

Enumeration of Equilibria for Bimatrix Games

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Nash equilibria of bimatrix games

$A =$

3	3
2	5
0	6

$B =$

1	0
0	2
4	3

Nash equilibrium =

pair of strategies x , y with

x best response to y and

y best response to x .

Mixed equilibria

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 3 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$y^T = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$$

$$Ay = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$$

$$x^T B = \begin{bmatrix} 8/3 & 8/3 \end{bmatrix}$$

only pure best responses can have probability > 0

Best response polyhedron H_2 for player 2

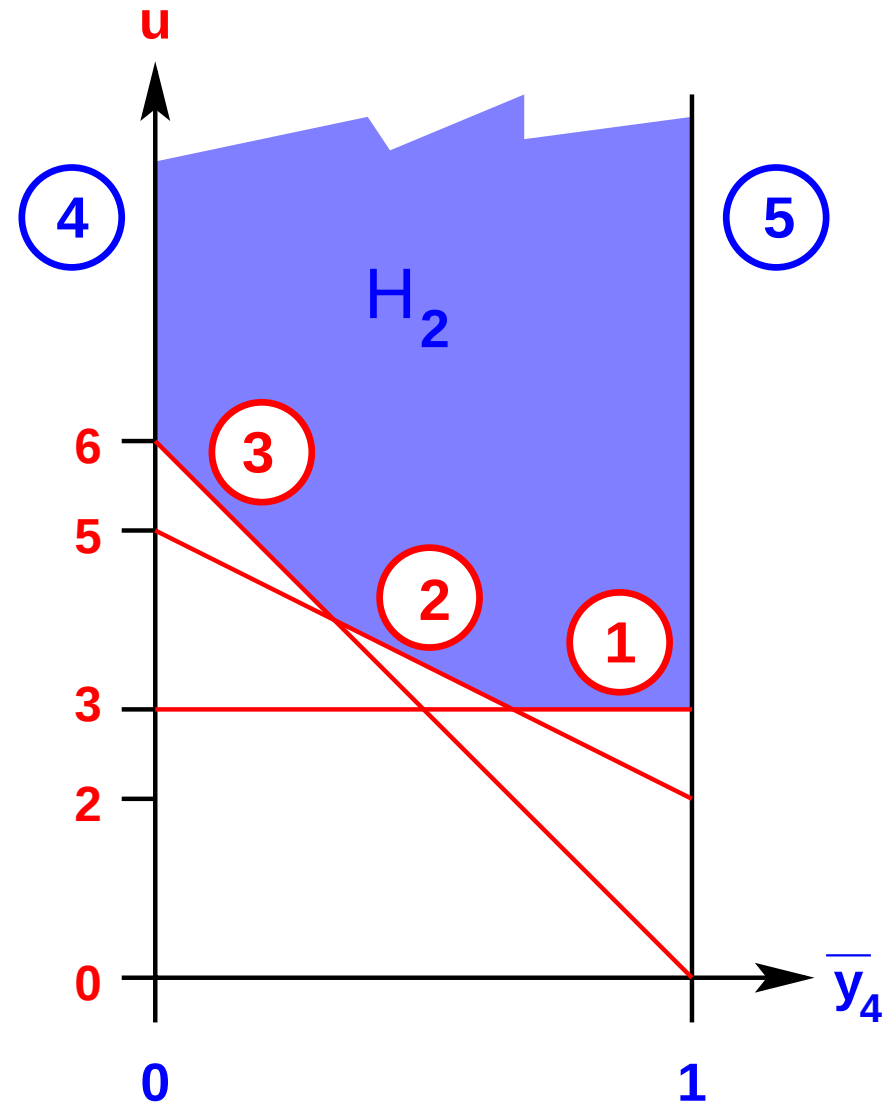
$$\begin{array}{c} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{array} \begin{array}{cc} \bar{y}_4 & \bar{y}_5 \\ \hline 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{array} = A$$

$$H_2 = \{ (\bar{y}_4, \bar{y}_5, u) \mid$$

$$\begin{array}{l} \textcircled{1} : 3\bar{y}_4 + 3\bar{y}_5 \leq u \\ \textcircled{2} : 2\bar{y}_4 + 5\bar{y}_5 \leq u \\ \textcircled{3} : 6\bar{y}_5 \leq u \end{array}$$

$$\bar{y}_4 + \bar{y}_5 = 1$$

$$\begin{array}{l} \textcircled{4} : \bar{y}_4 \geq 0 \\ \textcircled{5} : \bar{y}_5 \geq 0 \end{array} \}$$



Best response polytope Q for player 2

$$\begin{array}{c} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{array} \begin{array}{cc} y_4 & y_5 \\ \hline 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{array} = A$$

$$Q = \{ y \mid Ay \leq 1, y \geq 0 \}$$

$$Q = \{ (y_4, y_5) \mid$$

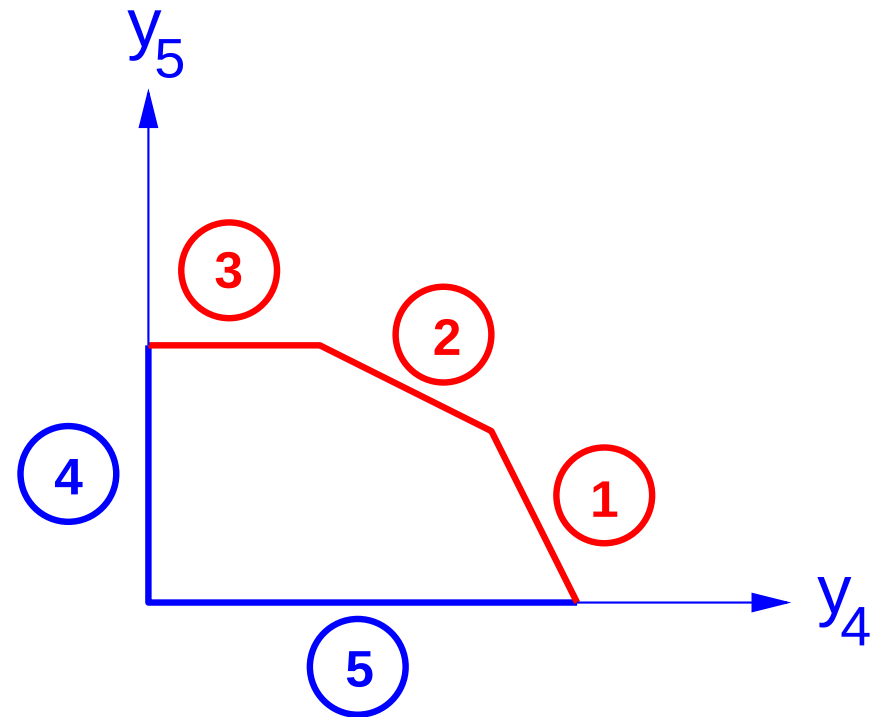
$$\textcircled{1} : 3y_4 + 3y_5 \leq 1$$

