Finding Nash Equilibria in Certain Classes of 2-Player Game

Adrian Vetta

McGill University

Introduction

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Are there general classes of game in which finding a NE is easier?

Our Results

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Planar Win-Lose Games

(Addario-Berry, Olver and Vetta 2006)

There is a polytime algorithm for finding a NE in a planar win-lose 2-player game.

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- $p^* = \text{argmax}_p \, p^T(Aq^*)$ and $q^* = \text{argmax}_q \, q^T(B^T p^*)$

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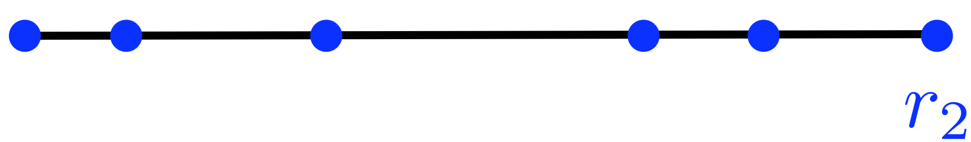

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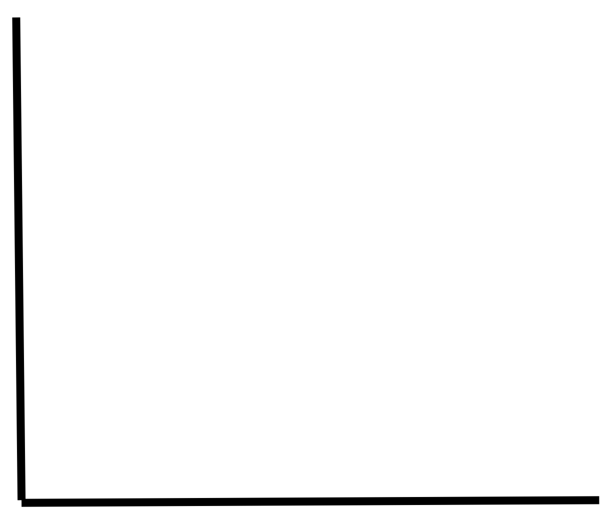

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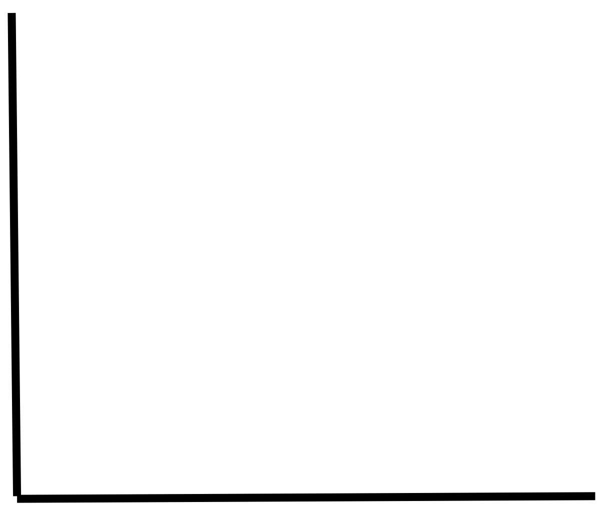

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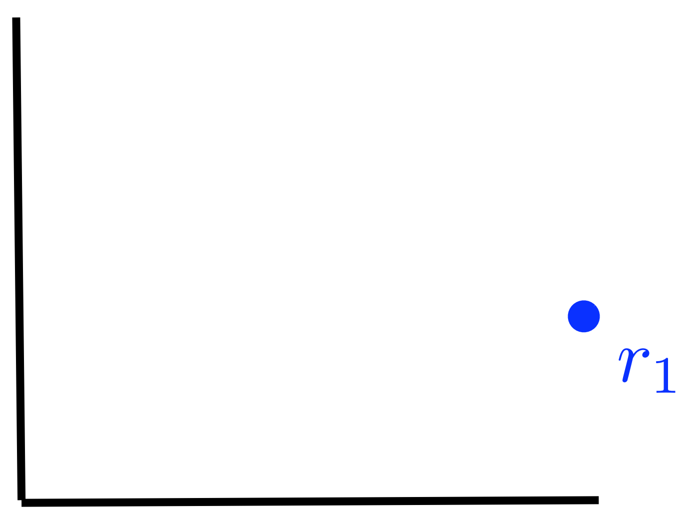

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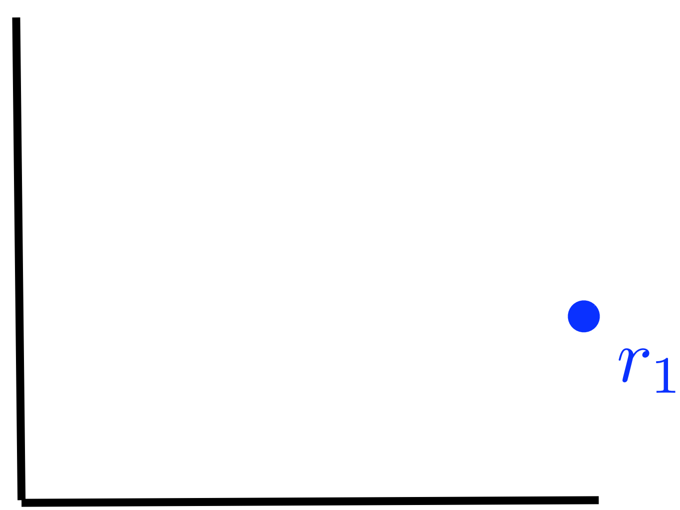

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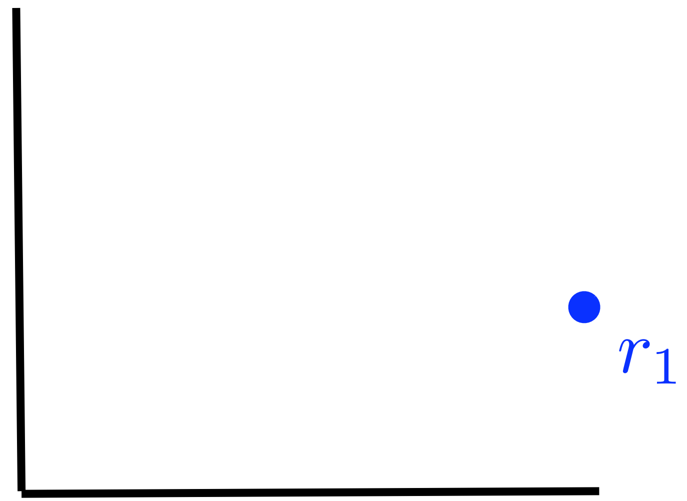

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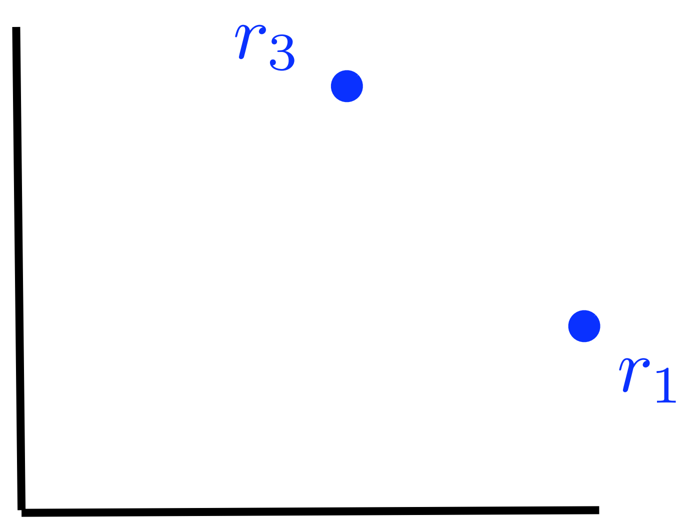

