Well Supported Approximate Equilibria in Bimatrix Games

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Skeleton

Well Supported Approximate Nash Equilibria in Bimatrix Games

1. Introduction
    - Bimatrix Games Notation
    - Approximations of Nash Equilibria
    - Recent Advances in Approximations of NE
2. Existence and Construction of non-trivial SuppNE
    - A Subexponential Scheme for SuppNE
    - A Graph Theoretic Construction of SuppNE
    - An LP Based Construction of SuppNE
    - SuppNE in Random Games
3. Recap and Open Problems

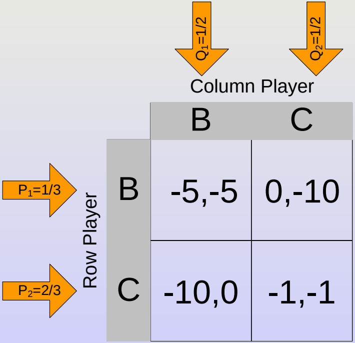
What are the Bimatrix Games?

Definition (Bimatrix Games)

An $m \times n$ bimatrix game $\langle A, B \rangle$ is a $2-$player game in strategic form in which the payoffs of the two players are determined by a pair of $m \times n$ real matrices A, B (aka the bimatrix (A, B)).

The two players choose rows and columns:

- either deterministically (pure strategy)...
- or probabilistically (mixed strategy)...
- and get expected payoffs $\mathbf{p}^T A \mathbf{q}$ and $\mathbf{p}^T B \mathbf{q}$.



	B	C
B	-5,-5	0,-10
C	-10,0	-1,-1

Some Special Cases of Bimatrix Games

- $[a, b]-$Bimatrix Game: A bimatrix game $\langle A, B\rangle$ whose payoff matrices get values from the real interval $[a, b]$.
- Normalized Bimatrix Game: A $[0, 1]-$bimatrix game.
- Win Lose Bimatrix Game: A bimatrix game $\langle A, B\rangle$ whose payoff matrices get values from the set $\{0, 1\}$.
- $\lambda-$Sparse Win Lose Bimatrix Game: A win lose bimatrix game having at most λ $(0, 1)-$elements per column and at most λ $(1, 0)-$elements per row of the bimatrix.

What is the Outcome of the Game?

- The two players
  - ... choose their strategy selfishly.
  - ... are aware of the bimatrix, and of the selfishness of the opponent.
  - ... do not cooperate their actions.
  - ⇒ This leads to hope for existence of equilibrium points.
- **What is the solution of the bimatrix game?**

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- **What is the solution of the bimatrix game?**

Definition (Nash Equilibrium (NE))

A strategies profile $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is Nash Equilibrium of $\langle A, B\rangle$ iff:

$$\bar{\mathbf{x}} \in \arg\max_{\mathbf{x}}\{\mathbf{x}^T A\bar{\mathbf{y}}\} \text{ and } \bar{\mathbf{y}} \in \arg\max_{\mathbf{y}}\{\bar{\mathbf{x}}^T B\mathbf{y}\}$$

or equivalently,

$$\forall i, r \in [m], \bar{x}_i > 0 \Rightarrow A^i\bar{\mathbf{y}} \geq A^r\bar{\mathbf{y}} \text{ and } \forall j, s \in [n], \bar{y}_j > 0 \Rightarrow B_j^T\bar{\mathbf{x}} \geq B_s^T\bar{\mathbf{x}}.$$

How about Approximate Solutions?

Definition (Approximations of NE in Normalized Games)

- Approximate NE ($\varepsilon-$ApproxNE): Each player cannot have a positive additive gain strictly larger than ε, by unilaterally changing her own strategy.
- Well Supported Approximate NE ($\varepsilon-$SuppNE): Each player adopts with positive probability only actions that are at most a positive additive term ε worse than their optimal choice of an action, given the opponent's strategy:

$$(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \varepsilon-\mathrm{SuppNE}(A, B) \Leftrightarrow$$

$$\Leftrightarrow \left\{ \begin{array}{lcl} \forall i, r \in [m], \bar{x}_i > 0 & \Rightarrow & A^i\bar{\mathbf{y}} \geq A^r\bar{\mathbf{y}} - \varepsilon \\ \forall j, s \in [n], \bar{y}_j > 0 & \Rightarrow & B_j^T\bar{\mathbf{x}} \geq B_s^T\bar{\mathbf{x}} - \varepsilon \end{array} \right.$$

What's the difference again?

