Congestion control algorithms

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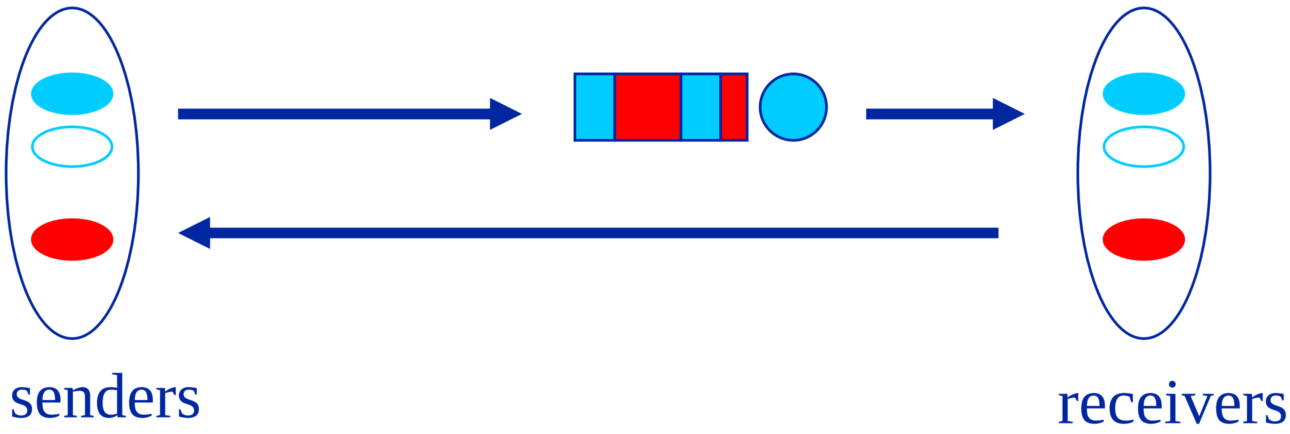
Fluid model for a network operating under a fair bandwidth-sharing policy K & W *Ann Appl Prob 2004*
On fluid and Brownian approximations for an Internet congestion control model. W. Kang, K, N.H. Lee & W *CDC 2004*
State space collapse and diffusion approximation…
W. Kang, K, N.H. Lee & W *forthcoming*

End-to-end congestion control



Senders learn (through feedback from receivers) of congestion at queue, and slow down or speed up accordingly. With current TCP, throughput of a flow is proportional to

$$1/(T\sqrt{p})$$

T = round-trip time, p = packet drop probability.

(Jacobson 1988, Mathis, Semke, Mahdavi, Ott 1997, Padhye, Firoiu, Towsley, Kurose 1998, Floyd and Fall 1999)

Model definition

- We want to describe a network model, with fluctuating numbers of flows
- We first need
  - notation for network structure
  - abstraction of rate allocation
- Then we need to define the random nature of flow arrivals and departures

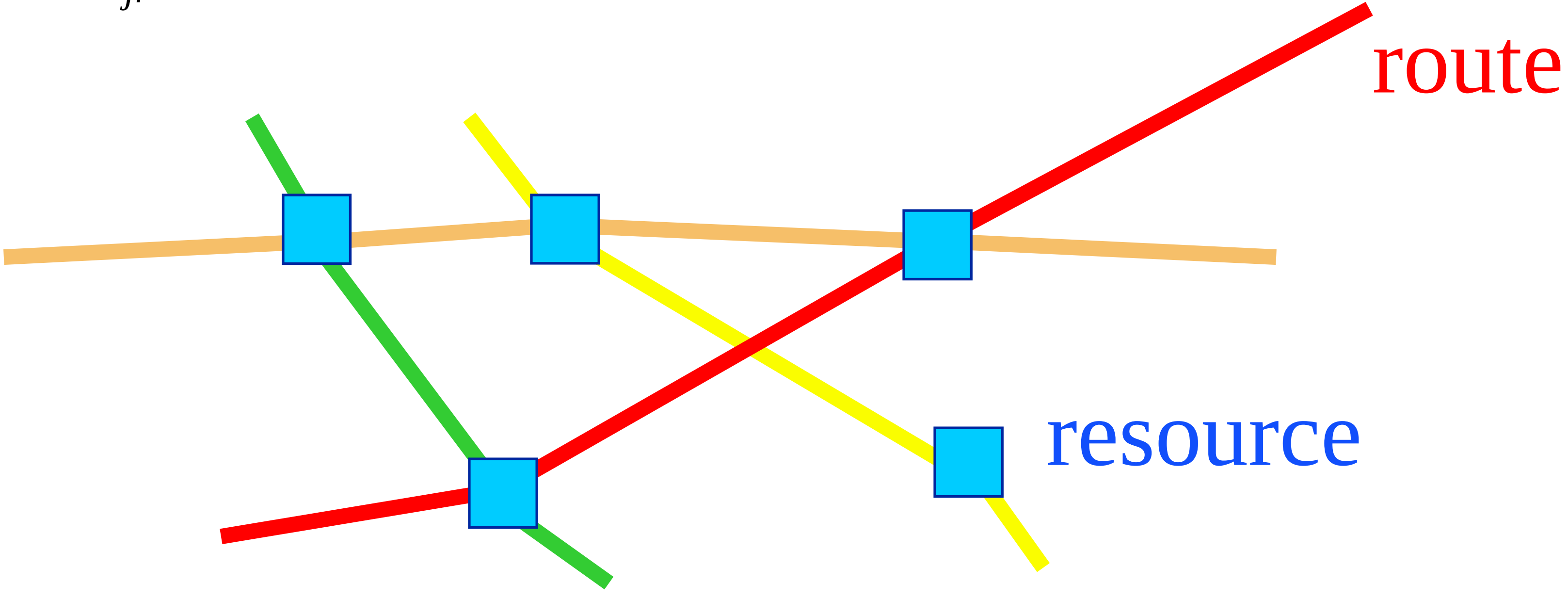
Network structure (J, R, A)