$$\textcircled{2} : 2y_4 + 5y_5 \leq 1$$

$$\textcircled{3} : 6y_5 \leq 1$$

$$\textcircled{4} : y_4 \geq 0$$

$$\textcircled{5} : y_5 \geq 0 \}$$

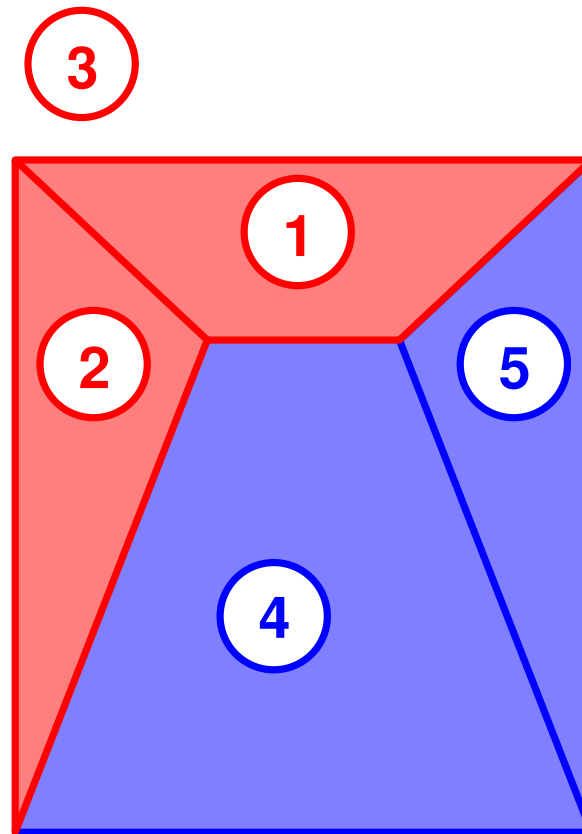
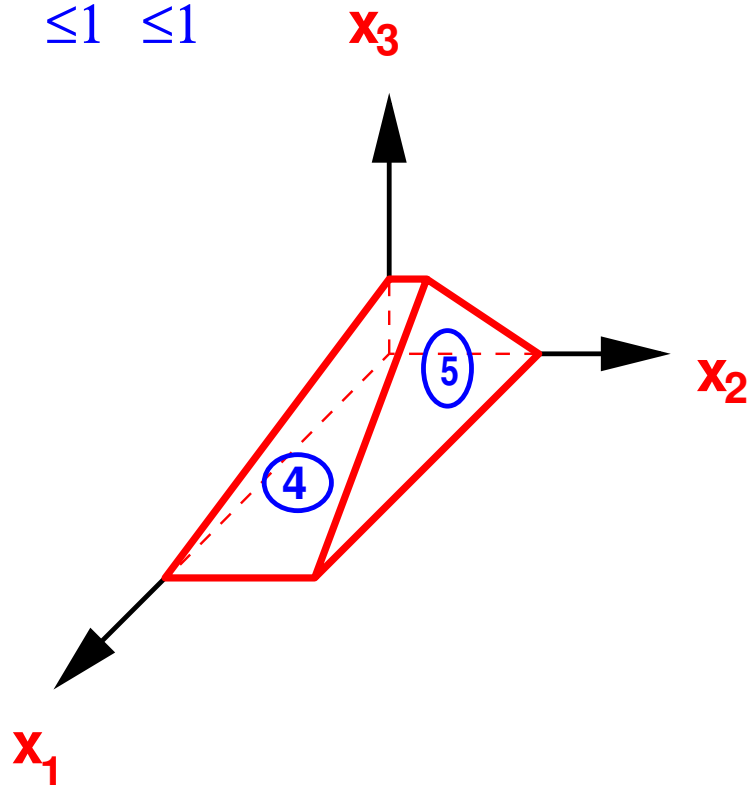


Best response polytope P for player 1

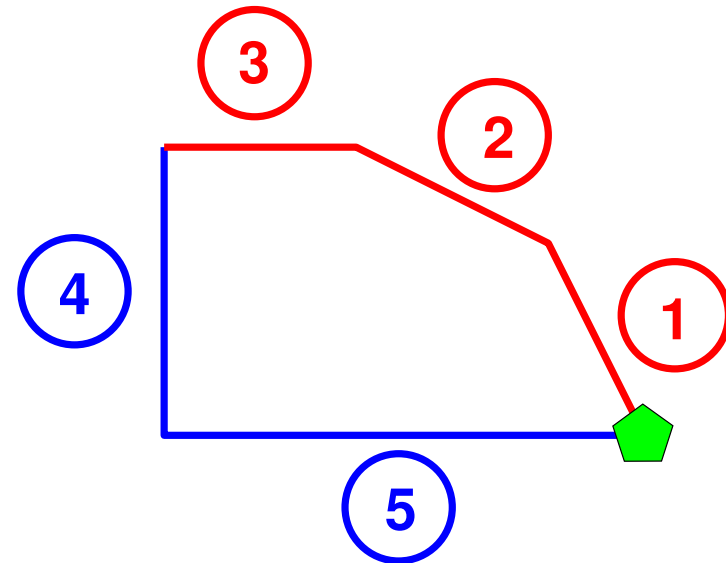
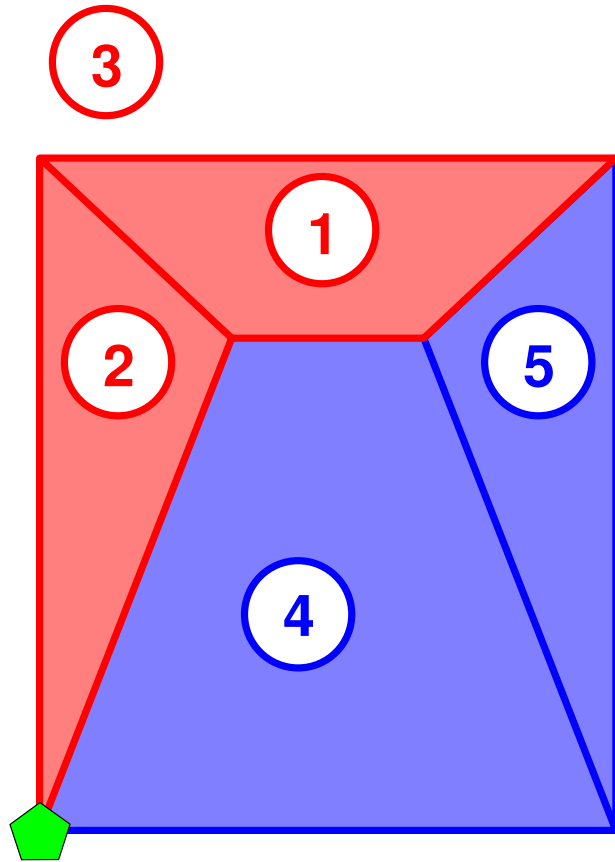
$$\begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{array} \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 2 \\ \hline 4 & 3 \\ \hline \end{array} = \mathbf{B}$$

$\leq 1 \quad \leq 1$

$$P = \{ x \mid x \geq 0, x^\top B \leq 1 \}$$

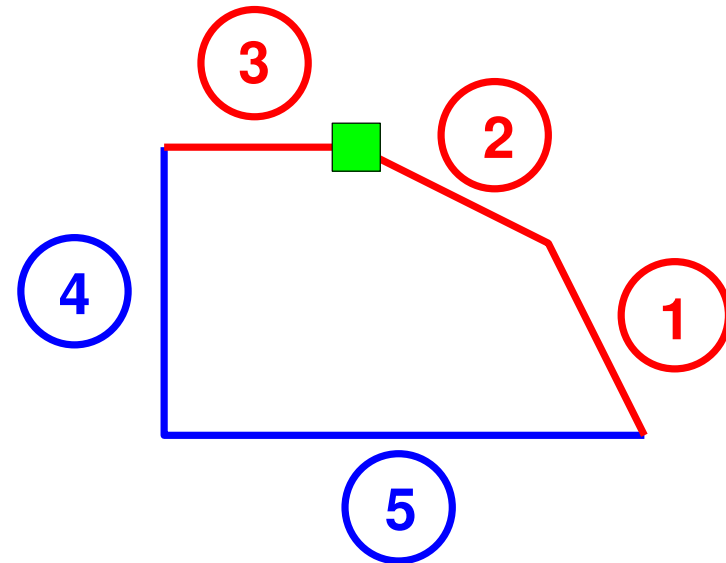
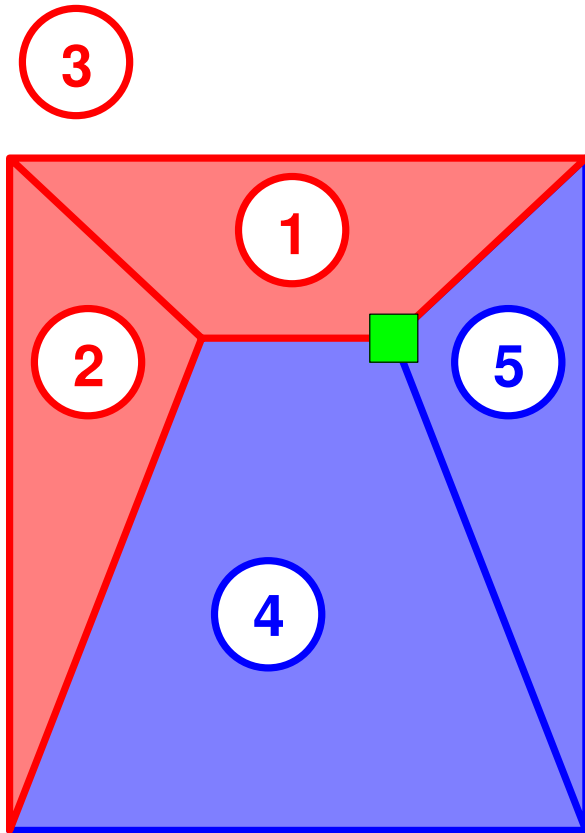


Equilibrium = completely labeled pair



pure equilibrium

Equilibrium = completely labeled pair



mixed equilibrium

Convex equilibrium components

[Winkels 1979 / Jansen 1980]