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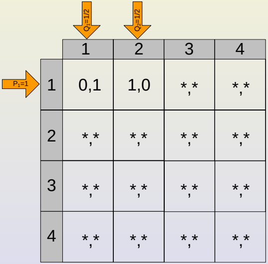
What's the difference again?

ApproxNE vs. SuppNE

- Both are generalizations of NE: Each $0-$ApproxNE and each $0-$SuppNE are (exact) NE.
- Every $\varepsilon-$SuppNE is also a $\varepsilon-$ApproxNE (trivial observation).
- From any $\frac{\varepsilon^2}{8n}-$ApproxNE we can construct in polynomial time an $\varepsilon-$SuppNE {Chen,Deng,Teng 2006} .
- SuppNE seem to be better motivated by selfish behavior: Each player (rather than choosing best response actions), chooses approximate best response actions with positive probability.
- It seems much harder to provide SuppNE.

For the normalized game:



	1	2	3	4
1	0,1	1,0	*,*	*,*
2	*,*	*,*	*,*	*,*
3	*,*	*,*	*,*	*,*
4	*,*	*,*	*,*	*,*

the profile $\left(\mathbf{e}_1, \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)\right)$ is $0.5-$ApproxNE but $1-$SuppNE.

What do we know about (exact) NE?

- The problem $k-$NASH of computing any NE of an arbitrary $k-$person strategic game, is one of the most important algorithmic questions at the boundary between $\mathcal{P}$ and $\mathcal{NP}$. {Papadimitriou (ESA 1996, STOC 2001)} .
- $k-$NASH is $\mathcal{PPAD}$–complete, even for...

  ... $k = 4$ {Daskalakis,Goldberg,Papadimitriou (STOC 2005)} ,

  ... $k = 3$ {Daskalakis,Papadimitriou (ECCC 2005)} ,

  ... or even $k = 2$ {Chen,Deng (FOCS 2006)} !!!
- The correlation of $\mathcal{PPAD}$ with other complexity classes is not clear.

A Useful(?) Tool

- {Lemke, Howson 1964} : A combinatorial algorithm based on pivots, that computes (exact) NE for arbitrary bimatrix games.
- {Savani, von Stengel (FOCS 2004)} : The algorithm of Lemke and Howson takes an exponential number of pivots to converge to a NE, independently of the initial choice it makes, even in win lose instances.
- How about approximations of NE?

Advances in ApproxNE

- {Chen,Deng,Teng (FOCS 2006b)} : Unless $\mathcal{PPAD} \subseteq \mathcal{P}$, there is no algorithm for $\varepsilon-$ApproxNE with time complexity $poly(n, 1/\varepsilon)$, for any $\varepsilon = n^{-\Theta(1)}$ $\Rightarrow$ (probably) there is no FPTAS!!!
- {Chen,Deng,Teng (FOCS 2006b)} : Unless $\mathcal{PPAD} \subseteq \mathcal{RP}$, there is no algorithm for $2-$NASH with time complexity $poly(n, 1/\sigma)$ (σ = the size of the perturbations of the elements in the bimatrix).
- So far we have no Polynomial Time Approximation Scheme for computing $\varepsilon-$ApproxNE for any constant $\varepsilon > 0$.
- Important Observation: For any constant $\varepsilon > 0$, there exist (uniform) profiles with with support sizes $O(\log(m+n)/\varepsilon^2)$, which are $\varepsilon-$ApproxNE {Lipton, Markakis, Mehta (EC 2003)} .
$\Rightarrow$ Subexponential computational time!!!

How about Constant ApproxNE?

- {Kontogiannis,Panagopoulou,Spirakis (WINE 2006)} Polynomial time construction of $\frac{2+\lambda}{4}-$ApproxNE ($\lambda =$ smallest equilibrium payoff to a player).

- {Daskalakis,Mehta,Papadimitriou (WINE 2006)} Polynomial time construction of $\frac{1}{2}-$ApproxNE.