J - set of resources

R - set of routes

$A_{jr} = 1$ - if resource j is on route r

$A_{jr} = 0$ - otherwise



Rate allocation

w_r - weight of route r

n_r - number of flows on route r

x_r - rate of each flow on route r

Given the vector $n = (n_r, r \in R)$

how are the rates $x = (x_r, r \in R)$

chosen ?

Optimization formulation

Suppose $x = x(n)$ is chosen to

maximize $\sum_r w_r n_r \frac{x_r^{1-\alpha}}{1-\alpha}$

subject to $\sum_r A_{jr}\, n_r\, x_r \le C_j \quad j \in J$

$$x_r \ge 0 \quad r \in R$$

(weighted α-fair allocations, Mo and Walrand 2000)

$0 < \alpha < \infty$ (replace $\frac{x_r^{1-\alpha}}{1-\alpha}$ by $\log(x_r)$ if $\alpha = 1$)

Solution

$$x_r = \left(\frac{w_r}{\sum_j A_{jr} p_j(n)} \right)^{1/\alpha} \qquad r \in R$$

$p_j(n)$ - shadow price (Lagrange multiplier) for the resource j capacity constraint

Observe alignment with square-root formula when

$$\alpha = 2, \quad w_r = 1/T_r^2, \quad p_r \approx \sum_j A_{jr} p_j$$

Examples of α-fair allocations

maximize $\sum_r w_r n_r \frac{x_r^{1-\alpha}}{1-\alpha}$

subject to $\sum_r A_{jr} n_r x_r \leq C_j \quad j \in J$

$x_r \geq 0 \quad r \in R$

$$x_r = \left(\frac{w_r}{\sum_j A_{jr} p_j(n)} \right)^{1/\alpha} \quad r \in R$$

$\alpha \to 0 \quad (w = 1)$ - maximum flow

$\alpha \to 1 \quad (w = 1)$ - proportionally fair

$\alpha = 2 \quad (w_r = 1/T_r^2)$ - TCP fair

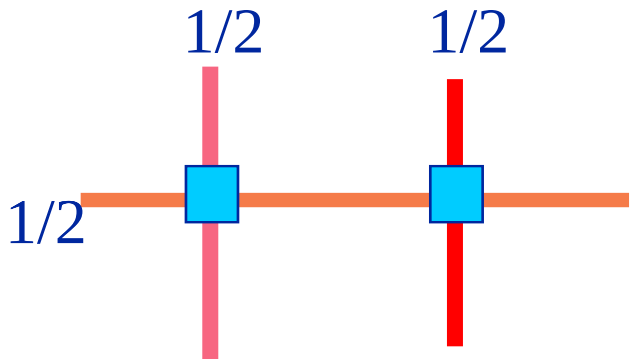
$\alpha \to \infty \quad (w = 1)$ - max-min fair