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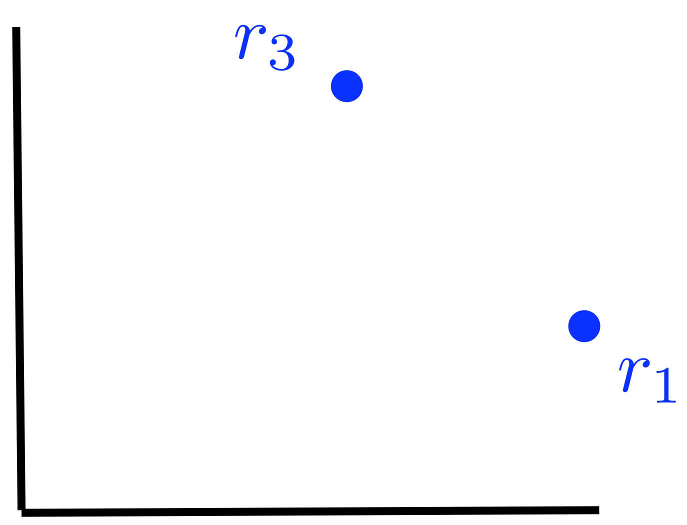

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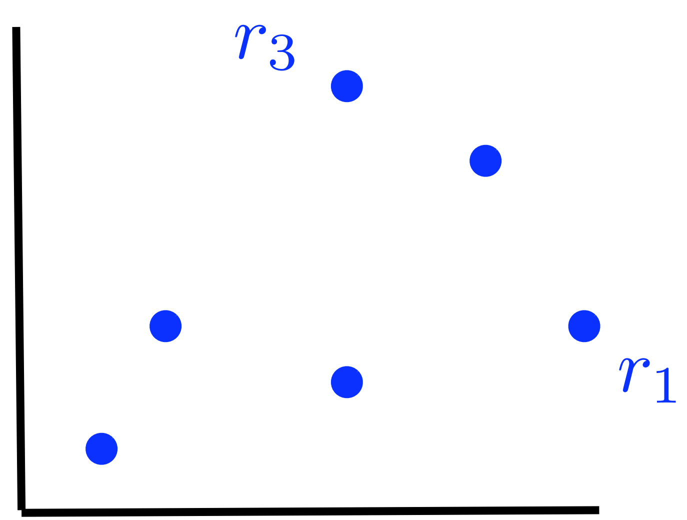

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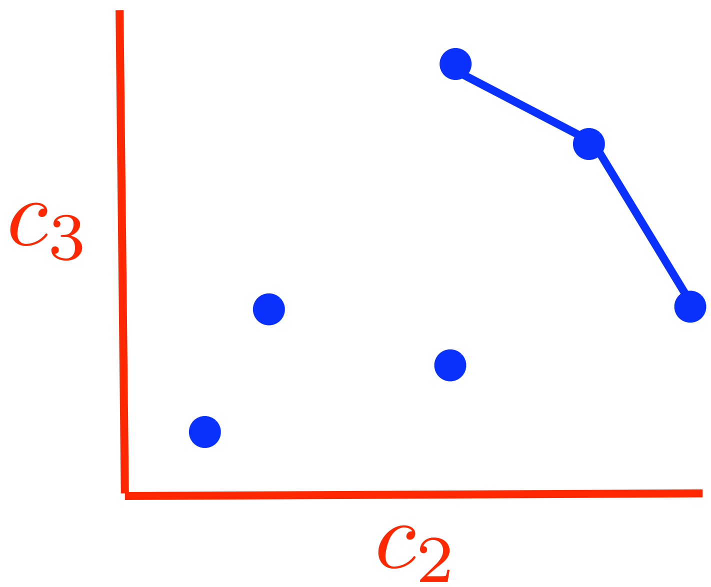

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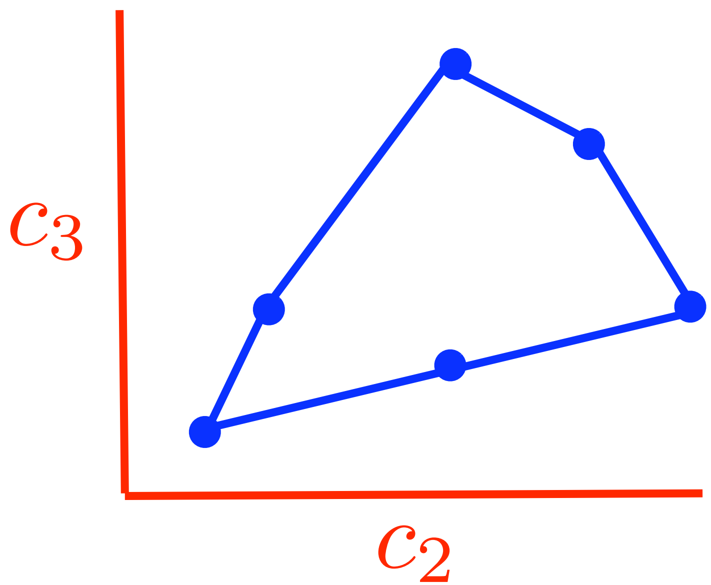

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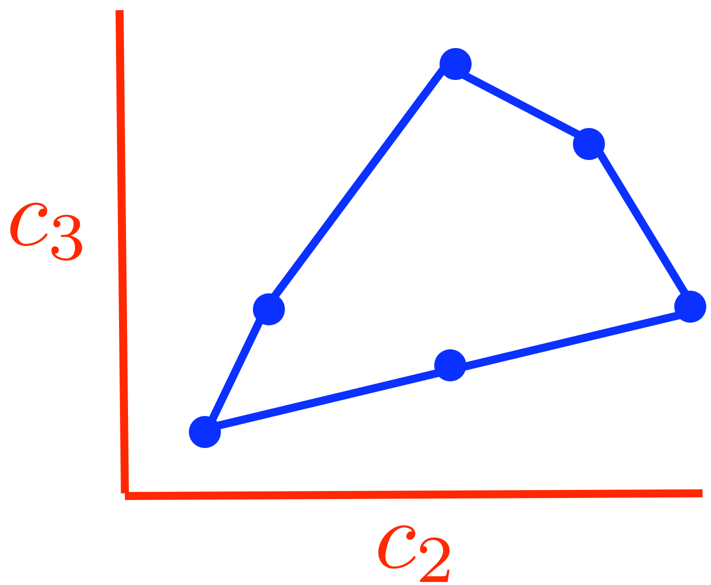

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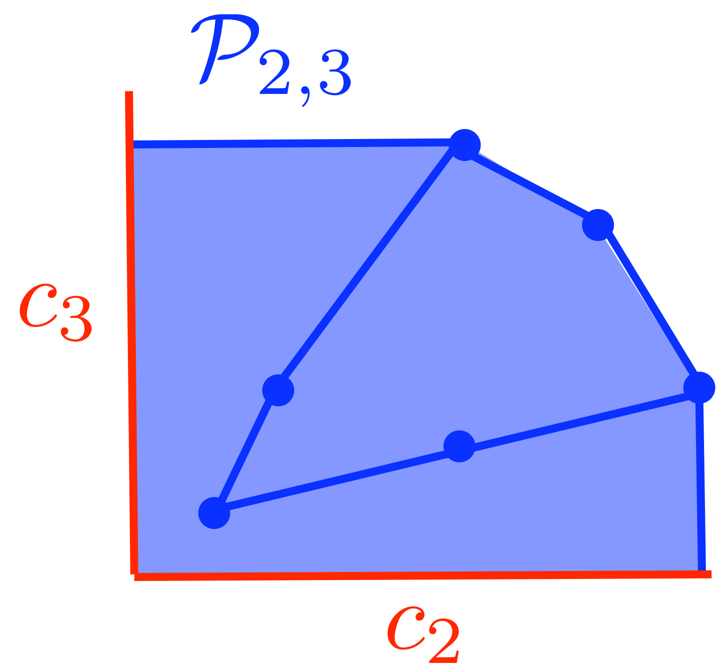