  Recent Development: They improved this to $0.38-$ApproxNE.

- {Daskalakis,Mehta,Papadimitriou (WINE 2006)} Construction of some $\varepsilon-$SuppNE in polynomial time, for some (non-constant) $1 > \varepsilon > 0$, if a graph theoretic conjecture holds (not true for small values!!!).

- Remark: Nothing is known about non-trivial SuppNE!!!

Skeleton

Well Supported Approximate Nash Equilibria in Bimatrix Games

Existence of SuppNE (I)

Theorem

For any $m \times n$ $[0,1]-$bimatrix game $\langle A, B\rangle$, and any constant $\varepsilon \in (0,1)$, there is an $\varepsilon-$SuppNE with support sizes $\left\lceil \frac{\log(2n)}{2\varepsilon^2} \right\rceil$.

WHY?

- Althoefer's Approximation Lemma:
  Assume C is any $m \times n$ matrix over the real numbers, with $0 \leq C_{i,j} \leq 1, \forall (i,j) \in [m] \times [n]$. Let $\mathbf{p} \in \Delta_m$ be any $m-$probability vector. Fix arbitrary positive constant $\varepsilon > 0$. Then, there exists another probability vector $\hat{\mathbf{p}} \in \Delta_m$ with $|supp(\hat{\mathbf{p}})| \leq k \equiv \left\lceil \frac{log(2n)}{2\varepsilon^2} \right\rceil$, such that $|\mathbf{p}^T C_j - \hat{\mathbf{p}}^T C_j| \leq \varepsilon, \ \forall j \in [n]$. Moreover, $\hat{\mathbf{p}}$ is a $k-$uniform strategy.

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For any $m \times n$ $[0,1]-$*bimatrix game* $\langle A, B \rangle$*, and any constant* $\varepsilon \in (0,1)$*, there is an* $\varepsilon-$*SuppNE with support sizes* $\left\lceil \frac{\log(2n)}{2\varepsilon^2} \right\rceil$.

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Existence of SuppNE (II)

WHY? (contd.)

- Application of Approximation Lemma: Wrt arbitrary $(\mathbf{p},\mathbf{q}) \in NE(A,B)$, consider $(\hat{\mathbf{p}},\hat{\mathbf{q}})$ s.t. $\forall j \in [n],\ |\mathbf{p}^T B_j - \hat{\mathbf{p}}^T B_j| \leq \varepsilon$, and $\forall i \in [m],\ |A^i\mathbf{q} - A^i\hat{\mathbf{q}}| \leq \varepsilon$.
- Proposition: Since $\hat{\mathbf{p}}$ is produced via a hypothetical sampling of $\mathbf{p}$, it holds that $support(\hat{\mathbf{p}}) \subseteq support(\mathbf{p})$.

$$\begin{array}{lcl}
\forall i \in [m], \hat{p}_i > 0 & \overset{\text{/* Sampling */}}{\Longrightarrow} & p_i > 0 \\
 & \overset{\text{/* Nash Prop. */}}{\Longrightarrow} & A^i\mathbf{q} \geq A^r\mathbf{q},\ \forall r \in [m] \\
 & \overset{\text{/* Approx. Lemma */}}{\Longrightarrow} & A^i\hat{\mathbf{q}} + \varepsilon \geq A^r\hat{\mathbf{q}} - \varepsilon,\ \forall r \in [m] \\
 & \Longrightarrow & A^i\hat{\mathbf{q}} \geq A^r\hat{\mathbf{q}} - 2\varepsilon,\ \forall r \in [m]
\end{array}$$

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SuppNE for Win Lose Games (I)

Theorem

For any win lose bimatrix game, there exists a polynomial time constructible $\left(1-\frac{2}{g}\right)-$*SuppNE, where* g *is the girth of the Nash Dynamics graph (*$g = 2$*, if there is no cycle).*

SuppNE for Win Lose Games (II)

WHY? (Step 1)

- Cut off win lose games with PNE.
- The following structures are forbidden in the bimatrix:

$$\begin{bmatrix} (0,\star) \\ \vdots \\ (0,\star) \\ (0,1) \\ (0,\star) \\ \vdots \\ (0,\star) \end{bmatrix} \qquad \begin{bmatrix} (\star,0) & \cdots & (\star,0) & (1,0) & (\star,0) & \cdots & (\star,0) \end{bmatrix}$$

- Proposition: Any row (column) of (A,B) with a $(1,0)-$element ($(0,1)-$element) must also have a $(0,1)-$element ($(1,0)-$element).