Example

$$n_r = 1, \;\; w_r = 1 \;\; r \in R,$$
$$C_j = 1 \;\; j \in J$$

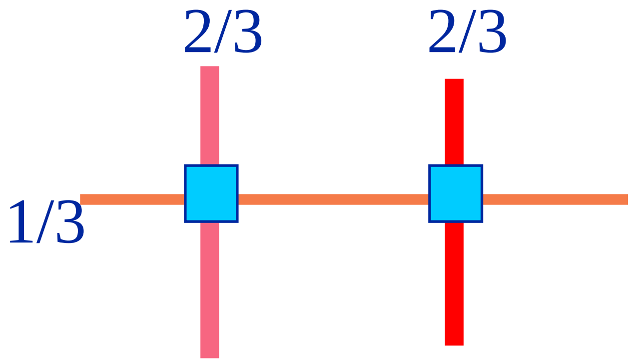
max-min fairness:

$$\alpha \to \infty$$



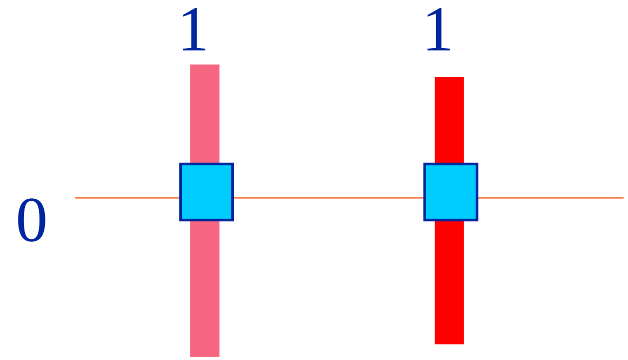
proportional fairness:

$$\alpha = 1$$



maximum flow:

$$\alpha \to 0$$



Flow level model

Define a Markov process $n(t) = (n_r(t), r \in R)$

with transition rates

$$n_r \to n_r + 1 \quad \text{at rate} \quad \nu_r \qquad r \in R$$

$$n_r \to n_r - 1 \quad \text{at rate} \quad n_r \, x_r(n) \, \mu_r \quad r \in R$$

- Poisson arrivals, exponentially distributed file sizes

- let

$$\rho_r = \frac{\nu_r}{\mu_r} \qquad r \in R,$$

the *load* on route r

Stability? (i.e. positive recurrence?)

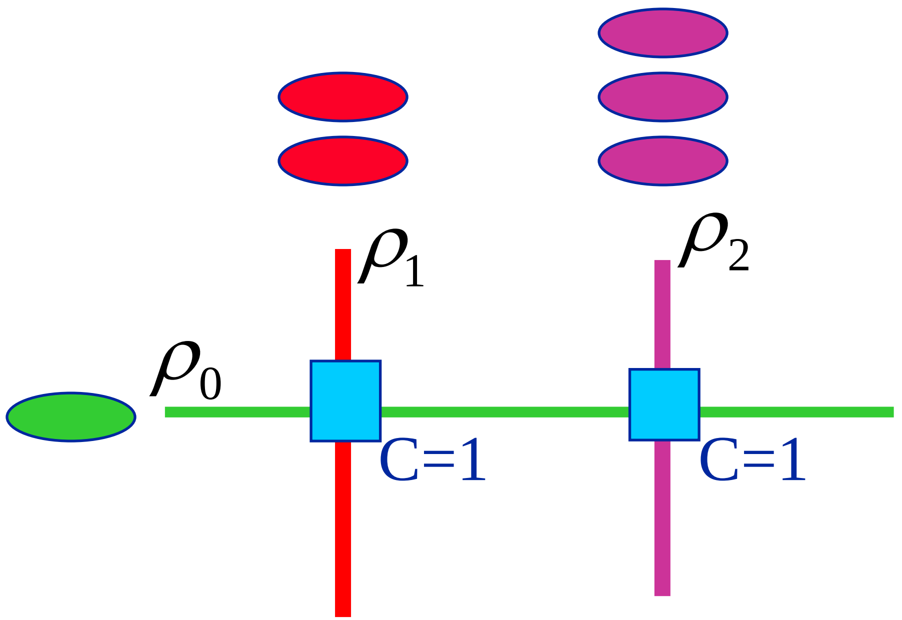
Suppose vertical streams have priority: then condition for stability is

$$\rho_0 < (1-\rho_1)\ (1-\rho_2)$$

and *not*

$$\rho_0 < \min\{1-\rho_1,\ 1-\rho_2\}$$

(Bonald and Massoulie 2001)



Fairness leads to stability

Suppose $\sum_r A_{jr}\ \rho_r \quad < \quad C_j \qquad j \in J$

and resource allocation is weighted α-fair.

Then the Markov process $n(t) = (n_r(t), r \in R)$

is positive recurrent

(De Veciana, Lee and Konstantopoulos 1999;

Bonald and Massoulie 2001).

Heavy traffic

We're interested in what happens when we approach the edge of the achievable region, when

$$\sum_r A_{jr}\,\rho_r \;\approx\; C_j \qquad j \in J$$

Balanced fluid model

Suppose $\sum_r A_{jr}\,\rho_r = C_j \quad j \in J$

and consider differential equations

$$\frac{\mathrm{d}n_r(t)}{\mathrm{d}t} = \nu_r - n_r x_r(n)\mu_r \qquad (n_r > 0) \qquad r \in R$$

First let's substitute for the values of $x_r(n),\ \ r \in R$, to give:

$$\frac{\mathrm{d}n_r(t)}{\mathrm{d}t} = \nu_r - n_r \mu_r \left(\frac{w_r}{\sum_j A_{jr} p_j(n)} \right)^{1/\alpha} \qquad r \in R$$

(care needed when $n_r = 0$).