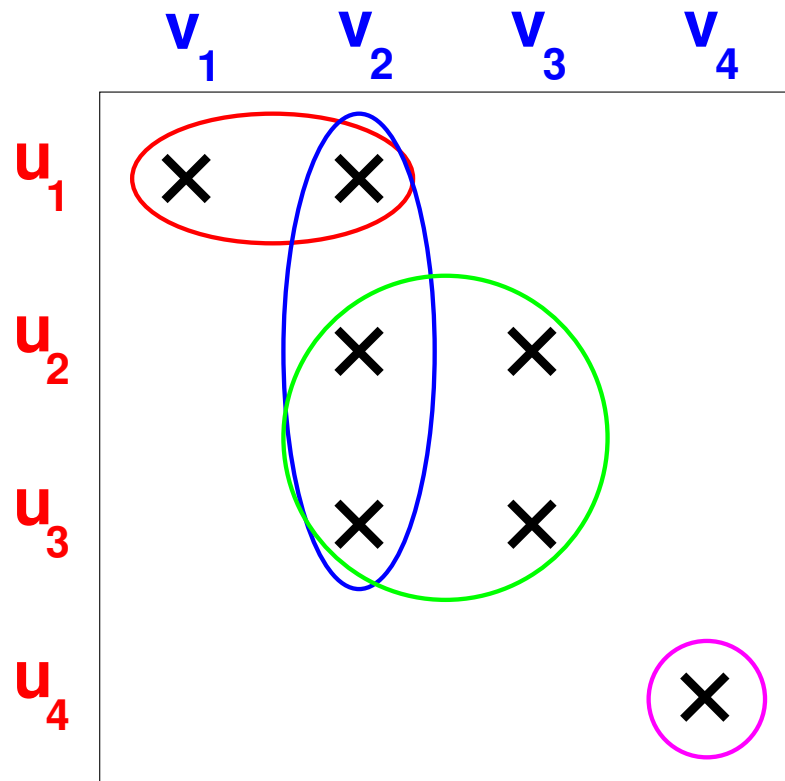
$(\mathbf{x}, \mathbf{y})$ is an equilibrium of $(\mathbf{A}, \mathbf{B}) \Leftrightarrow$

$(\mathbf{x}, \mathbf{y})$ is in the convex hull of $\mathbf{U} \times \mathbf{V}$, where all $(\mathbf{u}, \mathbf{v}) \in \mathbf{U} \times \mathbf{V}$ are completely labelled vertex pairs of $\mathbf{P} \times \mathbf{Q} - (\mathbf{0}, \mathbf{0})$

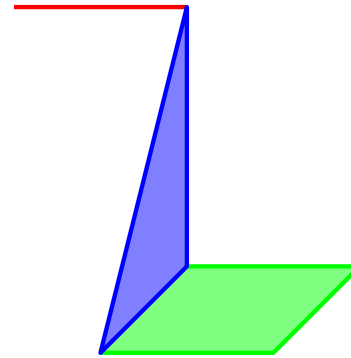
Convex equilibrium components $\mathbf{U} \times \mathbf{V}$

= maximal cliques of bipartite graph

Convex equilibrium components



Geometry:



Clique enumeration

[Bron & Kerbosch 1973]

Recursive bottom-up generation of maximal cliques by elegant backtracking and branch and bound.

Adapted to bipartite graphs, outputs 2000 cliques / second independent of graph size.

So far

Problem:

Given: bimatrix game (A, B) . What are its Nash equilibria?

Overview:

Any equilibrium is a convex combination of extreme equilibria = certain vertices of polytopes derived from A , B .

Enumerate extreme equilibria (finitely many).

Output convex equilibrium components.

No efficient equilibrium enumeration

[Gilboa & Zemel 1989]

Uniqueness of Nash equilibrium is NP -hard.

⇒ No output efficient enumeration of equilibria possible
(unless $P = NP$).

The EEE algorithm

Audet, C., P. Hansen, B. Jaumard, and G. Savard (2001),
Enumeration of all Extreme Equilibria of bimatrix games.
SIAM Journal on Scientific Computing **23**, 323-338.

- depth-first exploration of search tree for both polyhedra via parameterized LPs

Our improvements:

- stand-alone instead of using CPLEX
- exact arithmetic instead of floating-point
- much faster degenerate handling

Equilibria via LP duality

x best response to y

$\Leftrightarrow$ solves **primal LP**:

$$\max x^T(Ay)$$

$$\text{s.t. } \mathbf{1}^T x = 1$$

$$x \geq 0$$

dual LP: $\min u$

$$\text{s.t. } \mathbf{1}u \geq Ay$$

$$\text{optimal } \Leftrightarrow u = x^T Ay$$

complementary slackness:

$$\Leftrightarrow x \geq 0 \perp \mathbf{1}u \geq Ay$$

$$\text{i.e. } x^T (\mathbf{1}u - Ay) = 0$$

Equilibria via LP duality

x best response to y

$\Leftrightarrow$ solves **primal LP**:

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$$\text{s.t. } \mathbf{1}^T x = 1$$

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complementary slackness:

$$\Leftrightarrow x \geq 0 \perp \mathbf{1}u \geq Ay$$

$$\text{i.e. } x^T (\mathbf{1}u - Ay) = 0$$

y best response to x

$$\max (x^T B)y$$

$$\text{s.t. } \mathbf{1}^T y = 1$$

$$y \geq 0$$

$\min v$

$$\text{s.t. } \mathbf{1}v \geq B^T x$$

$$v = x^T B y$$

$$y \geq 0 \perp \mathbf{1}v \geq B^T x$$

$$y^T (\mathbf{1}v - B^T x) = 0$$

Equilibria via Linear Complementarity

x best response to y

$\Leftrightarrow$ solves **primal LP**:

$$\max x^T(Ay)$$

$$\text{s.t. } \boxed{1^T x = 1}$$

$$x \geq 0$$

dual LP: $\min u$

$$\text{s.t. } 1u \geq Ay$$

$$\text{optimal } \Leftrightarrow u = x^T Ay$$

complementary slackness:

$$\Leftrightarrow \boxed{x \geq 0 \perp 1u \geq Ay}$$

$$\text{i.e. } x^T (1u - Ay) = 0$$

y best response to x

$$\max (x^T B)y$$

$$\text{s.t. } \boxed{1^T y = 1}$$

$$y \geq 0$$

$\min v$

$$\text{s.t. } 1v \geq B^T x$$

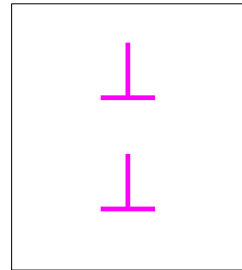
$$v = x^T B y$$

$$\boxed{y \geq 0 \perp 1v \geq B^T x}$$

$$y^T (1v - B^T x) = 0$$

Best response polyhedra

$$\mathbf{P} = \{ (\mathbf{x}, \mathbf{v}) \in \mathbf{R}^{m+1} \mid$$
$$\mathbf{x} \geq \mathbf{0}$$
$$\mathbf{B}^T \mathbf{x} \leq \mathbf{1} \mathbf{v}$$
$$\mathbf{1}^T \mathbf{x} = 1 \}$$



$$\mathbf{Q} = \{ (\mathbf{y}, \mathbf{u}) \in \mathbf{R}^{n+1} \mid$$
$$\mathbf{A} \mathbf{y} \leq \mathbf{1} \mathbf{u}$$
$$\mathbf{y} \geq \mathbf{0}$$
$$\mathbf{1}^T \mathbf{y} = 1 \}$$

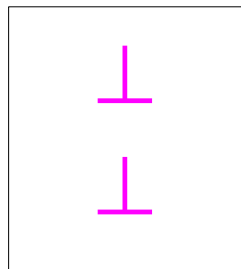
Best response polyhedra

$$\mathbf{P} = \{ (\mathbf{x}, \mathbf{v}) \in \mathbf{R}^{m+1} \mid$$

$$\mathbf{x} \geq \mathbf{0}$$

$$\mathbf{B}^T \mathbf{x} \leq \mathbf{1} \mathbf{v}$$

$$\mathbf{1}^T \mathbf{x} = 1 \}$$



$$\mathbf{Q} = \{ (\mathbf{y}, \mathbf{u}) \in \mathbf{R}^{n+1} \mid$$

$$\mathbf{A} \mathbf{y} \leq \mathbf{1} \mathbf{u}$$

$$\mathbf{y} \geq \mathbf{0}$$

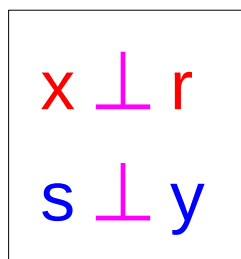
$$\mathbf{1}^T \mathbf{y} = 1 \}$$

with slack variables $\mathbf{r}, \mathbf{s}$:

$$\mathbf{x}, \mathbf{s} \geq \mathbf{0}$$

$$\mathbf{B}^T \mathbf{x} + \mathbf{s} = \mathbf{1} \mathbf{v}$$

$$\mathbf{1}^T \mathbf{x} = 1$$



$$\mathbf{r} + \mathbf{A} \mathbf{y} = \mathbf{1} \mathbf{u}$$

$$\mathbf{r}, \mathbf{y} \geq \mathbf{0}$$

$$\mathbf{1}^T \mathbf{y} = 1$$

Faces of **P**, **Q** via label sets **K**

Let $K, L \subseteq M \cup N$ = set of labels (pure strategies)

$$P(K) = \{ (x, v) \in P \mid x_i = 0, (B^T x)_j = v \text{ for } i, j \in K \} \\ [\Leftrightarrow s_j = 0]$$