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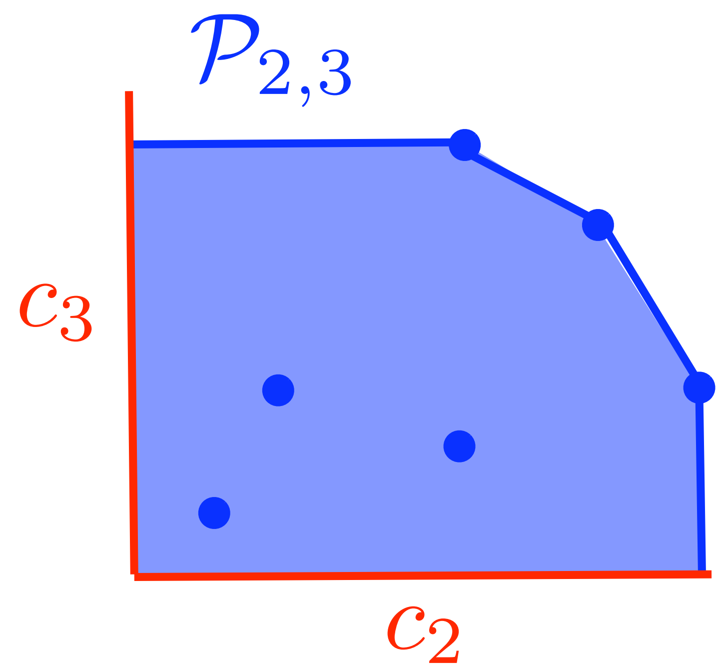

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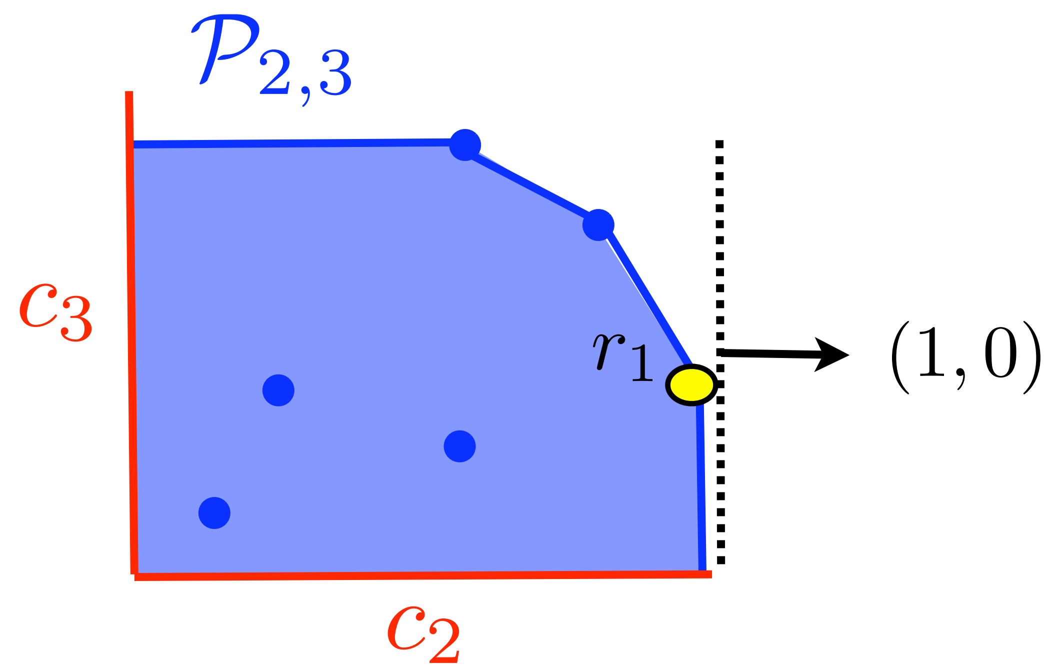

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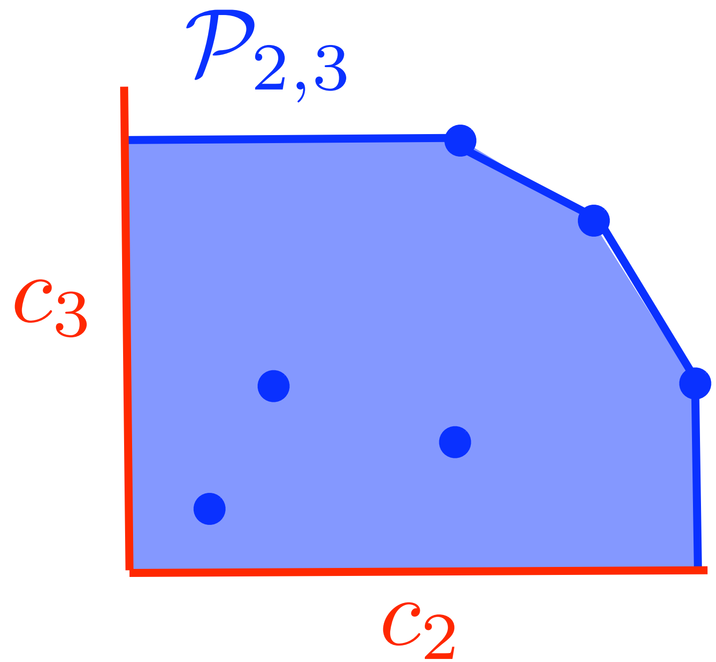

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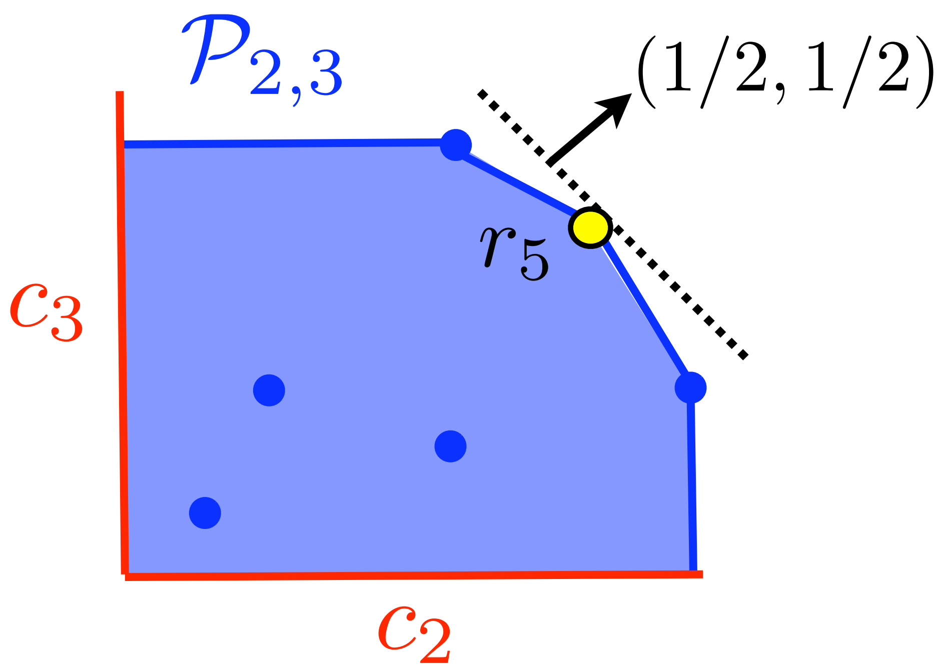

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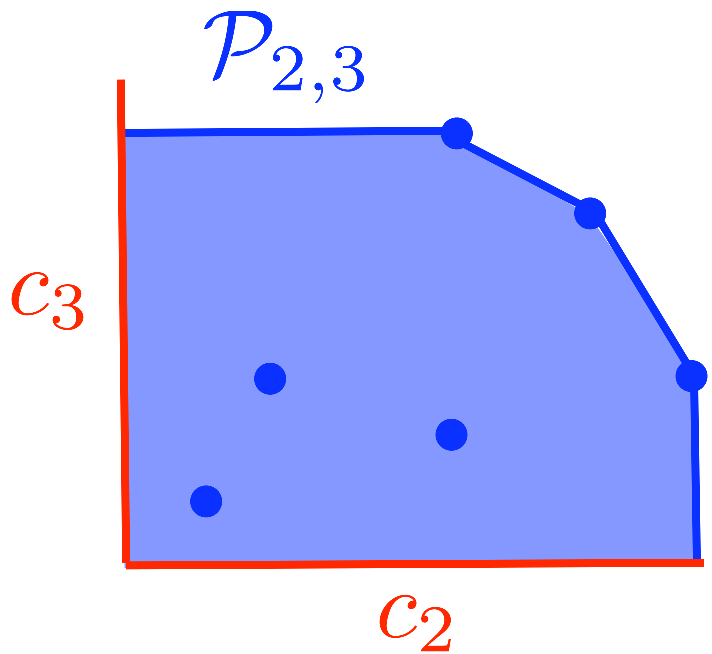

