aX_1v , $v\varphi w$ and wY_2b for some v and w . The fact that aX_1p and qY_2b are in Γ suggest $v=p$ and $q=w$. Thus it remains to derive $p\varphi q$ from Γ . Since

$$\varphi = \check{X}_1Y_1 \cap X_2\check{Y}_2 \cap \psi_1\psi_2$$

we must derive $p\check{X}_1Y_1q$, $pX_2\check{Y}_2q$ and $p\psi_1\psi_2q$ from Γ . The first two are immediate: for example, since aX_1p and aY_1q are in Γ , we have $p\check{X}_1a$ by ($\check{\vee}$ E) and then $p\check{X}_1Y_1q$ by ($;$ I). To derive $p\psi_1\psi_2q$ we need $p\psi_1t$ and $t\psi_2q$ for some t . From the definition of ψ_1 it follows we must derive $p\check{X}_1Z_1t$ and $pX_2\check{Z}_2t$, and this follows easily if we take $t=r$. Since $r\psi_2q$ also follows in the same way, the proof is complete.

For our second example we verify Tarski's fifth axiom, i.e. show that

$$X;Y \cap Z \subseteq \Omega \vdash Y;\check{Z} \cap \check{X} \subseteq \Omega.$$

The proof is presented in straight line form as in first order logic, each assertion being derived from those above and not to the left.

- | | | |
|------|--------------------------------|--------------------------------|
| (1) | $X;Y \cap Z \subseteq \Omega$ | assumption |
| (2) | $aY;\check{Z} \cap \check{X}b$ | assumption |
| (3) | $aY;\check{Z}b$ | (\cap E) and (2) |
| (4) | $a\check{X}b$ | " |
| (5) | bXa | ($\check{\vee}$ E) |
| (7) | aYv | assumption |
| (8) | $v\check{Z}b$ | assumption |
| (9) | bZv | ($\check{\vee}$ E) and (8) |
| (10) | $bX;Yv$ | ($;$ I) using (5) and (7) |
| (11) | $bX;Y \cap Zv$ | (\cap I) using (9) and (10) |

- | | | |
|------|--------------------------------|--|
| (12) | $b \cap v$ | ($\subseteq E$) using (1) and (11) |
| (13) | $a \cap b$ | ($\cap E$) using (12) |
| (14) | $a \cap b$ | ($;\exists E$) using (13) and (3) cancelling (7),(8) |
| (15) | $Y; Z \cap X \subseteq \Omega$ | ($\subseteq I$) using (14) cancelling (2) |

4. Completeness

In this section we show that \vdash and \models are equivalent, i.e. that $\Gamma \models F$ iff $\Gamma \vdash F$ for all Γ and F . The 'if' direction, soundness, is straight forward.

Theorem I. For any set V of variables, any set Γ of V -formulas and any V -formula F :

if $\Gamma \vdash F$ then $\Gamma \models F$.

Proof. Since \vdash was defined to be the least relation for which our rules are valid, it is sufficient to verify that each of the rules is sound (this is omitted).

Our proof of the 'only if' direction is an adaptation of the Henkin completeness proof for first order logic - we show that every consistent theory has an interpretation. By a V -theory (V a set of variables) we mean a set T of V -formulas such that $T \vdash F$ implies $F \in T$ for every V -formula F . A V -theory T is consistent iff T does not contain the negation of any formula in T , and T is complete iff T contains the negation of every formula not in T . Finally, T is Henkin-complete (H-complete) iff for any variables x and y and any terms R and S , if $xR; Sy \in T$ then there is a v in T such that $xRv \in T$ and $vSy \in T$. The strategy of the proof is to show that every consistent theory has a consistent, complete, H-complete extension, and that every such theory has an interpretation.

The first step is to show that every consistent theory has a consistent complete extension.

Theorem II. For any countable set V of variables and any V -theory T :

if T is consistent there is a consistent complete V -theory T' extending T .

Proof. The standard proof carries over. Let $\langle F_i \mid i \in \mathbb{N} \rangle$ be an enumeration of all V -formulas. Define a sequence $\langle T_i \mid i \in \mathbb{N} \rangle$ of V -theories extending T by setting $T_0 = T$ and $T_{i+1} = T_i$ if $F_i \in T_i$, otherwise T_{i+1} = the closure of $T \cup \{F_i\}$. It is easily verified that $\cup\{T_i \mid i \in \mathbb{N}\}$ is a consistent complete extension of T .

Next we show that every consistent theory has a consistent H -complete extension.

Theorem III. For any countable set V of variables and any V -theory T :

if T is consistent there is a countable set V' of variables extending V and a consistent H -complete V' -theory T' extending T .

Proof. Define a sequence $\langle V_i \mid i \in \mathbb{N} \rangle$ of sets of variables and a sequence $\langle T_i \mid i \in \mathbb{N} \rangle$ of theories, each T_i a V_i -theory, as follows:

$$(1) \quad T_0 = T;$$

(2) for any i , for each formula in T_i of the form xRy add to V_i a new variable v and add to T_i the formulas xRv and vSy . The resulting set of variables is V_{i+1} and the closure under \vdash of the resulting set of formulas is T_{i+1} .

Then the $\cup\{V_i \mid i \in \mathbb{N}\}$ -theory $T' (= \cup\{T_i \mid i \in \mathbb{N}\})$ is a consistent H -complete extension of T .