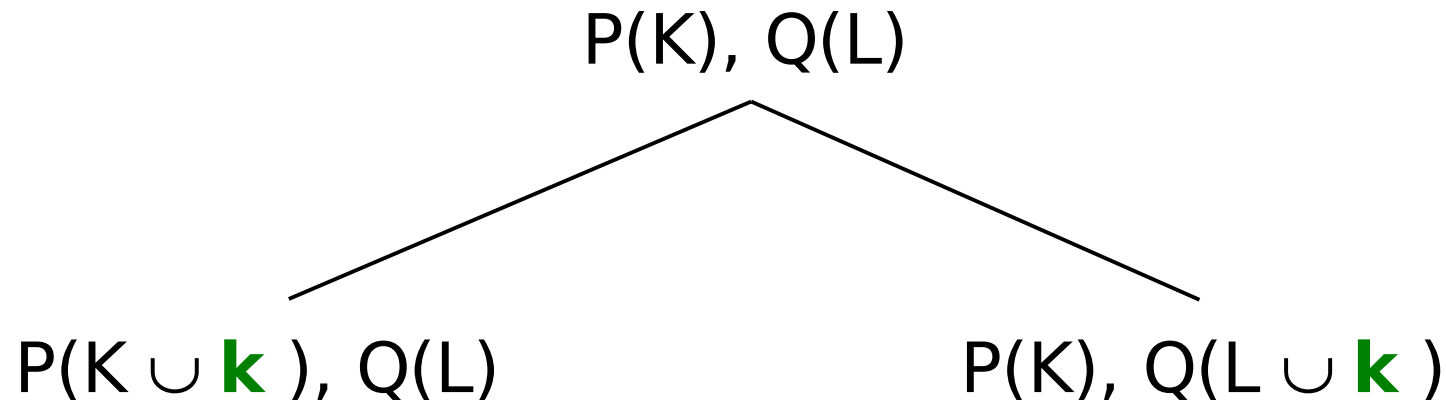
$$Q(L) = \{ (y, u) \in Q \mid (Ay)_i = u, y_j = 0 \text{ for } i, j \in L \} \\ [\Leftrightarrow r_i = 0]$$

EEE - binary search tree

root: $P, Q = P(\emptyset), Q(\emptyset)$

nodes: $P(K), Q(L)$ with disjoint K, L

binary branching: new label $\mathbf{k}$ added to K or L



Face representatives

face $P(K)$ of P represented by $x \in P(K)$,

face $Q(L)$ of L represented by $y \in Q(L)$.

$x \in P(K), y \in Q(L)$

$P(K \cup \mathbf{k}), Q(L)$

new x , same y

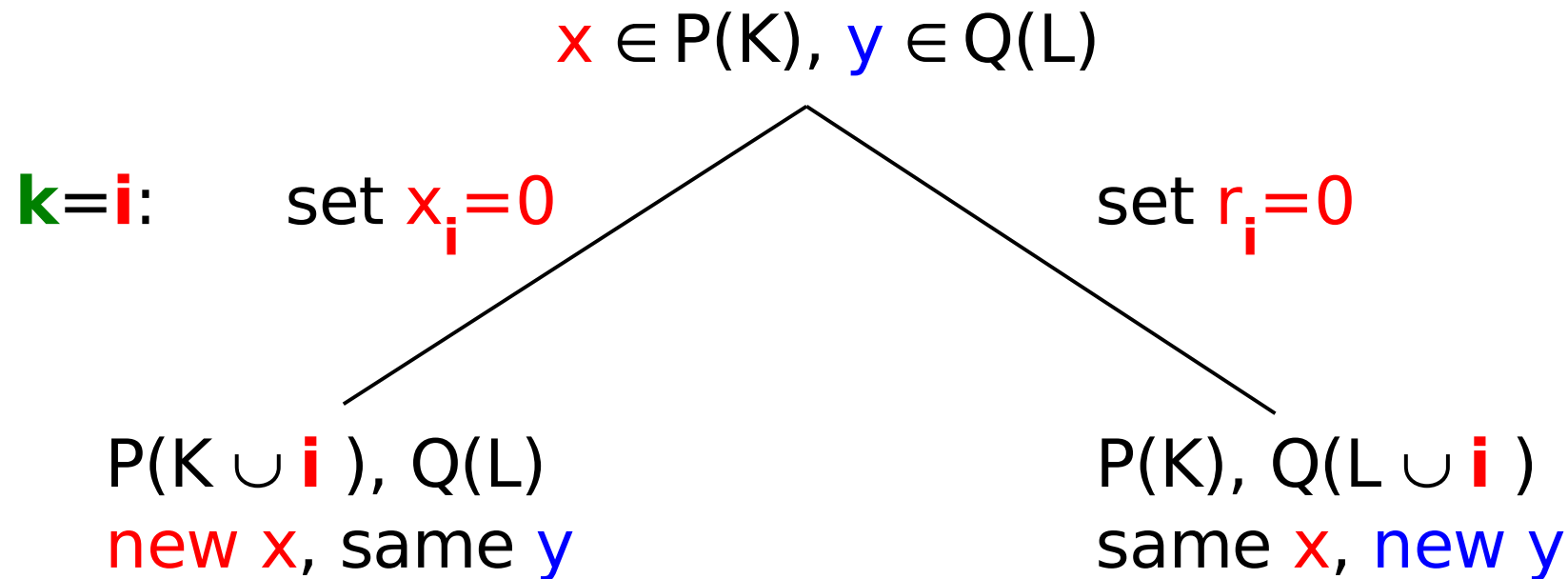
$P(K), Q(L \cup \mathbf{k})$

same x , new y

Face representatives

face $P(K)$ of P represented by $x \in P(K)$,

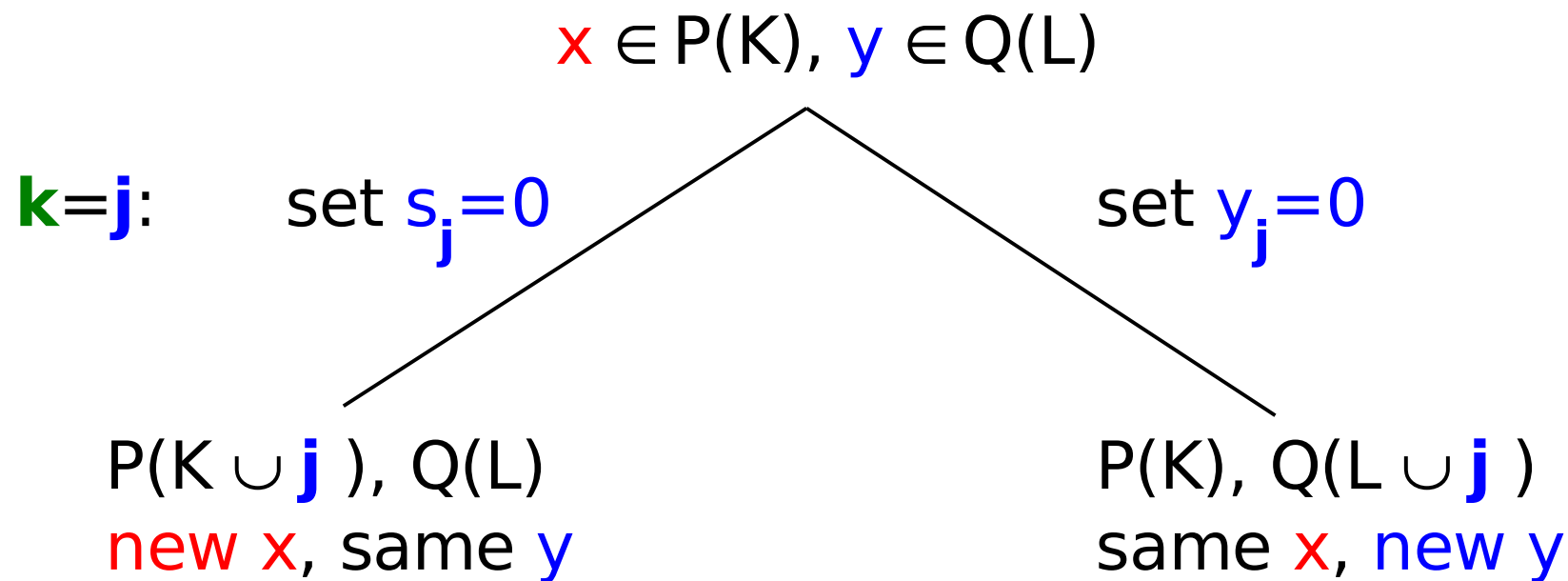
face $Q(L)$ of L represented by $y \in Q(L)$.