⇒ Each non$-(0,0)-$element belongs to a cycle of the Nash Dynamics graph.

SuppNE for Win Lose Games (III)

WHY? (Step 2)

- A shortest cycle in the Nash Dynamics graph defines a $\frac{g}{2}$–Matching Pennies subgame:

$$\begin{bmatrix} (1,0) & (0,1) & (0,0) & \cdots & (0,0) & (0,0) \\ (0,0) & (1,0) & (0,1) & \cdots & (0,0) & (0,0) \\ (0,0) & (0,0) & (1,0) & \cdots & (0,0) & (0,0) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (0,0) & (0,0) & (0,0) & \cdots & (1,0) & (0,1) \\ (0,1) & (0,0) & (0,0) & \cdots & (0,0) & (1,0) \end{bmatrix} \quad \begin{bmatrix} (0,1) \\ (0,1) \\ (0,1) \\ \vdots \\ (0,1) \\ (0,1) \end{bmatrix}$$

- The uniform profile on the rows and columns comprising a $(g/2)$–GMP is a $(1-2/g)$–SuppNE of the win lose game.

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SuppNE for $[0, 1]-$Bimatrix Games

Theorem

For any normalized bimatrix game, there exists a polynomial time constructible $\left(1-\frac{1}{g}\right)-$*SuppNE, where* g *is the girth of the Nash Dynamics graph (*$g = 2$*, if there is no cycle).*

WHY?

- {Daskalakis,Mehta,Papadimitriou (WINE2006)} :
  - Create a win lose image by rounding up to 1 values greater than $\frac{1}{2}$ and down to 0 values lower than $\frac{1}{2}$.
  - Any $\varepsilon-$SuppNE of the win lose image is a $\frac{1+\varepsilon}{2}-$SuppNE of the initial game.
- Simple application of the above observation to our result for win lose games.

SuppNE for $[0, 1]-$Bimatrix Games

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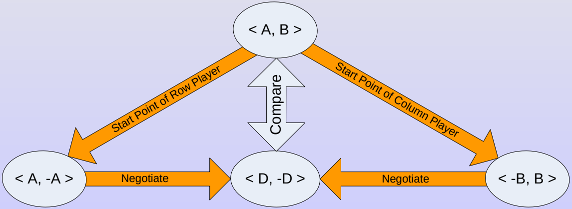
Applications of the Graph Theoretic Approach

- There is a polynomial time constructible $\varepsilon-$SuppNE, for some constant $1 > \varepsilon > 0$, for any normalized bimatrix game that maps to a win lose game of constant girth.
- For $\lambda-$sparse win lose games with non-constant girth, our construction gives an $o(1)-$SuppNE!!!
- For normalized games mapping to $\lambda-$sparse win lose games of large girth, our construction provides a $\left(\frac{1}{2} + o(1)\right)-$SuppNE.

Exploitation of Zero Sum Games

Main Idea: Fix arbitrary (normalized) game $\langle A, B\rangle$.

- The row (column) player would never accept a profit less than the one assured by maximin plays in $\langle A, -A\rangle$ (resp. $\langle -B, B\rangle$).
- What if the row player mimics the behavior of a player closer to the opponent of the column player?
- Find the proper zero sum game to solve, and compare the values of its solution in the real game.



A Simple Observation

We prove that:

Lemma

Fix arbitrary (normalized) $[0,1]-$*bimatrix game* $\langle A, B\rangle$ *and any real matrices* $R, C \in \mathbb{R}^{m\times n}$, *such that* $\forall i \in [m], R^i = \mathbf{r}^T \in \mathbb{R}^n$ *and* $\forall j \in [n], C_j = \mathbf{c} \in \mathbb{R}^m$. *Then,* $\forall 1 > \varepsilon > 0$ *and any profile* $(\mathbf{x}, \mathbf{y})$, *if* $(\mathbf{x}, \mathbf{y})$ *is an* $\varepsilon-$*SuppNE for* $\langle A, B\rangle$ *then it is also an* $\varepsilon-$*SuppNE for* $\langle A+R, B+C\rangle$.