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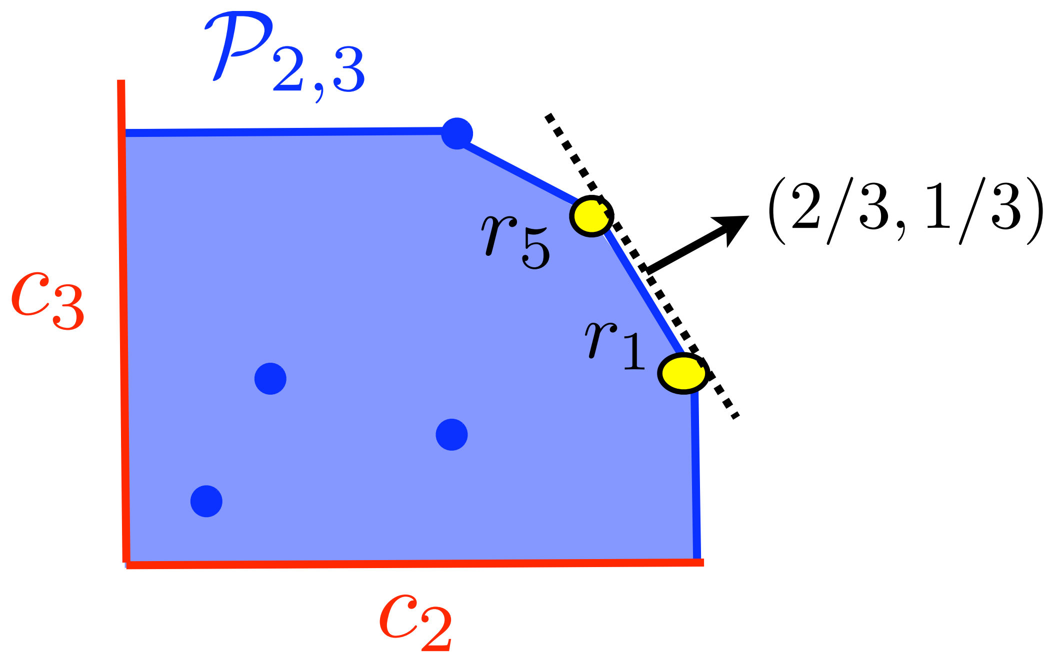

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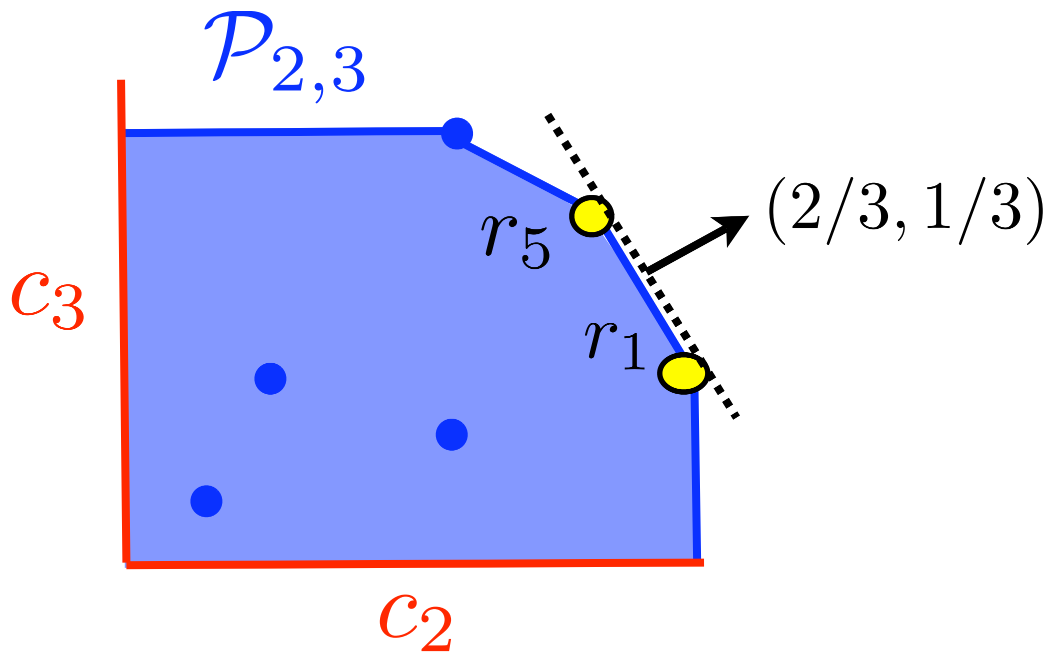

Theorem. (r_1, r_5) and (c_2, c_3) form a NE

if and only if

(r_1, r_5) is a facet of $\mathcal{P}_{2,3}$ and (c_2, c_3) is a facet of $\mathcal{P}_{1,5}$.

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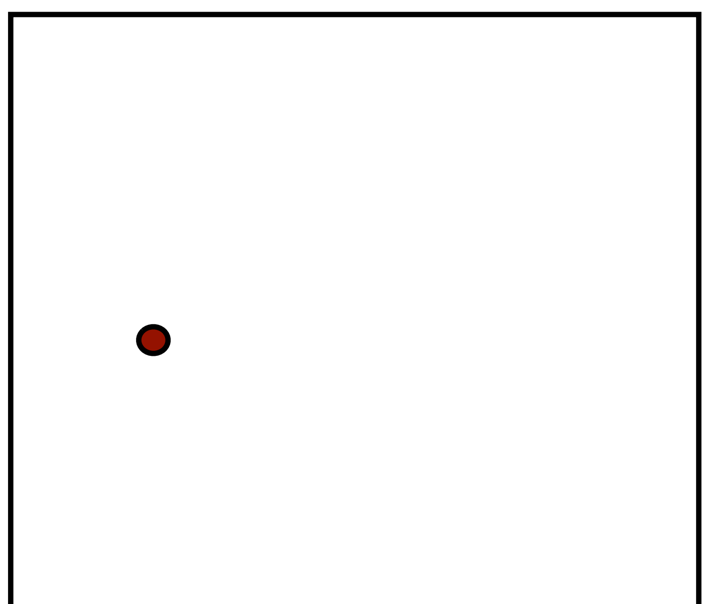

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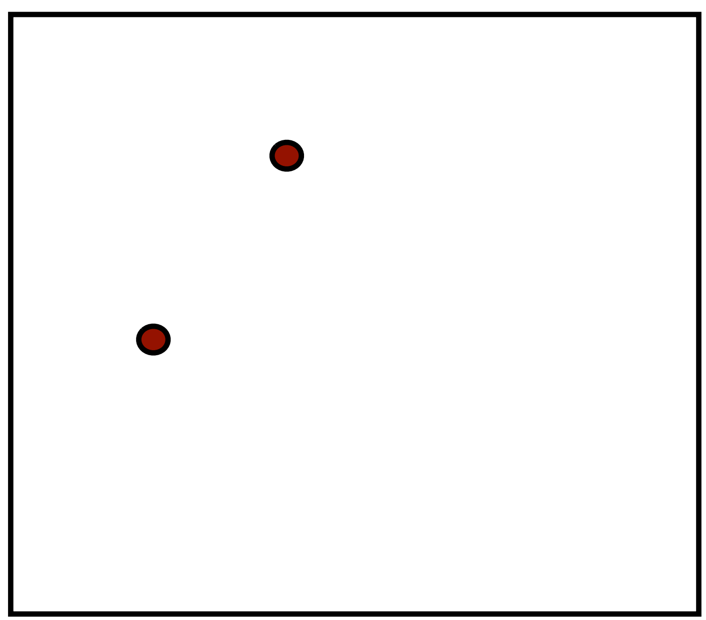