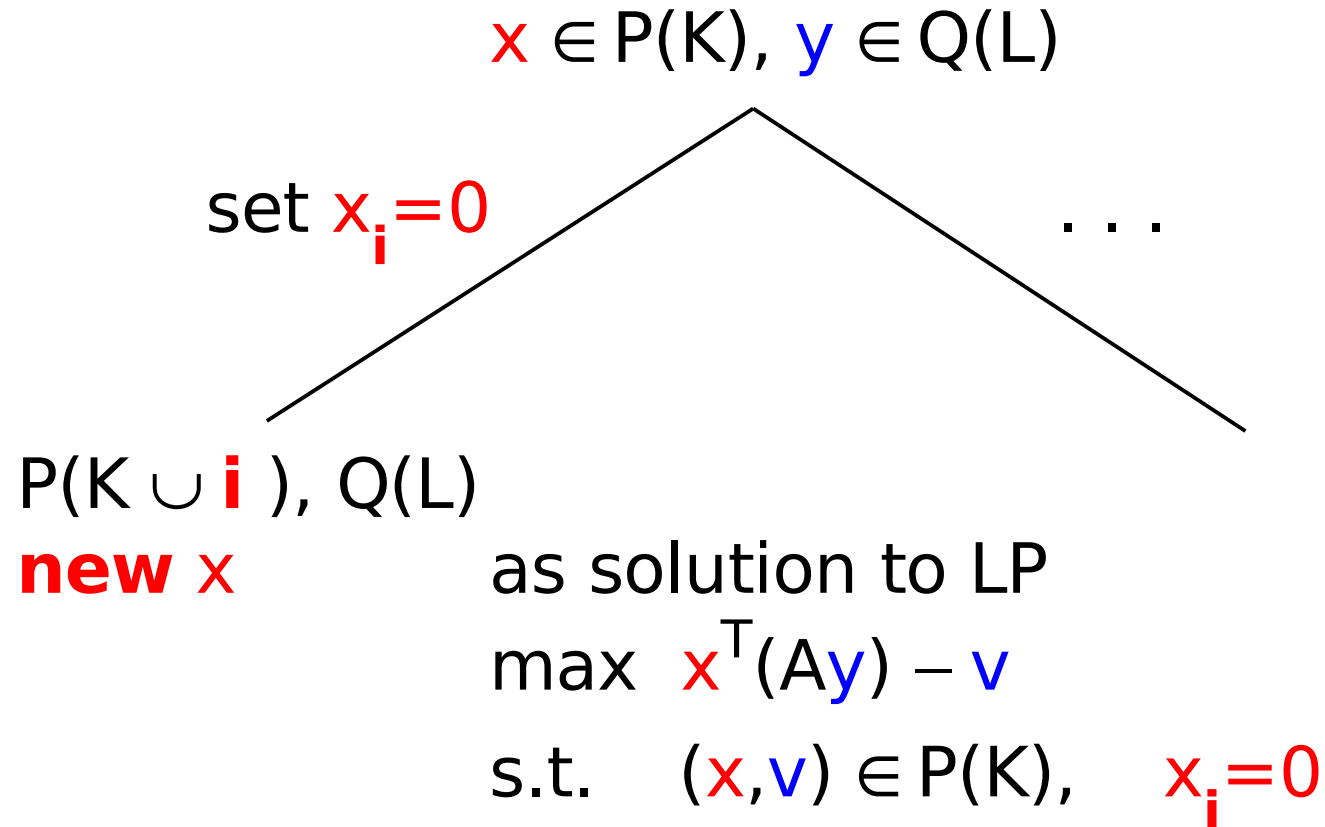
Face representatives

face $P(K)$ of P represented by $x \in P(K)$,

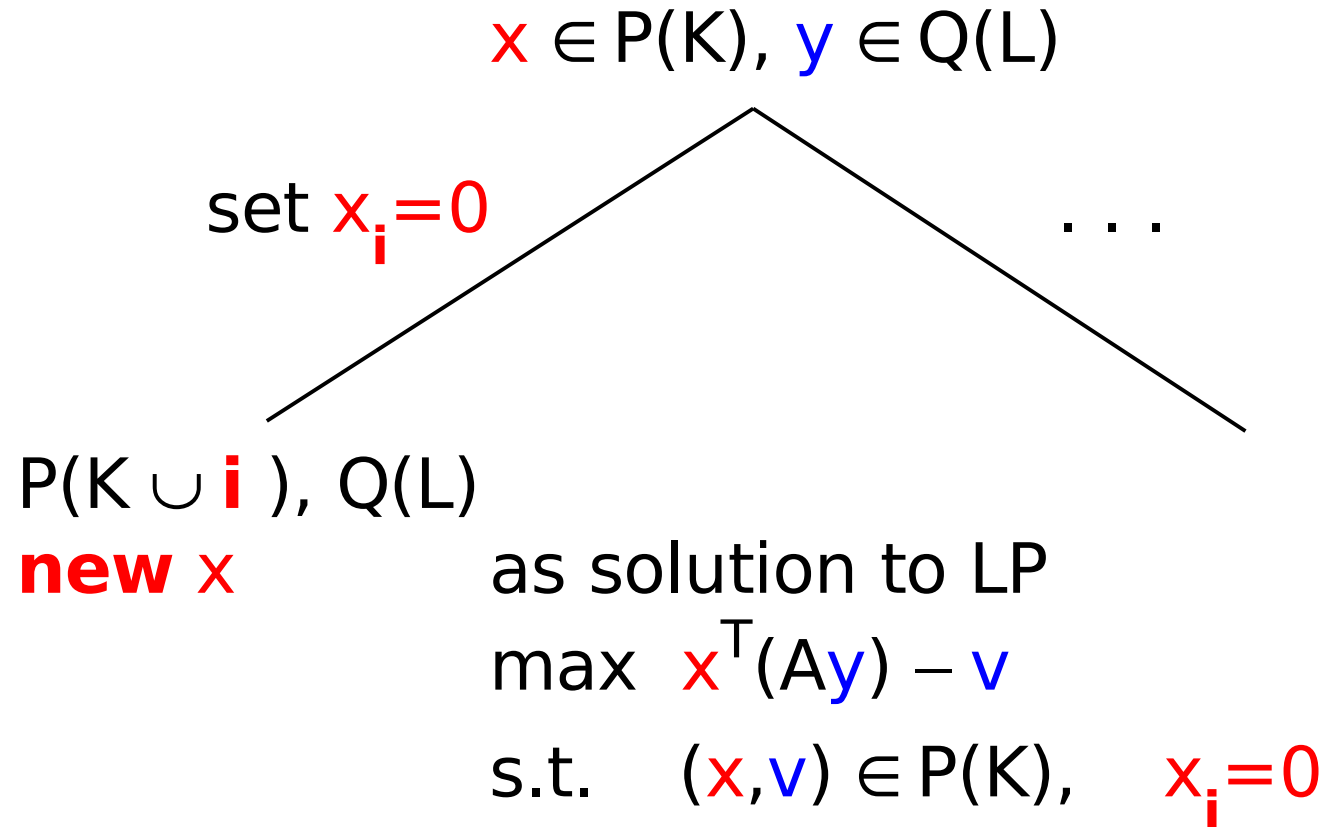
face $Q(L)$ of L represented by $y \in Q(L)$.



