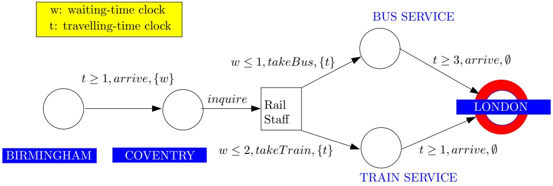
Games on Timed Automata

Ashutosh Trivedi
(Joint-work with Dr. Marcin Jurdziński.)

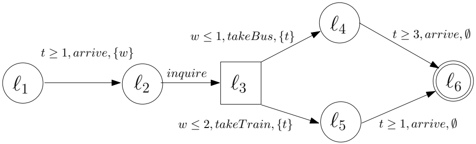
Warwick Postgraduate Colloquium in Computer Science, 2008

July 01, 2008

Timed Game Automaton

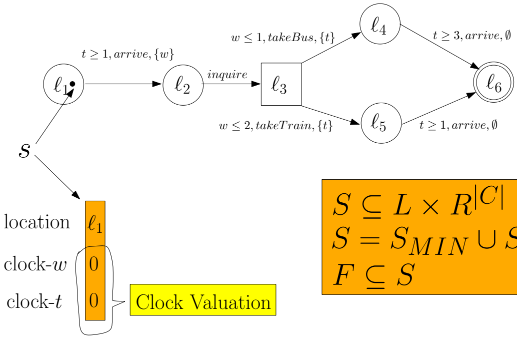


Timed Game Automaton



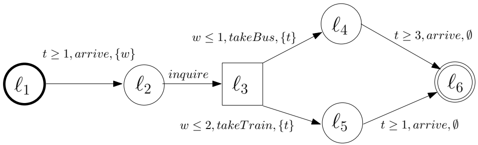
$$\Gamma = \{L, C, \rightarrow, A\}$$
$$L = L_{MAX} \cup L_{MIN}$$

Timed Game Automaton



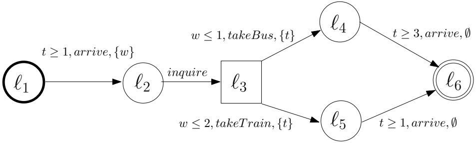
$S \subseteq L \times R^{|C|}$
$S = S_{MIN} \cup S_{MAX}$
$F \subseteq S$

Timed Game Automaton



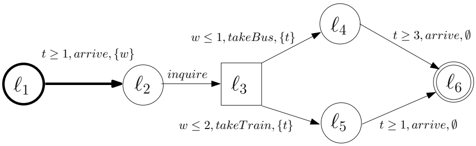
location	ℓ_1
clock-w	0
clock-t	0

Timed Game Automaton



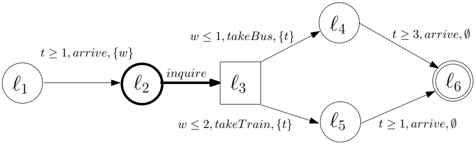
location	ℓ_1	ℓ_1
clock-w	0	0.5
clock-t	0	0.5

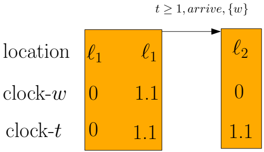
Timed Game Automaton



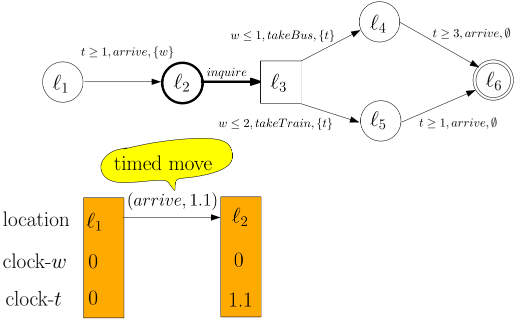
location	ℓ_1	ℓ_1
clock-w	0	1.1
clock-t	0	1.1