Thus, at an invariant state,

$$n_r = \frac{\nu_r}{\mu_r} \left(\frac{\sum_j A_{jr} p_j(n)}{w_r} \right)^{1/\alpha} \qquad r \in R$$

State space collapse

The following are equivalent:

- n is an invariant state
- there exists a non-negative vector p

with

$$n_r = \frac{\nu_r}{\mu_r}\left(\frac{\sum_j A_{jr} p_j}{w_r}\right)^{1/\alpha} \qquad r \in R$$

Thus the set of invariant states forms a J dimensional manifold, parameterized by p.

Workloads

Let

$$W_j(n) = \sum_r A_{jr} \frac{n_r}{\mu_r}$$

the *workload* for resource j, and let $\alpha = 1$

Define diagonal matrices

$$\rho = diag(\nu_r / \mu_r,\ r \in R),\ w = diag(w_r,\ r \in R)$$

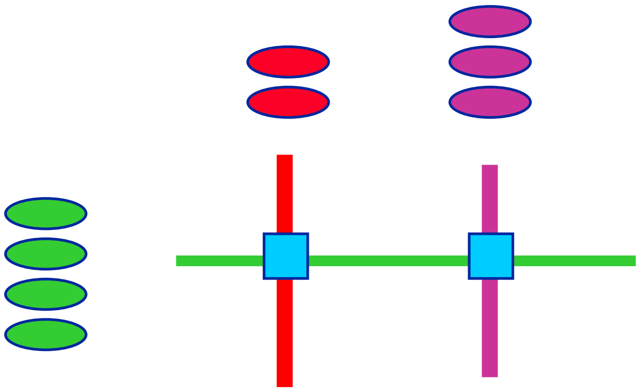
Then W lies in the polyhedral cone

$$\{W :\ W = A\mu^{-1}\rho\, w^{-1}A^T p,\ p \geq 0\}$$

Example

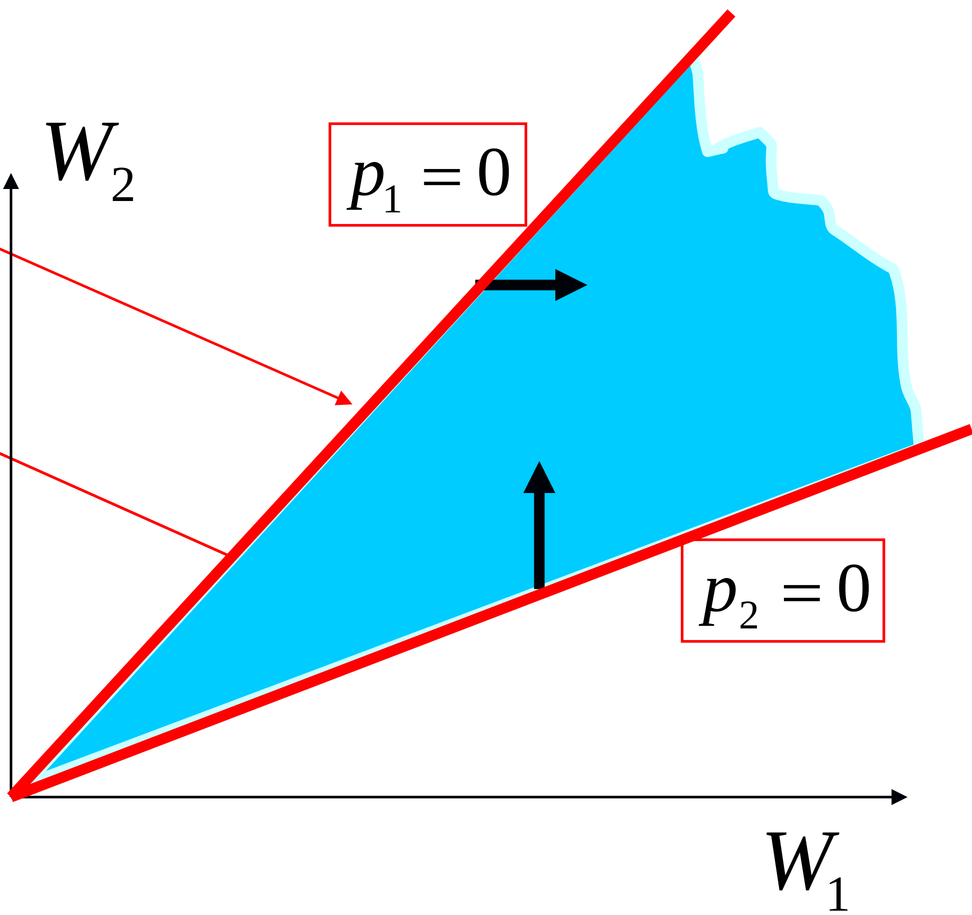
$$0 < \alpha < \infty$$

$$\mu_r = 1,\ w_r = 1,\ \ r \in R$$



slope $\dfrac{\rho_2 + \rho_0}{\rho_0}$

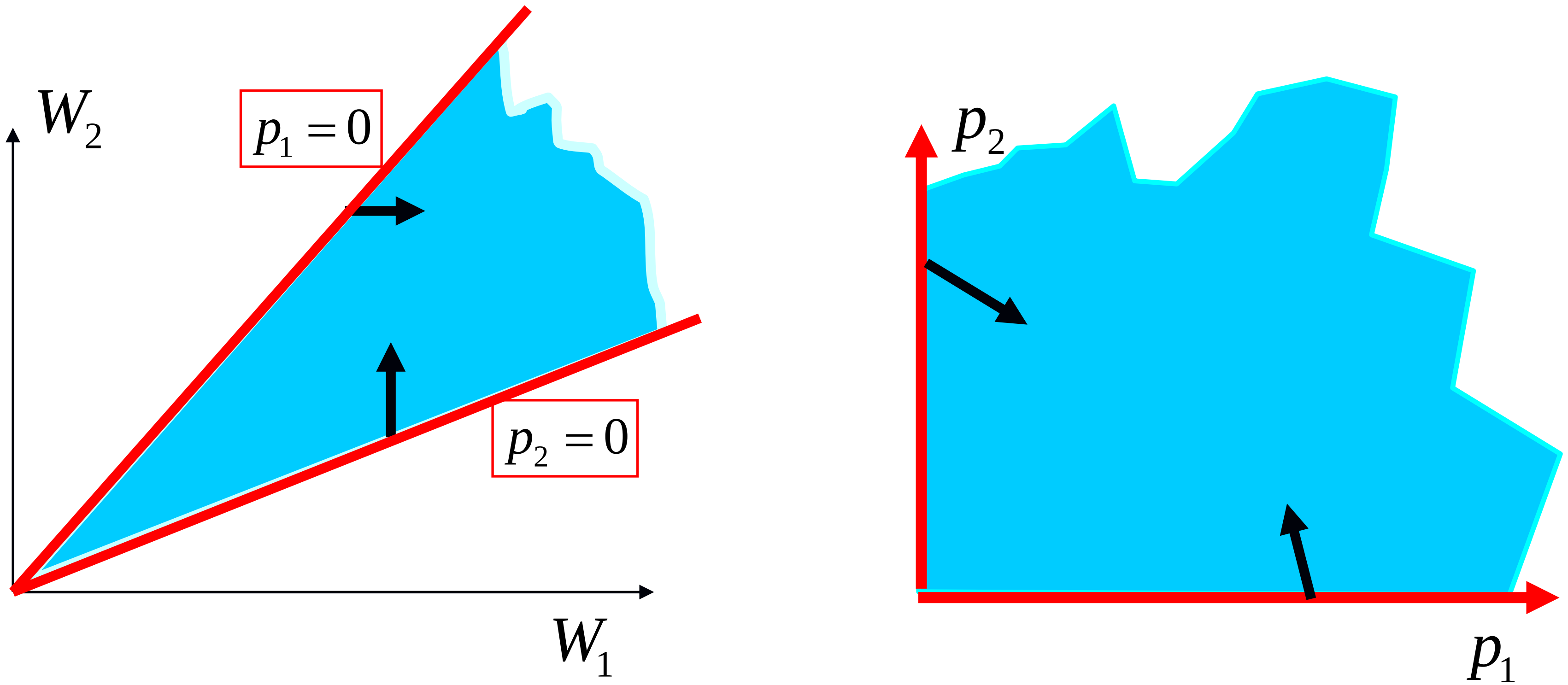
slope $\dfrac{\rho_0}{\rho_1 + \rho_0}$



Each bounding face corresponds to a resource not working at full capacity

Entrainment: congestion at some resources may prevent other resources from working at their full capacity.

Stationary distribution?