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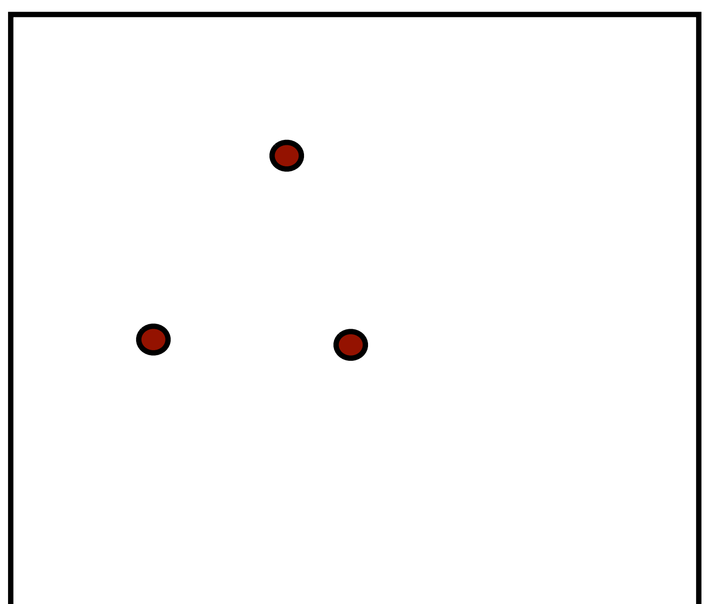

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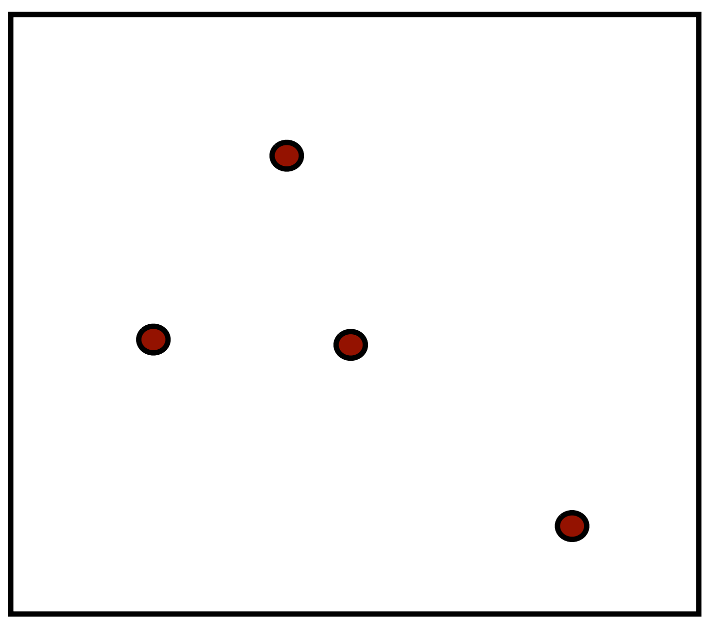

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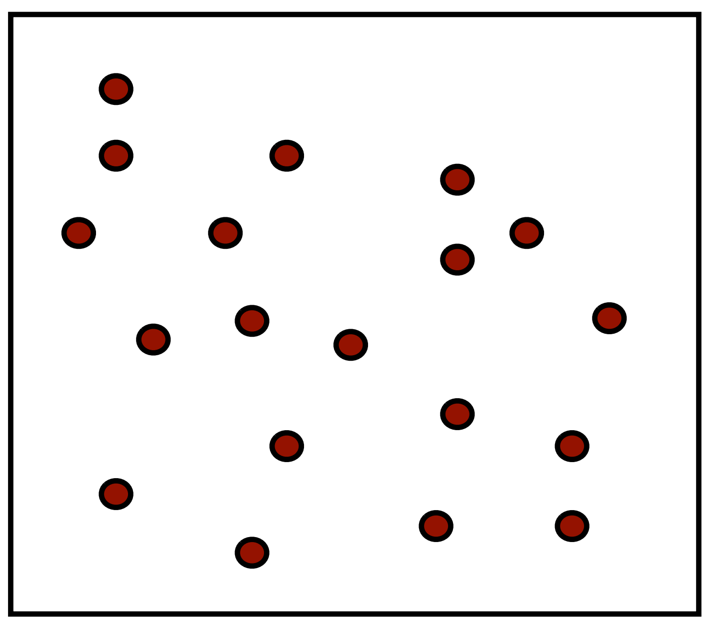

Random Games

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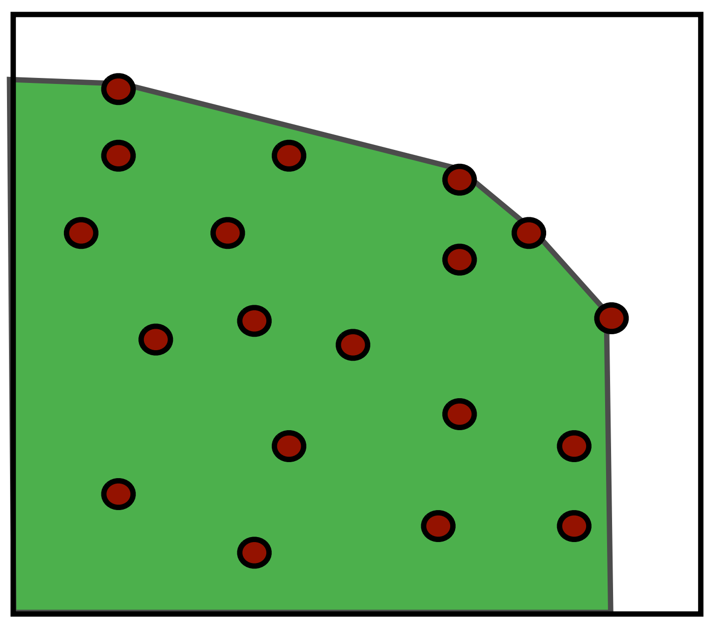

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Proof. Won't have `d+1` points on `(d−1)`-dimensional facet.

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Proof. Each facet has $\mathtt{d}$ points; each extreme point is on $\geq \mathtt{d}$ facets.

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Proof. A set *R* of d rows is a best response to a set *C* of d columns with probability

$$\frac{\#\text{facets}}{\binom{n}{d}}$$

and vice versa. $\square$

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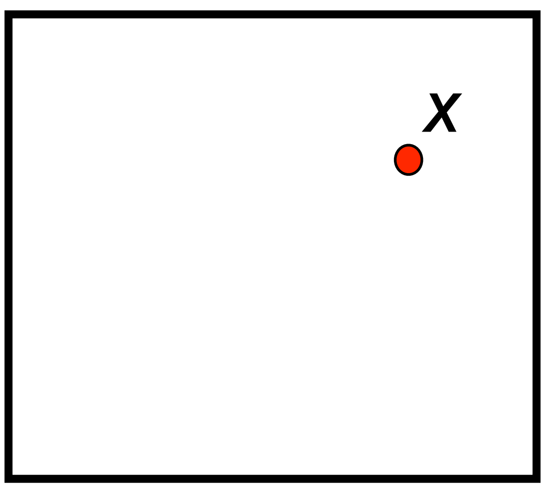

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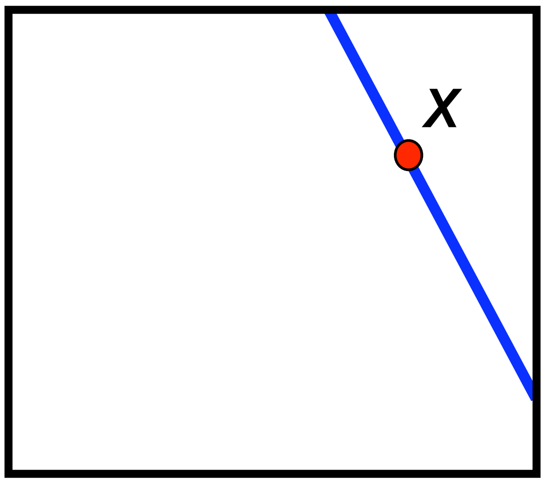

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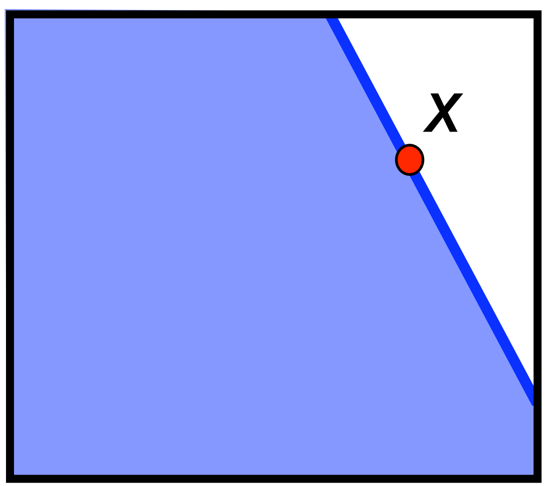