To see this note first that an easy induction on i shows that each T_i is consistent. In the case $i = 0$ the result is immediate. Now suppose that for some i , T_i is consistent but T_{i+1} is not. Then $T_{i+1} \vdash a \Omega a$ and so by the definition of T_{i+1} there is a finite set of the form

$$\{x_j R_j v_j, v_j S_j y_j \mid j \leq n\}$$

such that

$$T_i, x_0 R_0 v_0, v_0 S_0 y_0, \dots, x_n R_n v_n, v_n S_n y_n \vdash a \Omega a.$$

But then n applications of the (\exists E) rule give us $T_i \vdash aQa$ which contradicts the consistency of T_i .

Since each T_i is consistent, T' must be consistent.

Next, note that T' must be H-complete. For if $xR;Sy$ is in T' , it must be in T_i for some i . But then there is a v in V_{i+1} such that xRv and vSy are in T_{i+1} and therefore in T' .

Finally we put the last two theorems together to get the desired result on extensions.

Theorem IV. For any countable set V of variables and any V -theory T :
if T is consistent there is a set V' of variables extending V and a consistent, complete, H-complete theory T' extending T .

Proof. Define a sequence $\langle T_i \mid i \in \mathbb{N} \rangle$ of theories extending T as follows:

- (1) $T_0 = T$;
- (2) T_{2i+1} is a complete consistent extension of T_{2i} ;
- (3) T_{2i+2} is an H-complete consistent extension of T_{2i+1} .

It is easily verified that $\cup\{T_i \mid i \in \mathbb{N}\}$ is a consistent, complete, H-complete extension of T .

Theorem V. For any countable set V of variables and any V -theory T :
if T is consistent, complete and H-complete then there is an interpretation \mathcal{A} such that $\mathcal{A} \models F$ for every F in T .

Proof. For any variables x and y in V let $x \cong y$ iff $xEy \in T$. It is easily verified that \cong is an equivalence relation. Therefore for any v in V let \tilde{v} be the equivalence class of v , and let $\tilde{V} = \{\tilde{v} \mid v \in V\}$. Then for any variable v in V let $\mathcal{A}(v) = \tilde{v}$, and for any V -formula xRy let $\mathcal{A}(xRy) = \{ \langle \tilde{x}, \tilde{y} \rangle \mid xRy \in T \}$. The map \mathcal{A} so defined is then an interpretation of T .

We must first show that \mathcal{A} is in fact an interpretation, i.e. that \mathcal{A} is consistent with the intended meanings of the relation operations and

constants. We give only the proofs for ";" and "U", the others being similar.

(1) Union. We must show that $\mathcal{A}(RUS)$ is the union of $\mathcal{A}(R)$ and $\mathcal{A}(S)$ for any terms R and S . This amounts to showing that for any variables x and y in V , $xRUSy \in T$ iff $xRy \in T$ or $xSy \in T$.

If either xRy or xSy is in T the (UI) rule gives us $xRUSy \in T$.

Now suppose that $xRUSy \in T$ but that neither xRy nor xSy is in T . Since T is complete, both $\bar{x}Ry$ and $\bar{x}Sy$ must be in T . But this means that $\bar{x}R\bar{S}y \in T$ and therefore (omitting the proof) that $\bar{x}R\bar{S}y \in T$, which contradicts the consistency of T .

(1) Composition. We must show that $\langle \bar{x}, \bar{y} \rangle \in \mathcal{A}(R;S)$ iff $\langle \bar{x}, \bar{v} \rangle \in \mathcal{A}(R)$ and $\langle \bar{v}, \bar{y} \rangle \in \mathcal{A}(S)$ for some v . This amounts to showing that $xR;Sy \in T$ iff $xRv \in T$ and $vSy \in T$ for some v , and this is just the statement of H-completeness.

Finally, the fact that $\mathcal{A} \models F$ for every F in T is immediate.

Corollary. Every consistent set of formulas has an interpretation.

Proof. Extend the theory of the set to a consistent, complete, H-complete theory.

This gives us the main result.

Theorem V. For any countable set V of variables, any set Γ of V -formulas and any V -formula F :

if $\Gamma \models F$ then $\Gamma \vdash F$.

Proof. Suppose that F does not follow from Γ . Then Γ and the negation of F are consistent and so have an interpretation, impossible.

5. Typed Relational Calculi

This natural deduction system can be carried over quite easily to a typed system such as that of Hitchcock and Park. Formulas would be of the form $x_1x_2\dots x_nRy_1y_2\dots y_m$ with R a term of type $n \rightarrow m$. The rules of inference given on page 6 would remain valid if x , y and are interpreted as sequences of

variables of appropriate lengths. The rules for the extra operations are completely obvious, e.g.

$$(|I) \frac{x_0 \dots x_j R y_0 \dots y_k \quad v_0 \dots v_m S w_0 \dots w_n}{x_0 \dots x_j v_0 \dots v_m R | S y_0 \dots y_k w_0 \dots w_n}$$

$$(|E) \frac{x_0 \dots x_j v_0 \dots v_m R | S y_0 \dots y_k w_0 \dots w_n}{x_0 \dots x_j R y_0 \dots y_k} \quad \frac{x_0 \dots x_j v_0 \dots v_m R | S y_0 \dots y_k w_0 \dots w_n}{v_0 \dots v_m S w_0 \dots w_n}$$

The completeness proof for this typed system goes through with little change. The extra operations offer no difficulty because their rules are purely definitional, like the rules for \checkmark .

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