On Stability of the Core

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Co-operative Game with Transferable Utility

A **co-operative game with transferable utility** is a set of agents, N, and a function, v, from $2^N \rightarrow \Re$.

v asociates a real number for any coalition S.

$v(S)$ is called the **value** of S.

Imputation

An **imputation** is a $x \in \Re^{|N|}$ such that

$$\text{For all } \; i \in N, \;\; x_i \geq v(i) \; \text{ and}$$

$$\sum_{i \in N} x_i = v(N).$$

The set of imputations is denoted $I(v, N)$.

Solution Concept

A **solution concept** is a rule that associates with each (v, N) a subset of $I(v, N)$.

Stable Set

The very first solution concept for co-operative games, proposed by von Neumann and Morgentsern (1944), was called the **stable set**.

Lucas (1968) showed a co-operative game need not posess a stable set.

Question: Characterize which games have a stable set. Not known to be decidable even.

Subsequent work by others showed that a stable set could be quite complicated.

Core

Attention shifted to other solution concepts. The most important of these has been the **core** introduced by Gillies (1957).

Question: When is core stable? (i.e., when is the core a stable set too).

We provide a little bit of insight on this question. We observe that the question is decidable.

Notation

Given a vector $x \in \Re^{|N|}$, we write $\Sigma_{i \in S} x_i$ for any $S \subseteq N$ as $x(S)$.

For any $S \subseteq N$ define $C(v, S)$ to be:

$\{x \in \Re^{|S|} : x(S) = v(S), x(T) \geq v(T)\ \forall T \subset S\}$.

$C(v, N)$ is called the **core** of the game (v, N).

Balanced Game

A game (v, N) is **balanced** if and only if

$$v(N) \geq \max \sum_{T \subset N} \lambda_T v(T)$$

$$st \sum_{T \ni i} \lambda_T = 1 \; \forall i \in N$$

$$\lambda_T \geq 0 \; \forall T \subset N.$$

From LP duality a game is balanced iff it has a non-empty core.

Totally Balanced

A game is **totally balanced** if every subgame (v, S) is balanced.

From LP duality a game is totally balanced iff every subgame (v, S) has a non-empty core.

Strongly Totally Balanced

A game is **Strongly totally balanced** if $C(v, S)$ has non-empty relative interior, for every $|S| \geq 2$.

From any totally balanced game if we subtract any $\epsilon > 0$, from every $v(S)$ we get a strongly totally balanced game.

Stable Core

The core, $C(v, N)$ of a balanced game is **stable**

if for all $y \in I(v, N) \setminus C(v, N)$

there is an $x \in C(v, N)$ and $T \subset N$

such that $x(T) = v(T)$ and $x_i > y_i$ for all $i \in T$.

In this case x is said to **dominate** y via T.

Exact Games

A balanced game (v, N) is **exact**

if for all $T \subset N$

there is an $x \in C(v, N)$

such that $x(T) = v(T)$.

Exact games are totally balanced. Converse does not hold.

Supermodular games

A game is supermodular if any person adds more value to a larger coalition.

Take any agent i. Take two sets S and T not containing i. $S \subseteq T$.

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$$

Supermodular games have stable core. Supermodular game are also exact. None of the converses hold.

Games with Large Core

The core, $C(v, N)$ is called **large**

if for every $y \in \Re^{|N|}$ such that $y(S) \geq v(S)$ for all $S \subseteq N$

there is an $x \in C(v, N)$ such that $x \leq y$.

Supermodular $\Rightarrow$ Large $\Rightarrow$ Stable, Exact.

Reverse directions do not hold.

Games with Extendible Core

Extendability means that for each $S \subset N$ and $z \in C(v, S)$

there is an $x \in C(v, N)$

such that $x_i = z_i$ for all $i \in S$.

Supermodular $\Rightarrow$ Large $\Rightarrow$ Extendible $\Rightarrow$ Stable, Exact.

Reverse directions do not hold.

Possibilities

Supermodular $\Rightarrow$ Large $\Rightarrow$ Extendible $\Rightarrow$ Exact $\Rightarrow$ Stable.

Supermodular $\Rightarrow$ Large $\Rightarrow$ Extendible $\Rightarrow$ Stable $\Rightarrow$ Exact.

Both (proposed by Vohra)

None.

We do not resolve this question.

What we asked

Why the healthy sequence of finding weaker and weaker properties which imply stablity did not continue? Certainly there is a scope since the last converse does not hold. So we asked to find a property XYZ such that

Supermodular $\Rightarrow$ Large $\Rightarrow$ Extendible $\Rightarrow$ XYZ $\Rightarrow$ Stable.

or even

XYZ $\Rightarrow$ Stable.

What we found

It is a hard question. Finding such an XYZ will shed a non-trivial light on stability.

Violated Set

Given a $y \in I(v, N) \setminus C(v, N)$.

Call a set T such that $y(T) < v(T)$ a **violated** set.

Strongly Stable Core

The core, $C(v, N)$ of a balanced game is **strongly stable**

if for all $y \in I(v, N) \setminus C(v, N)$ and for any **minimally violated** set T,

there is an $x \in C(v, N)$

such that $x(T) = v(T)$ and $x_i > y_i$ for all $i \in T$.

We answered our question :-)

Strongly Stable $\Rightarrow$ Stable.

In fact,

Supermodular $\Rightarrow$ Large $\Rightarrow$ Extendible $\Rightarrow$ Strongly Stable $\Rightarrow$ Stable.

But not staisfactorily

Strongly Stable $\Rightarrow$ Extendible

The implication holds under Strongly Total Balancedness condition. There is an example which show that **Strongly** in Total Balancedness is required.

The message

Definition of Stability:

The core, $C(v, N)$ of a balanced game is **stable**

if for all $y \in I(v, N) \setminus C(v, N)$

there is an $x \in C(v, N)$ and $T \subset N$

such that $x(T) = v(T)$ and $x_i > y_i$ for all $i \in T$.

In this case x is said to **dominate** y via T.

The choice of T is the main non-trivial task. Once this choice is made then everything is a skillful use of Farkas Lemma at some high level.

Decidability of Stable Core

Define P_T as the set of vectors y in $I(v, N) \setminus C(v, N)$, which can use an $x \in C(v, N)$ to dominate y via T.

Check whether $I(v, N) \setminus C(v, N)$ is a union of P_T's for all $T \subset N$ and $|T| \geq 2$.