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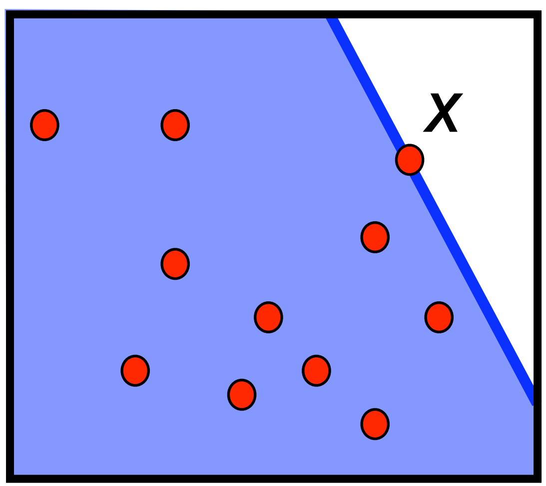

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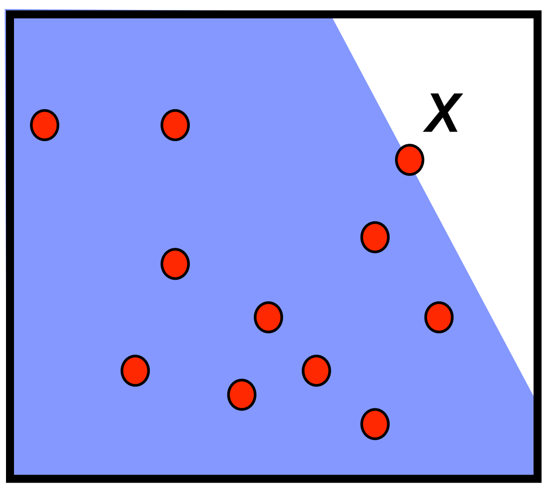

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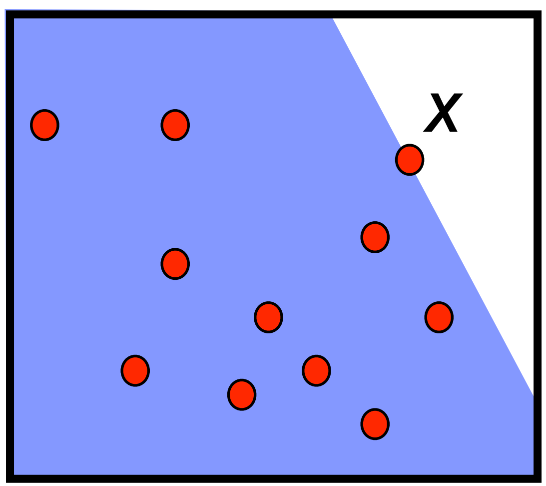

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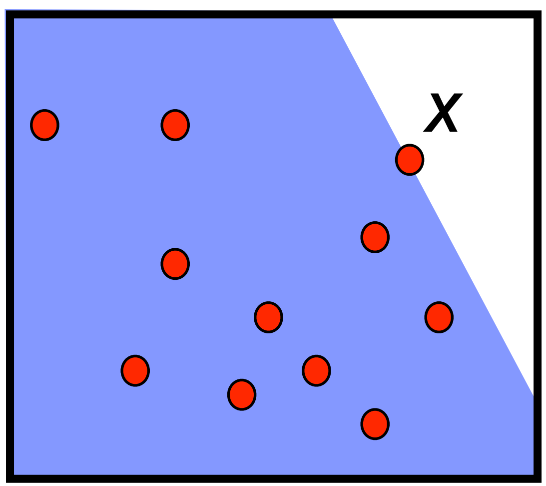

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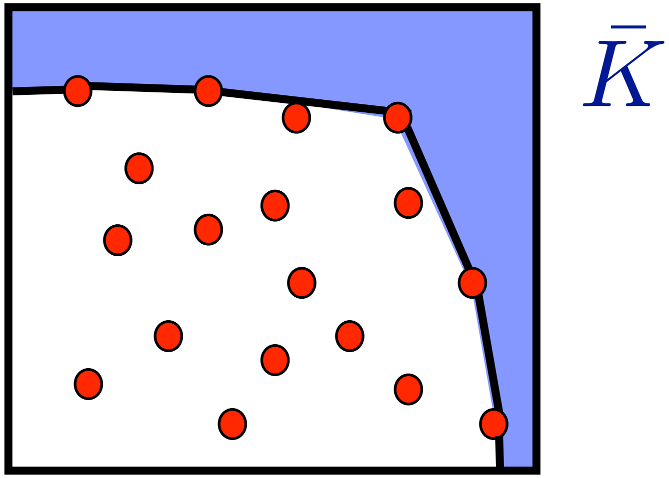

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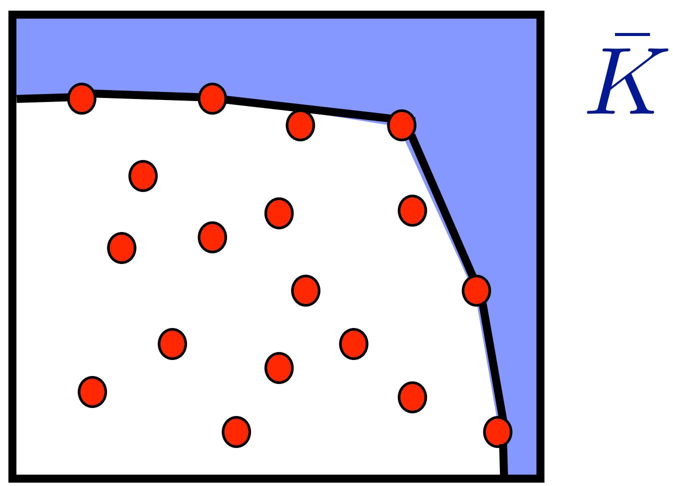

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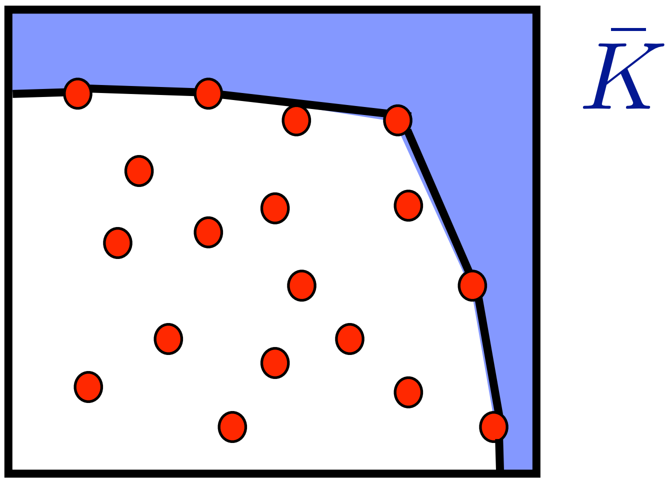

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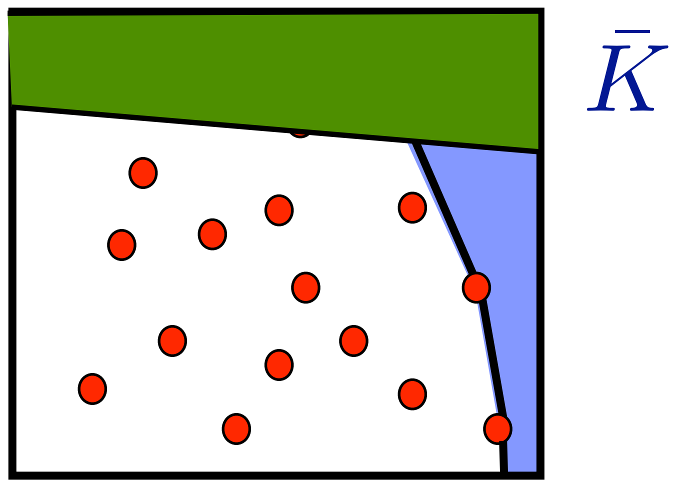