Timed Game Automaton

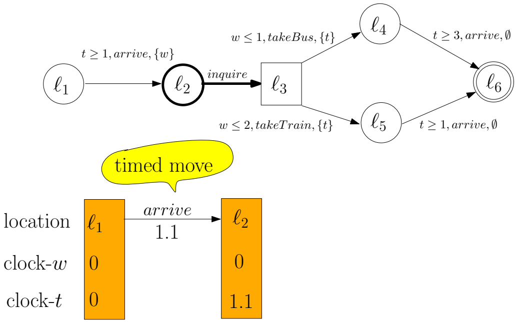




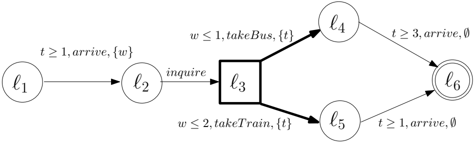
Timed Game Automaton

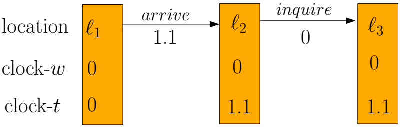


Timed Game Automaton

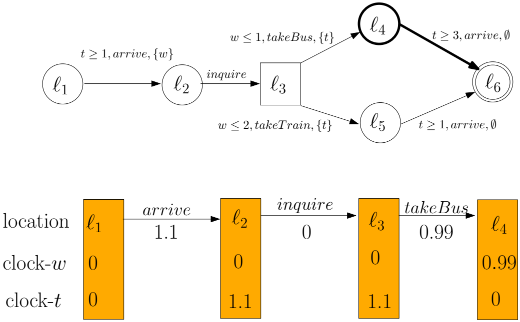


Timed Game Automaton

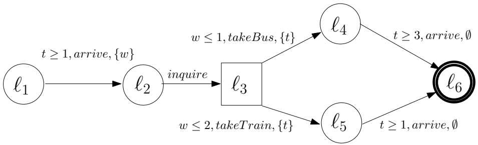


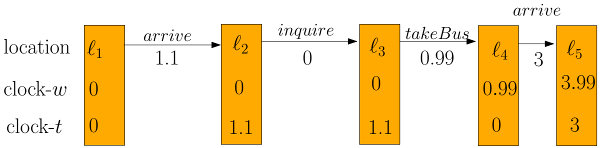


Timed Game Automaton

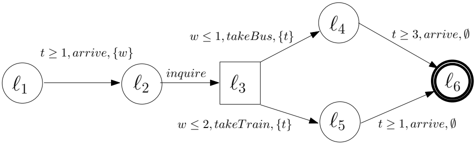


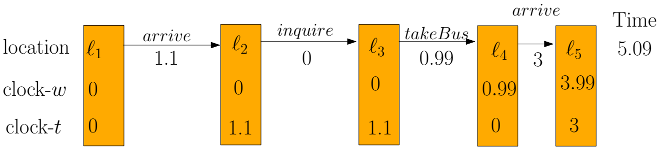
Timed Game Automaton



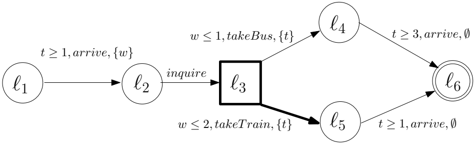


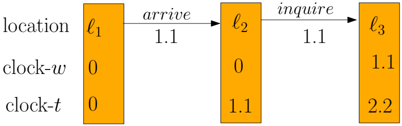
Timed Game Automaton



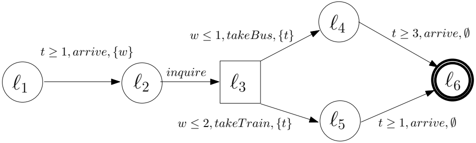


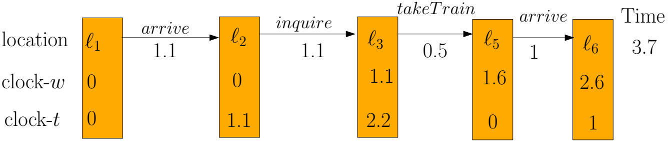
Timed Game Automaton



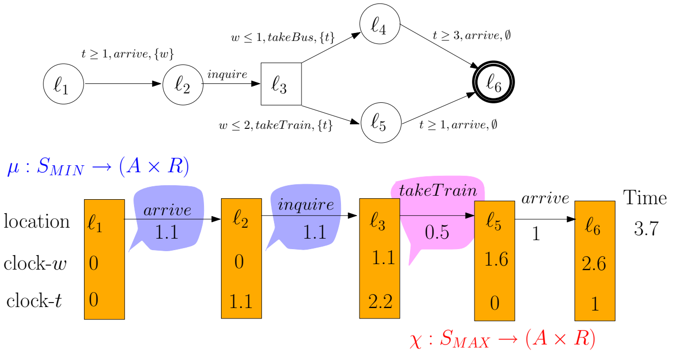


Timed Game Automaton





Timed Game Automaton



Payoff Functions

Path

- $\text{Path}(s_0, \mu, \chi) := \langle s_0 \xrightarrow[t_1]{a_1} s_1 \xrightarrow[t_2]{a_2} \dots \rangle$
- $\text{Final}(\text{Path}(s_0, \mu, \chi)) = \inf\{i : s_i \in F\}$.