To smaller-dimensional face via LP

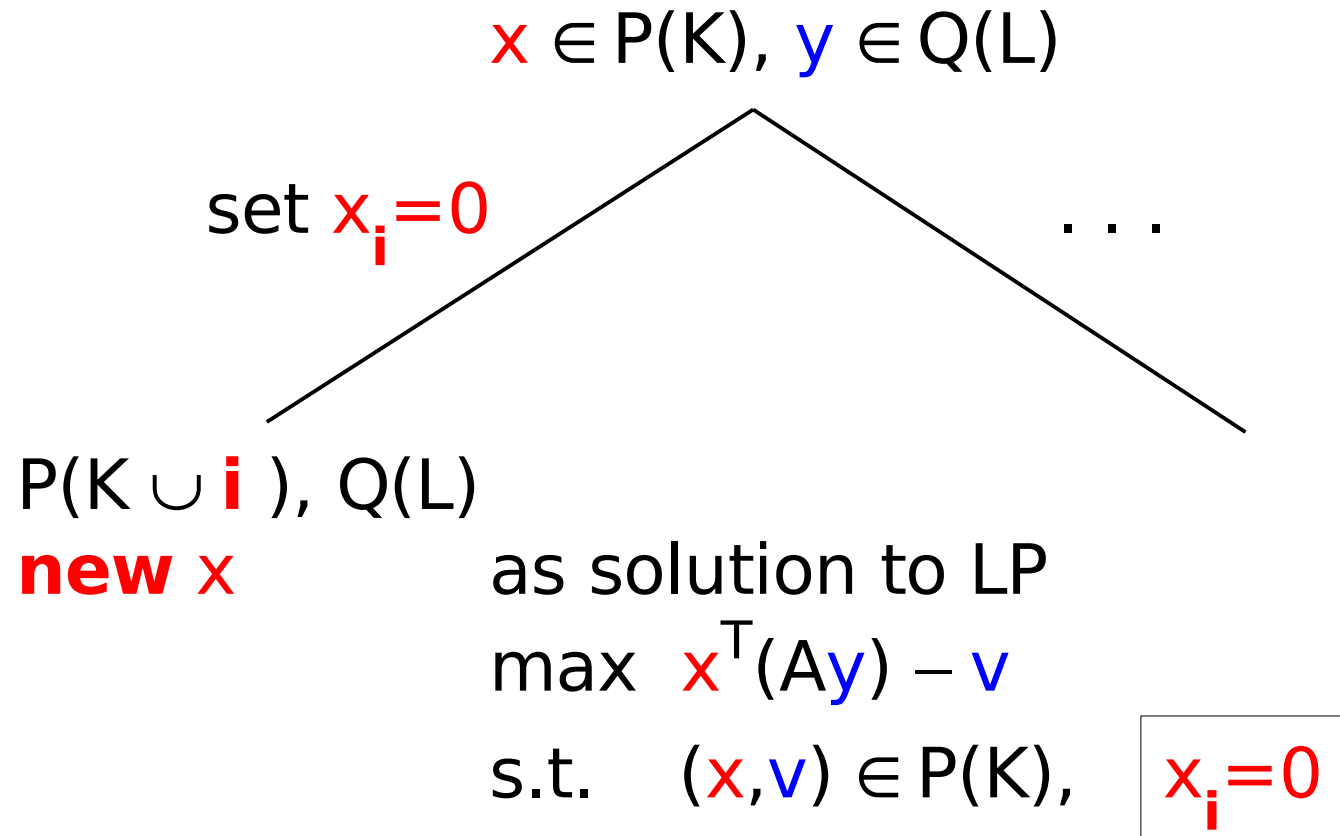


To smaller-dimensional face via LP



if infeasible: terminate search,
backtrack (depth-first search)

To smaller-dimensional face via LP



in LP tableau: drive x_i out of basis
(use tableau row to minimize x_i),
then **delete** cobasic column for x_i

Choice of label to branch on

given node with $x \in P(K)$, $y \in Q(L)$,

branch on k that violates complementarity most, i.e.

$$k = \arg \max \{ x_i r_i, s_j y_j \mid i, j \notin K \cup L \}$$

$\Rightarrow$ hope to discover infeasibility quickly
to prune search tree.

Choice of label to branch on

given node with $x \in P(K)$, $y \in Q(L)$,

branch on k that violates complementarity most, i.e.

$$k = \arg \max \{ x_i r_i, s_j y_j \mid i, j \notin K \cup L \}$$

possible: $\max \{ x_i r_i, s_j y_j \} = 0$
 $\Leftrightarrow (x, y)$ equilibrium,

but: is **ignored** unless $|K| = m$, $|L| = n$

(Nash equilibrium at leaf of search tree,
node at depth $m+n$).

Choice of objective function

given $y \in Q(L)$, $\mathbf{x} \in P(K \cup \mathbf{k})$ is solution to LP
 $\max \mathbf{x}^T(Ay) - v$
s.t. $(\mathbf{x}, v) \in P(K \cup \mathbf{k})$

given $x \in P(K)$, $y \in Q(L \cup \mathbf{k})$ is solution to LP
 $\max (x^T B)y - u$
s.t. $(y, u) \in Q(L \cup \mathbf{k})$

Choice of objective function

given $y \in Q(L)$, $x \in P(K \cup k)$ is solution to LP

$$\max x^T(Ay) - v$$

$$\text{s.t. } (x, v) \in P(K \cup k)$$

given $x \in P(K)$, $y \in Q(L \cup k)$ is solution to LP

$$\max (x^T B)y - u$$

$$\text{s.t. } (y, u) \in Q(L \cup k)$$

Why these objective functions? Their sum

$$x^T(Ay) + (x^T B)y - u - v \quad [\leq 0]$$

is negative of duality gap, try to make that zero!

Other objective functions

ignoring $y \in Q(L)$, $\mathbf{x} \in P(K \cup \mathbf{k})$ is solution to LP

$$\max \quad -v$$

$$\text{s.t.} \quad (\mathbf{x}, v) \in P(K \cup \mathbf{k})$$

ignoring $x \in P(K)$, $y \in Q(L \cup \mathbf{k})$ is solution to LP

$$\max \quad -u$$

$$\text{s.t.} \quad (y, u) \in Q(L \cup \mathbf{k})$$

find lowest point on upper envelope

Other objective functions

$x \in P(K \cup \mathbf{k})$ is solution to LP

max 0

s.t. $(x, v) \in P(K \cup \mathbf{k})$

$y \in Q(L \cup \mathbf{k})$ is solution to LP

max 0

s.t. $(y, u) \in Q(L \cup \mathbf{k})$

find feasible vertex

$\Rightarrow$ still on upper envelope

Run time comparisons [sec]

Objective function	Random (average) 17 x 17 57 NE	Guessing Game 22 x 22 3 NE	Dollar Game 10 x 10 91 NE
P: $\max \mathbf{x}^T \mathbf{A} \mathbf{y} - \mathbf{v}$ Q: $\max \mathbf{x}^T \mathbf{B} \mathbf{y} - \mathbf{u}$	186	44	126
P: $\max -\mathbf{v}$ Q: $\max -\mathbf{u}$	408	185	119
P: $\max 0$ Q: $\max 0$	397	169	117
P: $\max \mathbf{x}^T (\mathbf{A} + \mathbf{B}) \mathbf{y} - \mathbf{v}$ Q: $\max \mathbf{x}^T (\mathbf{A} + \mathbf{B}) \mathbf{y} - \mathbf{u}$	146	33	122

Extra multiple branching needed

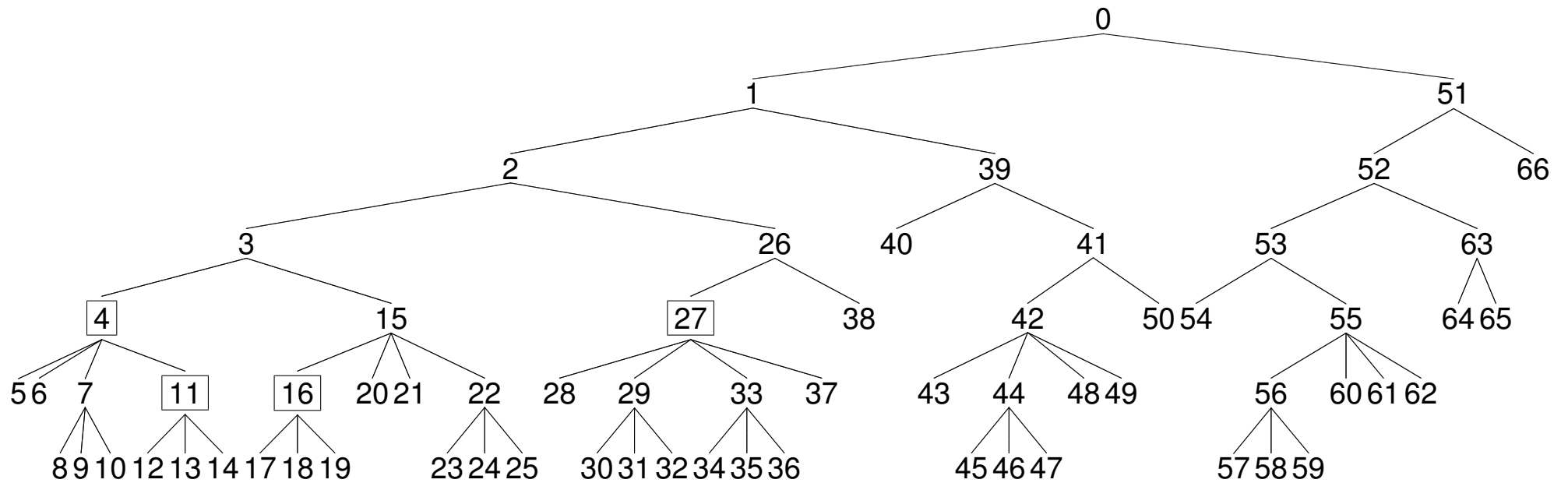
In degenerate games, **not all** extreme equilibria appear at level $m+n$ in the binary search tree.