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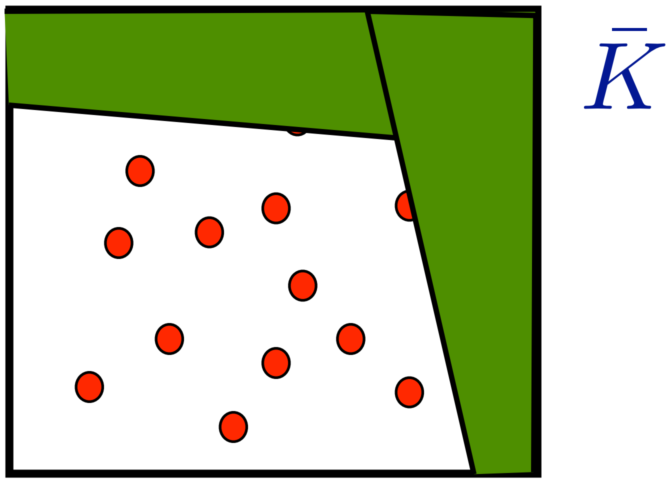

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Concentration Bounds

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Cap coverings give concentration bounds on:

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Combinatorially. For NE we examine the probability that a set *S* of rows forms a *facet* given that

(i) A set *T* of rows forms a *face*.

(ii) We *resample* some of the coordinates.

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Proof. There is a `2x2` NE w.h.p. □

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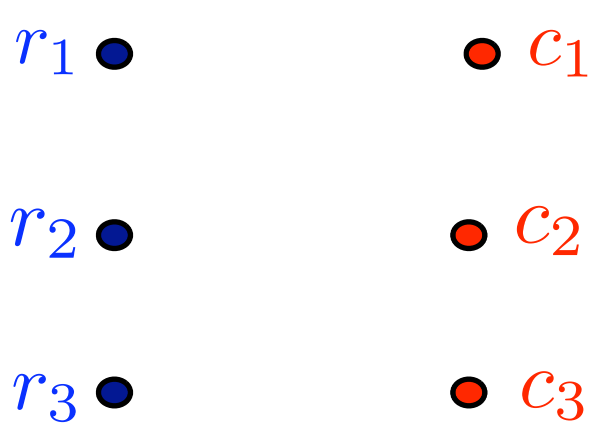

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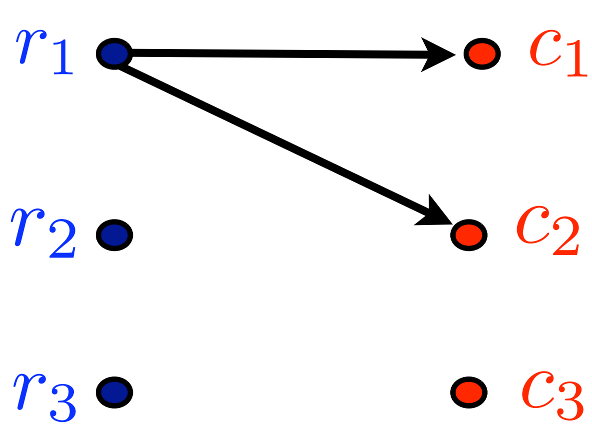

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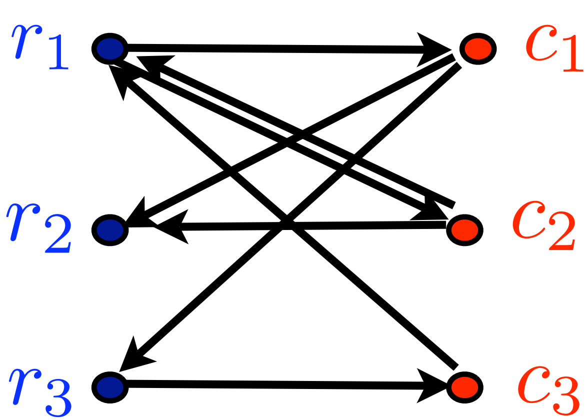

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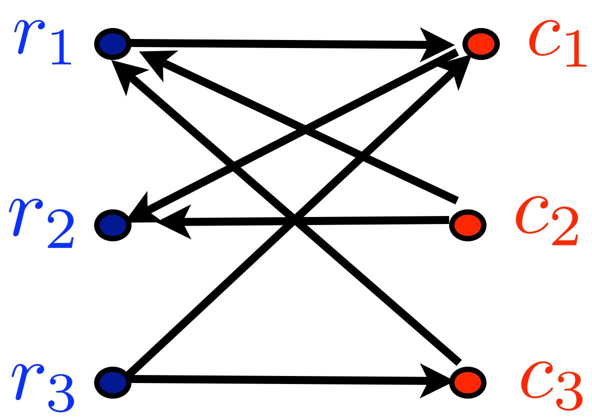

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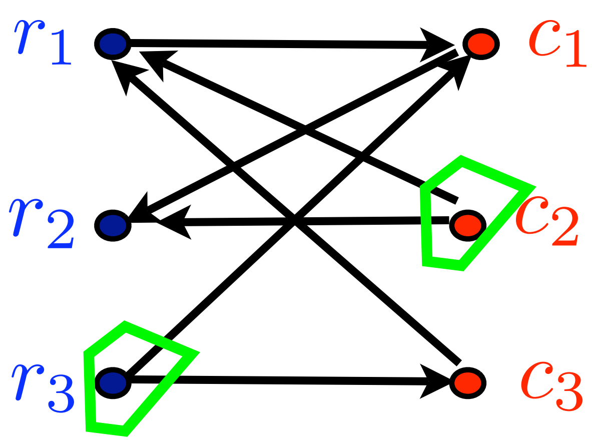

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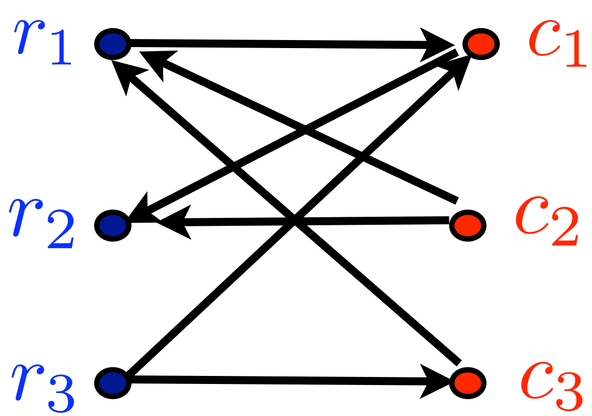

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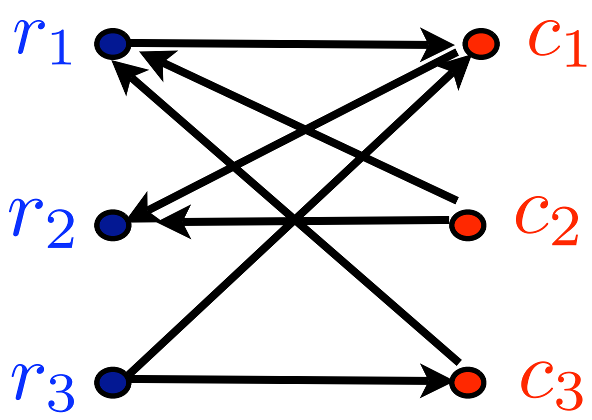

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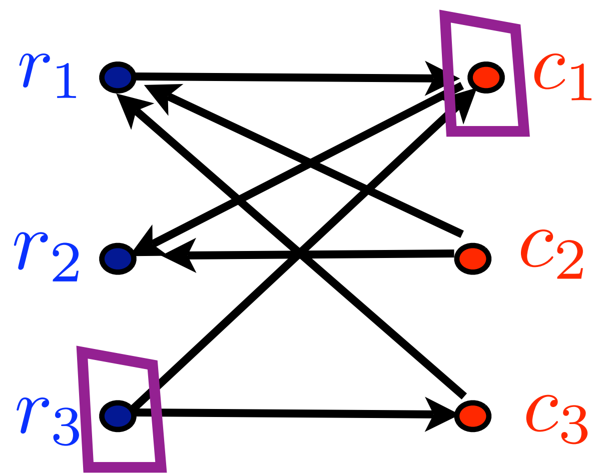