Payoff Functions

Path

- $\text{Path}(s_0, \mu, \chi) := \langle s_0 \xrightarrow[t_1]{a_1} s_1 \xrightarrow[t_2]{a_2} \ldots \rangle$
- $\text{Final}(\text{Path}(s_0, \mu, \chi)) = \inf\{i : s_i \in F\}$.

Reachability-time Game

$\text{Payoff}(\text{Path}(s, \mu, \chi)) := \sum_{i=1}^{\text{Final}(\text{Path}(s,\mu,\chi))} t_i$

Payoff Functions

Path

- $\mathrm{Path}(s_0, \mu, \chi) := \langle s_0 \xrightarrow[t_1]{a_1} s_1 \xrightarrow[t_2]{a_2} \dots \rangle$
- $\mathrm{Final}(\mathrm{Path}(s_0, \mu, \chi)) = \inf\{i : s_i \in F\}$.

Reachability-time Game

$\mathrm{Payoff}(\mathrm{Path}(s, \mu, \chi)) := \sum_{i=1}^{\mathrm{Final}(\mathrm{Path}(s,\mu,\chi))} t_i$

Average-time Game

$\mathrm{Payoff}(\mathrm{Path}(s, \mu, \chi)) := \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} t_i$.

Determinacy

Value

$\text{Val}^*(s) := \inf_{\mu} \sup_{\chi} \text{Payoff}(\text{Path}(s, \mu, \chi))$
$\text{Val}_*(s) := \sup_{\mu} \inf_{\chi} \text{Payoff}(\text{Path}(s, \mu, \chi))$

Determinacy

Value

$\mathrm{Val}^*(s) := \inf_{\mu} \sup_{\chi} \mathrm{Payoff}(\mathrm{Path}(s, \mu, \chi))$
$\mathrm{Val}_*(s) := \sup_{\mu} \inf_{\chi} \mathrm{Payoff}(\mathrm{Path}(s, \mu, \chi))$

Determinacy

$\mathrm{Val}(s) = \mathrm{Val}^*(s) = \mathrm{Val}_*(s)$

Determinacy

Value

$\mathrm{Val}^*(s) := \inf_{\mu} \sup_{\chi} \mathrm{Payoff}(\mathrm{Path}(s, \mu, \chi))$
$\mathrm{Val}_*(s) := \sup_{\mu} \inf_{\chi} \mathrm{Payoff}(\mathrm{Path}(s, \mu, \chi))$

Determinacy

$\mathrm{Val}(s) = \mathrm{Val}^*(s) = \mathrm{Val}_*(s)$

Determinacy Theorem

Every reachability-time game is determined.

Determinacy Theorem

Every average-time game is determined.

Algorithmic problem

Algorithmic problem

- Find the value of the game.
- Compute a pair of ε-optimal strategies for Max and Min.

Algorithmic problem

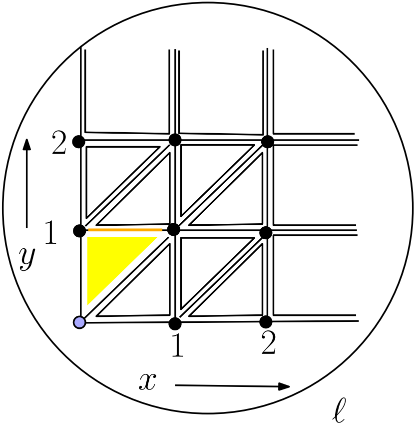
Algorithmic problem

- Find the value of the game.
- Compute a pair of ε-optimal strategies for Max and Min.

Bottleneck

- Infinitely many states.
- Every state has infinitely many successors.