Audet et al. add **multiple** branches at level $m+n$: each positive variable x_i, s_j, r_i, y_j at that level is attempted to be set to zero (typically infeasible, $m+n$ extra pivots), continuing search.

One observes a **very high number** of extra nodes for degenerate games.

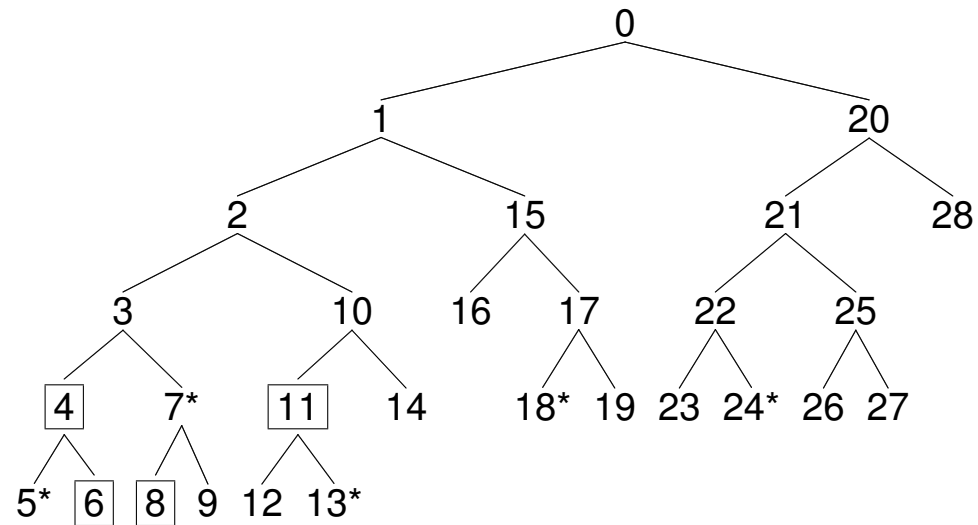
Example of original EEE search tree

$$A = \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 4 \\ 5 & 4 \end{pmatrix}$$



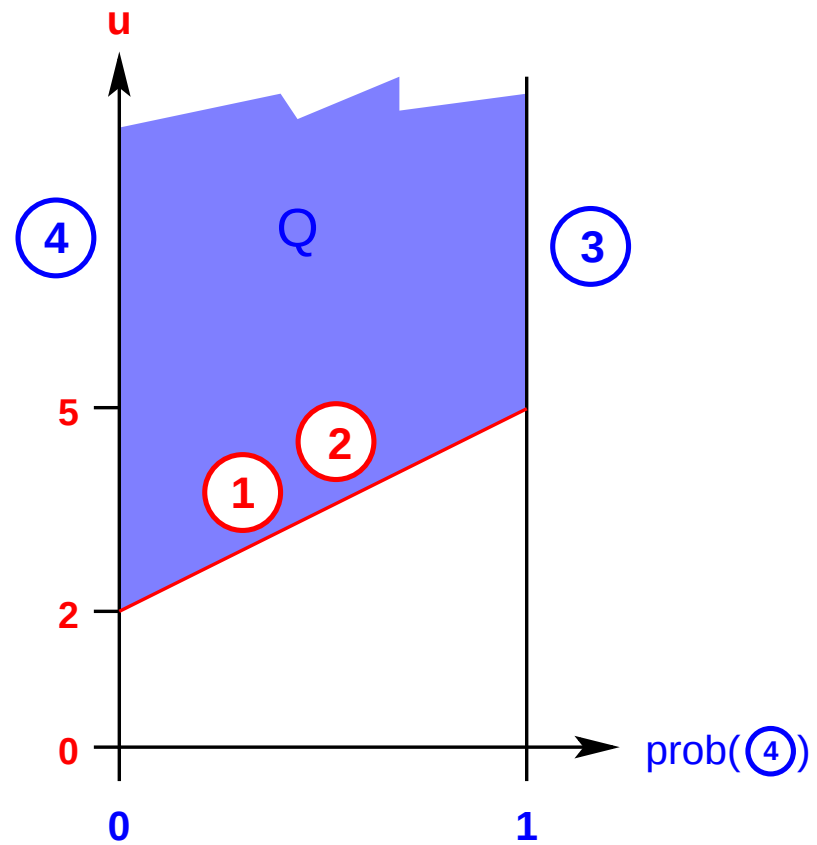
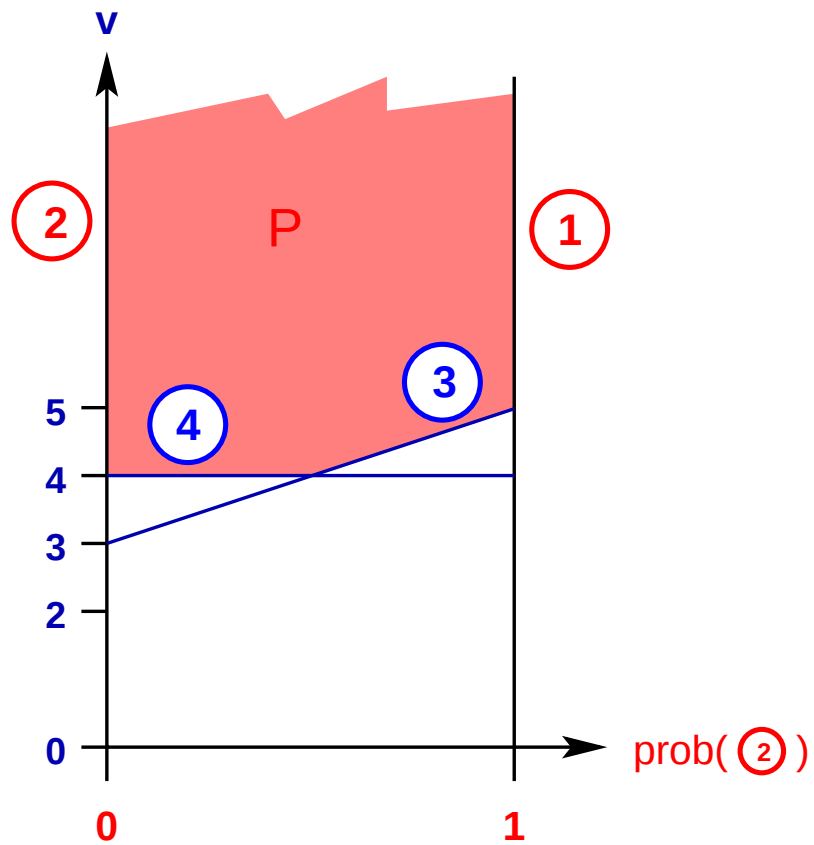
Improved EEE search tree

$$A = \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 4 \\ 5 & 4 \end{pmatrix}$$