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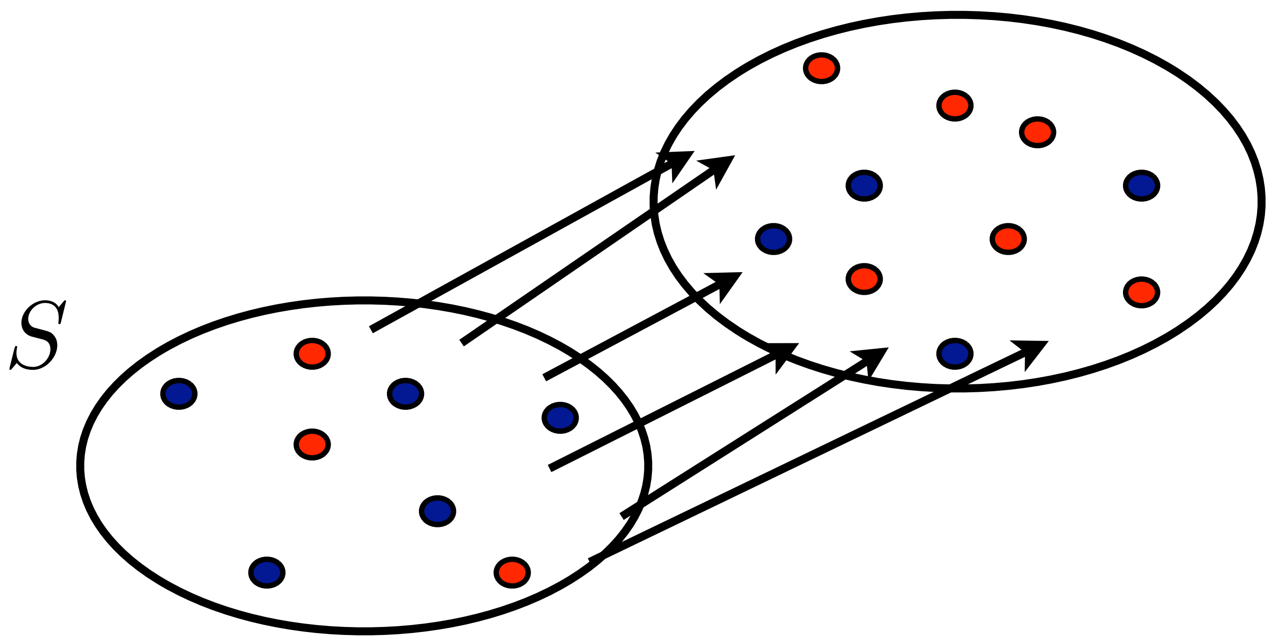

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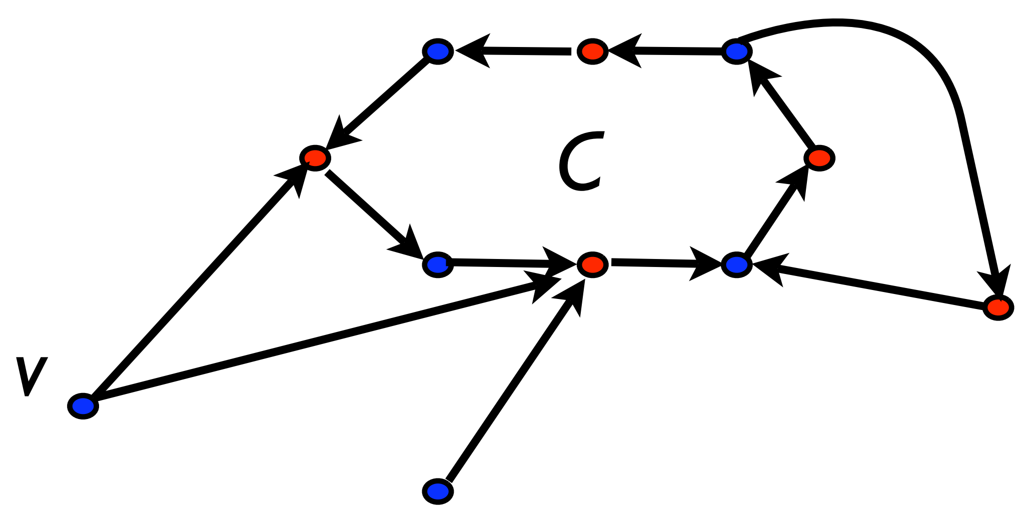

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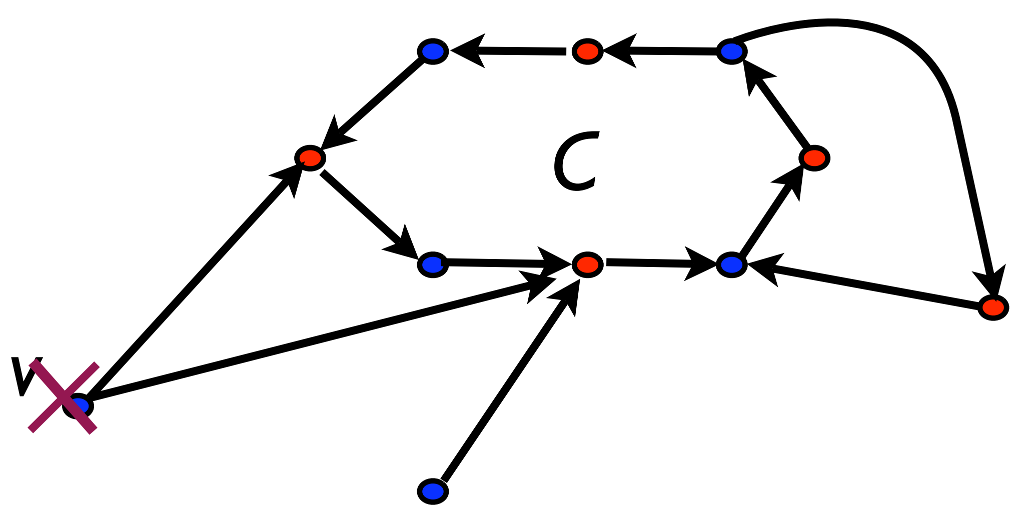
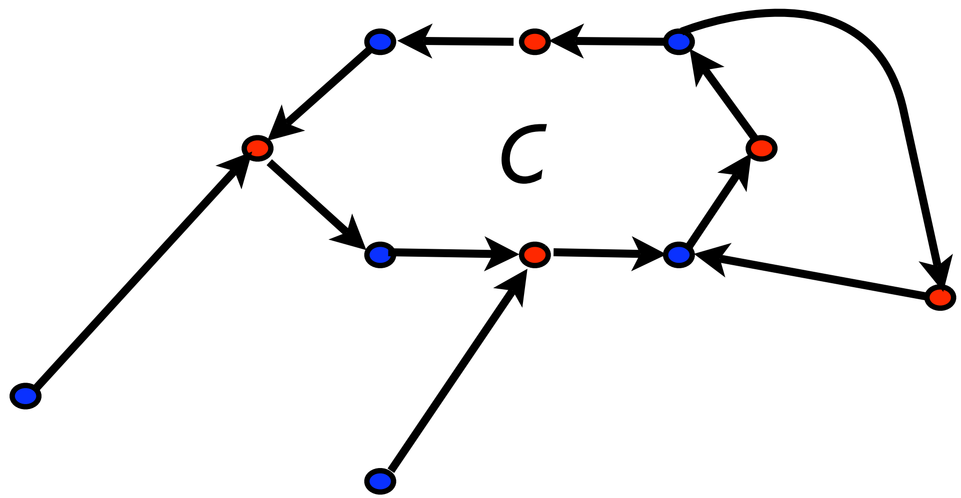


Undominated Induced Cycles

But an undominated, induced cycle gives a NE.



- Alice and Bob simply play the uniform distribution on their vertices in the cycle.

Theorem. There is a polytime algorithm to find a NE in a planar win-lose games. □

Open Problems

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- What other classes of game have polytime algorithms?