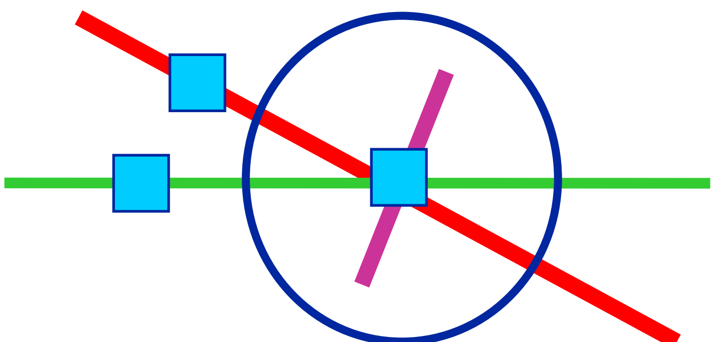
Look for a stationary distribution for W, or equivalently, p. Williams (1987) determined sufficient conditions, in terms of the reflection angles and covariance matrix, for a SRBM in a polyhedral domain to have a product form invariant distribution – a skew symmetry condition

Local traffic condition

Assume the matrix A contains the columns of the unit matrix amongst its columns:

$$A = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

i.e. each resource has some local traffic -



Product form under proportional fairness

$$\alpha = 1,\ w_r = 1,\ \ r \in R$$

Under the stationary distribution for the reflected Brownian motion, the (scaled) components of p are independent and exponentially distributed. The corresponding approximation for n is

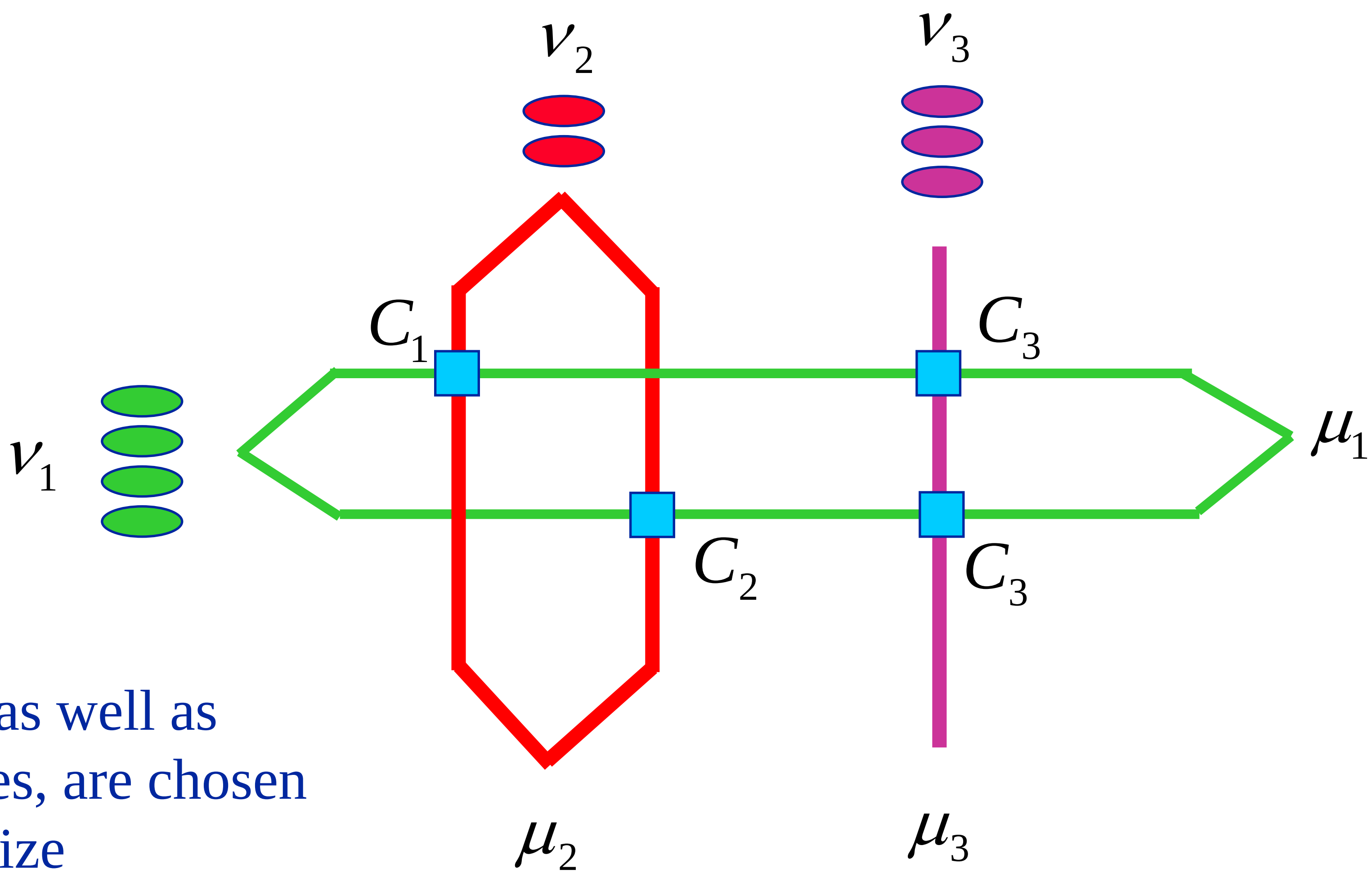
$$n_r \approx \rho_r \sum_j A_{jr} p_j \qquad r \in R$$

where

$$p_j \sim \mathrm{Exp}(C_j - \sum_r A_{jr} \rho_r) \qquad j \in J$$

Dual random variables are independent and exponential!