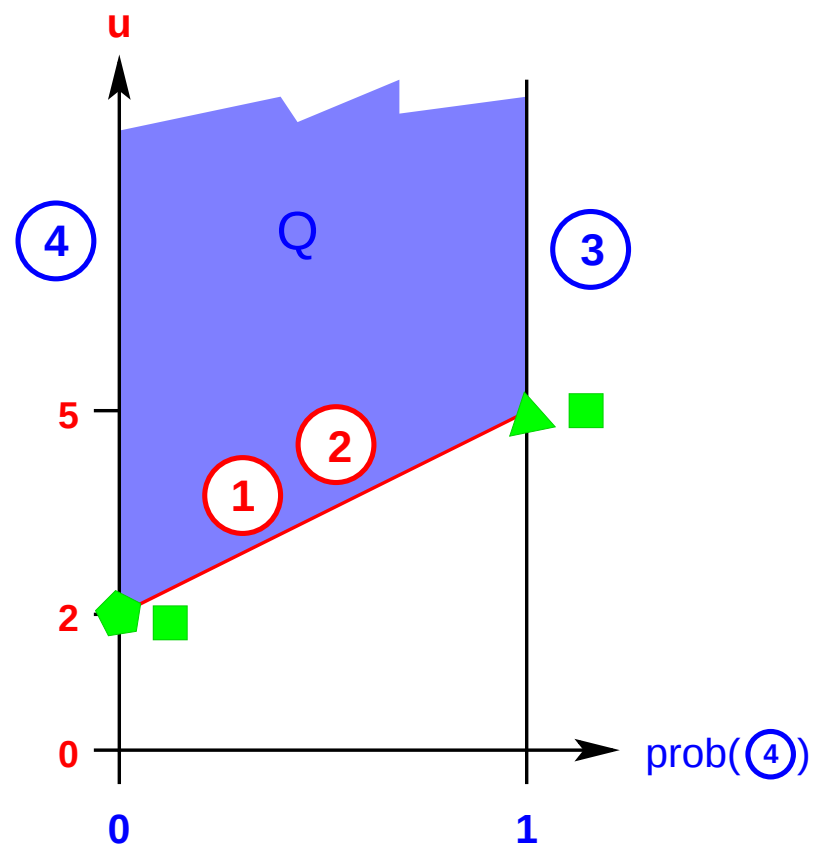
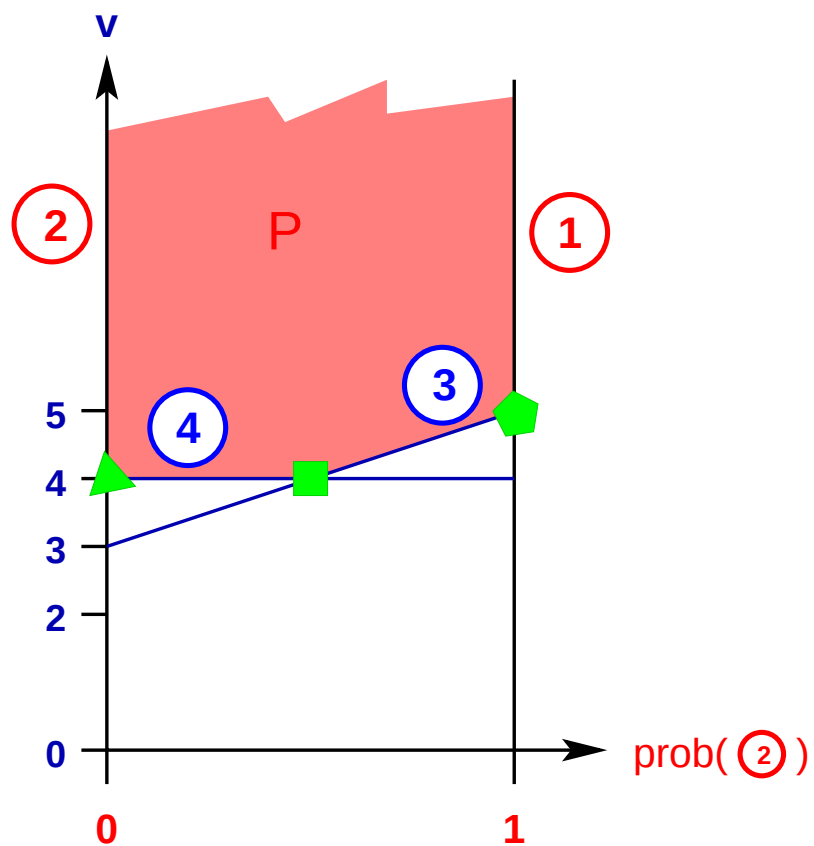
Polyhedra for example

$$A = \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 4 \\ 5 & 4 \end{pmatrix}$$



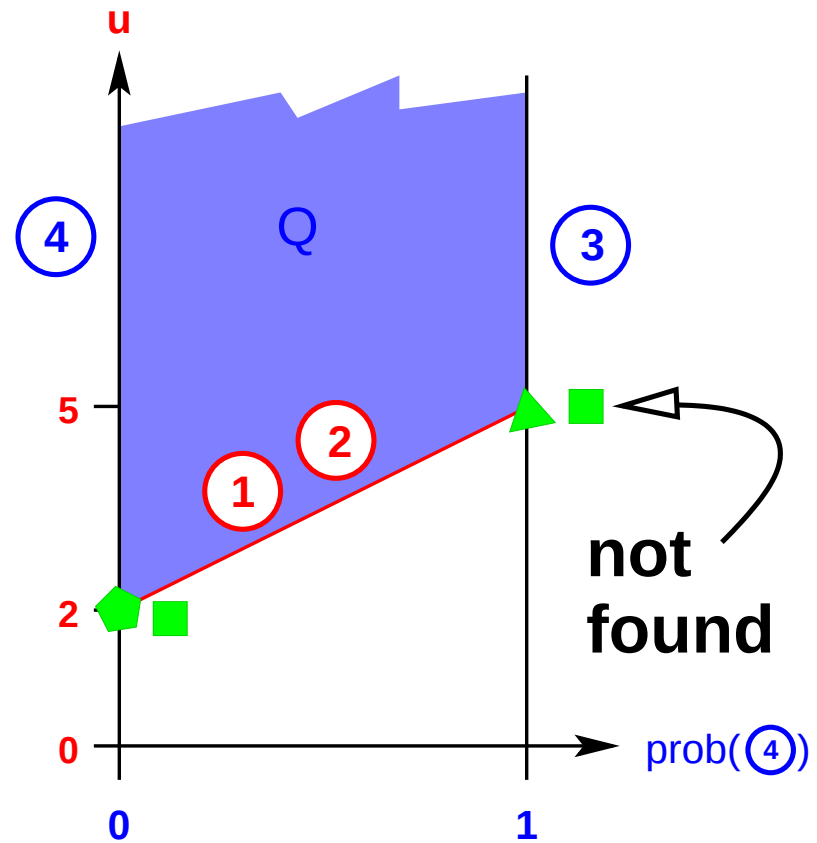
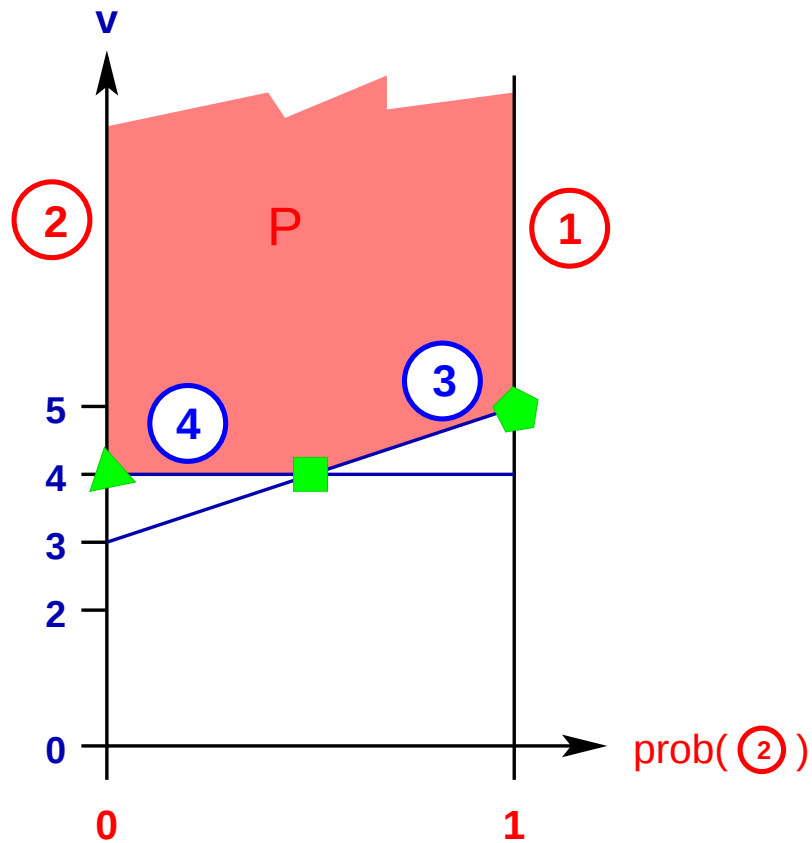
Extreme equilibria

$$A = \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 4 \\ 5 & 4 \end{pmatrix}$$



One equilibrium missed by objective function

$$A = \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 4 \\ 5 & 4 \end{pmatrix}$$



Improved degeneracy treatment

Vertices are missed due to **redundant equalities** (degeneracy in **description** of polyhedron!).

These appear as **zero rows** in tableaux and **cannot** be made **cobasic**.

⇒ **delete** zero row, keep extra **cobasic** variable;

only these extra cobasic variables are searched (eliminated) beyond search level **m+n**.

Run time comparisons

Degeneracy check	Guessing Game 5 NE	Dollar Game 73 NE
Original EEE time nodes pivots	0.36 427 530	14806.06 279020949 20326989
Improved EEE time nodes pivots	0.37 337 530	25.25 141531 43987