Boundary Region Automata $\widehat{\Gamma}$: motivation

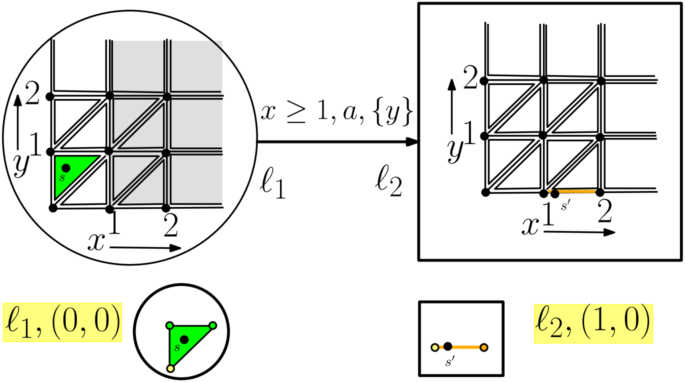


$x < 1, x > 0$
$y = 1$
THIN

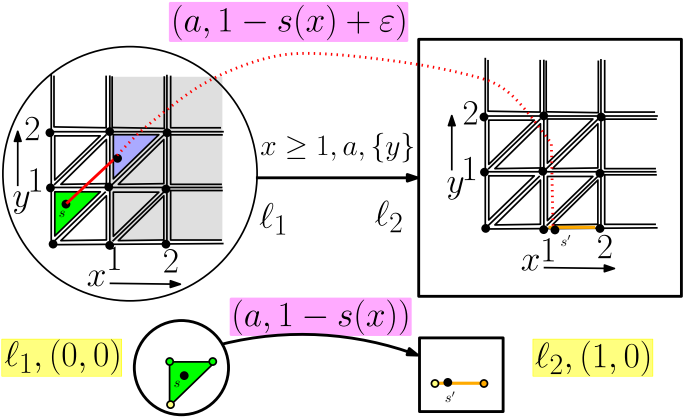
$x < 1, x > 0$
$y < 1, y > 0$
$y - x > 0$
THICK

$x = 0, y = 0$
THIN

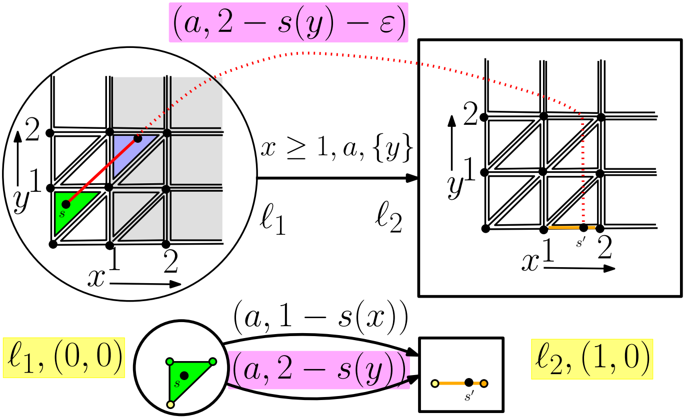
Boundary Region Automata $\widehat{\Gamma}$: motivation



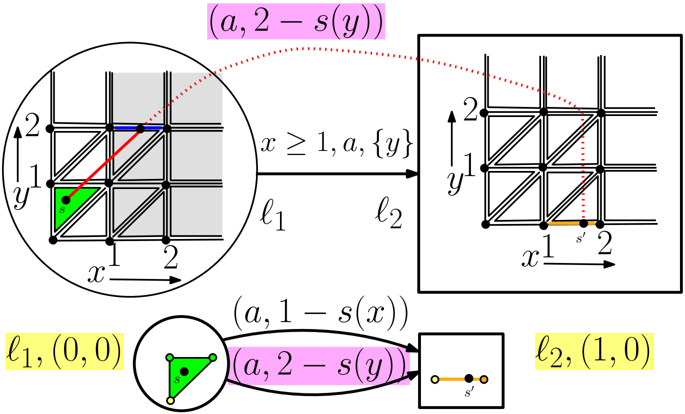
Boundary Region Automata $\widehat{\Gamma}$: motivation



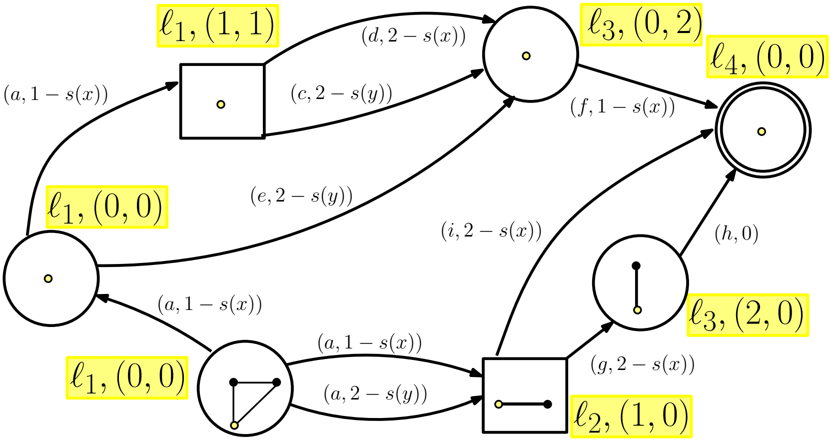
Boundary Region Automata $\widehat{\Gamma}$: motivation