Multipath routing

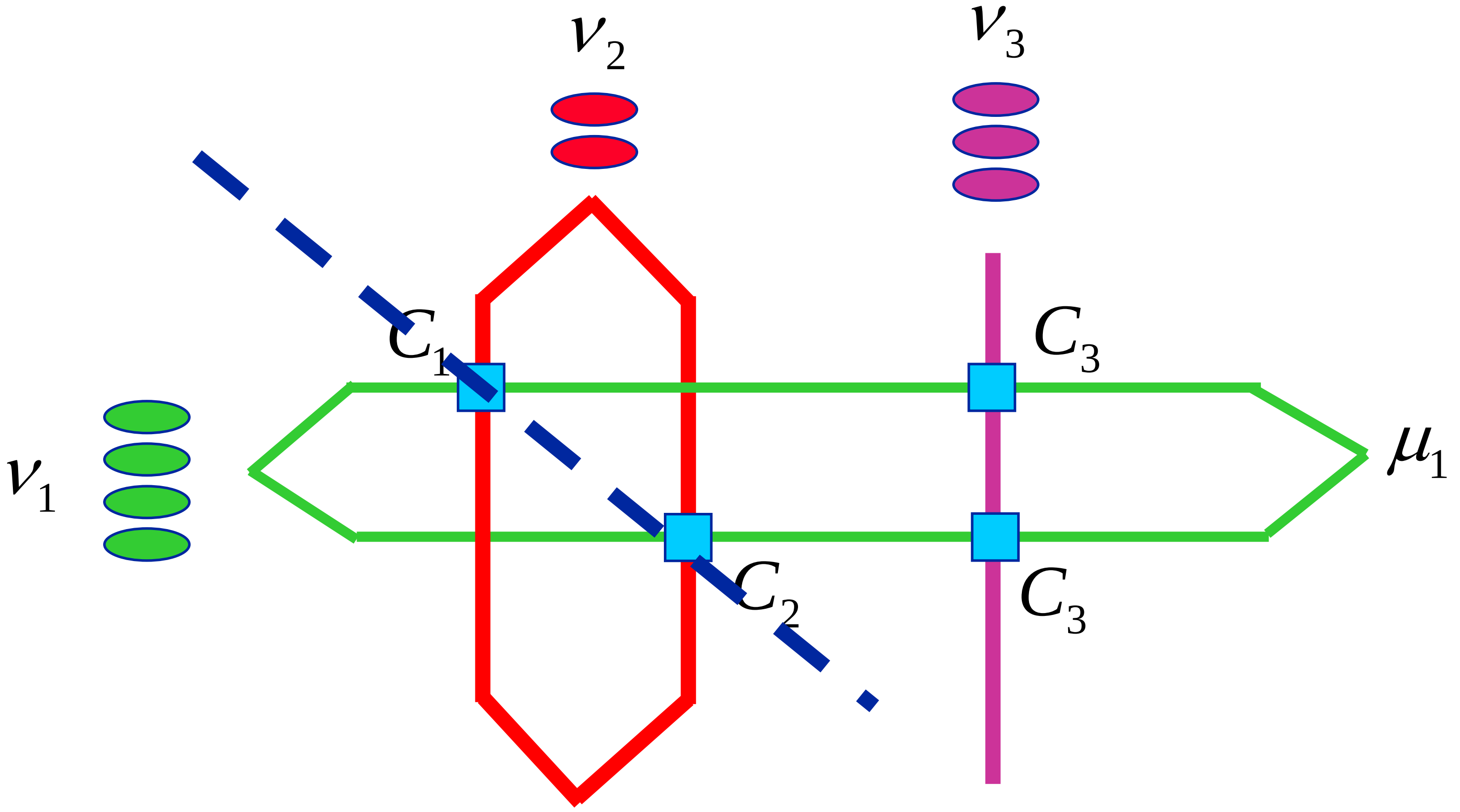


Routes, as well as flow rates, are chosen to optimize

$$\sum_s w_s n_s \frac{x_s^{1-\alpha}}{1-\alpha}$$

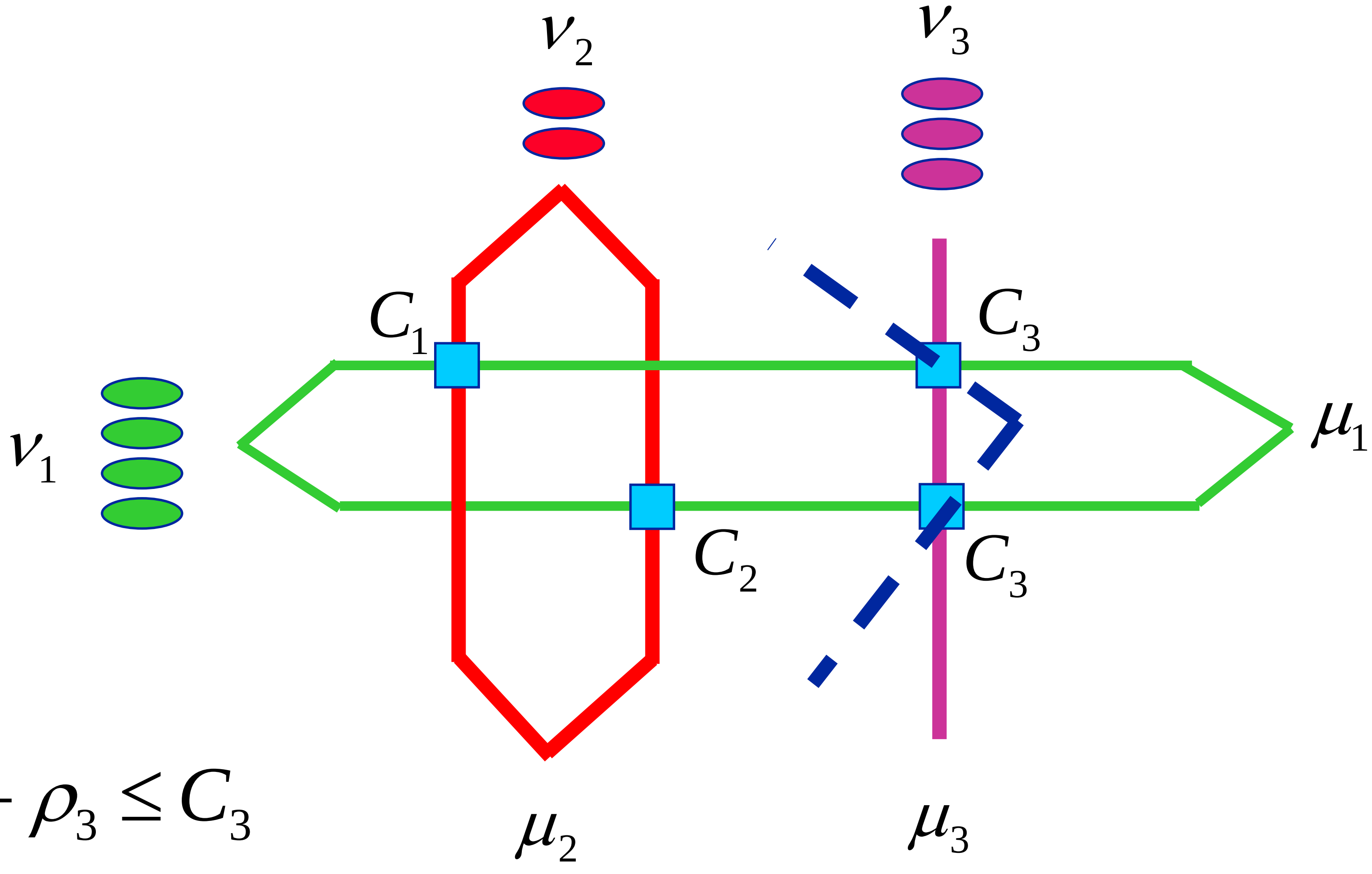
over source-sink pairs s

First cut constraint



$$\rho_1 + \rho_2 \leq C_1 + C_2$$

Second cut constraint



$$\frac{1}{2}\rho_1 + \rho_3 \leq C_3$$

Generalized cut constraints

In general, stability requires

$$\sum_{s} \overline{A}_{js}\ \rho_s \quad < \quad \overline{C}_j \qquad j \in \overline{J}$$

- a collection of *generalized cut constraints.*

Provided $\overline{A}$ contains a unit matrix, we again have the approximation

$$n_s \approx \rho_s \sum_{j \in \overline{J}} \overline{A}_{js}\, p_j \qquad s \in S$$

where

$$p_j \sim \mathrm{Exp}(\overline{C}_j - \sum_{s} \overline{A}_{js} \rho_s) \qquad j \in \overline{J}$$

Again independent dual random variables, now one for each generalized cut constraint!