Boundary Region Automata $\widehat{\Gamma}$: motivation



Boundary Region Automata $\widehat{\Gamma}$: motivation



Optimality equations for boundary region automata $\widehat{\Gamma}$.

Reachability-Time Games: Optimality equations $\mathrm{Opt}(\widehat{\Gamma})$

$$T(R)(s) \quad = \quad \begin{cases} 0 & \text{if } s \in F \\ opt_{(R,(a,b,c),R')}\{b - s(c) + T(R')(s') \,:\, s \xrightarrow[b-s(c)]{a} s'\} & \text{if } s \notin F \end{cases}$$

Optimality equations for boundary region automata $\widehat{\Gamma}$.

Reachability-Time Games: Optimality equations Opt($\widehat{\Gamma}$)

$$T(R)(s) = \begin{cases} 0 & \text{if } s \in F \\ opt_{(R,(a,b,c),R')}\{b - s(c) + T(R')(s') : s \xrightarrow[b-s(c)]{a} s'\} & \text{if } s \notin F \end{cases}$$

Theorem (Correctness of Abstraction)

For a timed game automaton Γ, one can construct a timed region graph $\widehat{\Gamma}$, such that a value of the reachability-time game can be obtained from a solution of Opt($\widehat{\Gamma}$).

Optimality equations for boundary region automata $\widehat{\Gamma}$.

Average-Time Games: Optimality equations $\mathrm{Opt}(\widehat{\Gamma})$

$$
\begin{aligned}
G(R)(s) &= opt_{([s],(a,b,c),R')}\{G(R')(s') \,:\, s \xrightarrow[b-s(c)]{a} s'\} \\
B(S)(s) &= opt_{([s],(a,b,c),R')}\{b - s(c) - G(R)(s) + B(R')(s') \\
&\quad : G(R)(s) = G(R')(s') \text{ and } s \xrightarrow[b-s(c)]{a} s'\}.
\end{aligned}
$$

Optimality equations for boundary region automata $\widehat{\Gamma}$.

Average-Time Games: Optimality equations $\mathrm{Opt}(\widehat{\Gamma})$

$$
\begin{aligned}
G(R)(s) &= opt_{([s],(a,b,c),R')}\{G(R')(s') : s \xrightarrow[b-s(c)]{a} s'\} \\
B(S)(s) &= opt_{([s],(a,b,c),R')}\{b-s(c)-G(R)(s)+B(R')(s') \\
&\quad : G(R)(s) = G(R')(s') \text{ and } s \xrightarrow[b-s(c)]{a} s'\}.
\end{aligned}
$$

Theorem (Correctness of Abstraction)

For a timed game automaton Γ, one can construct a timed region graph $\widehat{\Gamma}$, such that value of the average-time game can be obtained from a solution of $\mathrm{Opt}(\widehat{\Gamma})$.

Complexity Results

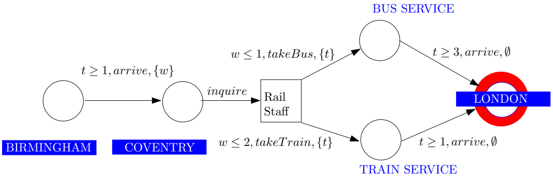
Theorem (Complexity)

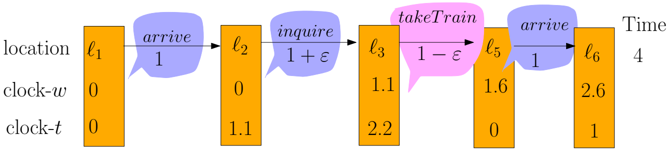
Reachability-Time Games are EXPTIME-complete.

Theorem (Complexity)

Average-Time Games are EXPTIME-complete.

Optimal-time to reach London





future Work

Future Work

- Games on more general *concavely-priced timed automata*.
- Games on *probabilistic timed automata*.
- Some Practical Work – Implementation / Symbolic Algorithms.