...which leads to the (folklore for exact NE) observation:

Corollary

SuppNE are immune to shifting operations of the payoff matrices.

that we shall use.

Application to Win Lose Games (I)

- Rather than working with $\{0,1\}-$bimatrix games, work with $\left\{-\frac{1}{2},\frac{1}{2}\right\}-$bimatrix games $\langle A,B\rangle$.
- Let $Z=-(A+B)$.
- Consider the (maximin) solution $(\bar{\mathbf{x}},\bar{\mathbf{y}})$ of the zero sum game $\langle A+\frac{1}{2}Z,-(A+\frac{1}{2}Z\rangle$.

Theorem

$(\bar{\mathbf{x}},\bar{\mathbf{y}})$ *is a (polynomial time computable)* $0.5-$*SuppNE for* $\langle A,B\rangle$.

Application to Win Lose Games (II)

WHY?

- Exclude $(1,1)-$elements (trivial PNE) from (A,B).
- Shift (A,B) to take $(R = A - \frac{1}{2}E, C = -\frac{1}{2}E)$.
- Consider the zero sum game $\langle D, -D\rangle$, s.t. $D = R + X \Leftrightarrow X = D - R$ and $-D = C + Y \Leftrightarrow Y = -(D+C)$ for arbitrary $m \times n$ bimatrix (X,Y).

$$(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in NE(D, -D) = NE(R+X, C+Y) \Leftrightarrow$$

$$\Leftrightarrow \quad \begin{cases} \forall i, r \in [m], & \bar{x}_i > 0 \Rightarrow R^i\bar{\mathbf{y}} \geq R^r\bar{\mathbf{y}} - [X^i - X^r]\bar{\mathbf{y}} \\ \forall j, s \in [n], & \bar{y}_j > 0 \Rightarrow C_j^T\bar{\mathbf{x}} \geq C_s^T\bar{\mathbf{x}} - [Y_j - Y_s]^T\bar{\mathbf{x}} \end{cases}$$

Application to Win Lose Games (III)

WHY? (contd.)

- Since $-Z \equiv R + C = -(X + Y)$, try $X = Y = \frac{1}{2}Z$:

$$(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in NE(D, -D)$$
$$\Leftrightarrow \left\{ \begin{array}{ll} \forall i, r \in [m], & \bar{x}_i > 0 \Rightarrow R^i\bar{\mathbf{y}} \geq R^r\bar{\mathbf{y}} - \frac{1}{2} \cdot [Z^i - Z^r]\bar{\mathbf{y}} \\ \forall j, s \in [n], & \bar{y}_j > 0 \Rightarrow C_j^T\bar{\mathbf{x}} \geq C_s^T\bar{\mathbf{x}} - \frac{1}{2} \cdot [Z_j - Z_s]^T\bar{\mathbf{x}} \end{array} \right.$$

- Any row or column of Z is a $\{0, 1\}-$vector.

$\Rightarrow$ $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is a $0.5-$SuppNE of $\langle R, C\rangle$, and thus also for $\langle A, B\rangle$.

■

Extension to Normalized Games (I)

Corollary

Any normalized bimatrix game has a polynomial time computable $0.75-$*SuppNE.*

- WHY? A simple application of the reduction of {Daskalakis,Mehta,Papadimitriou, 2006} .
- Question: Can we do better?
- Answer: Yes, if we parameterize our analysis for win lose games.

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Extension to Normalized Games (II)

Theorem

For any win lose bimatrix game $\langle R, C\rangle$ *and any* $0 < \delta < 1$, *the exact NE* $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ *of* $\langle R + \delta Z, -(R + \delta Z)\rangle$ *is an* $\varepsilon(\delta)-$*SuppNE, where:*

$$\begin{aligned} \varepsilon(\delta) &\equiv \max_{i,r\in[m],j,s\in[n],\mathbf{x},\mathbf{y}} \left\{ \delta \cdot \left[Z^i - Z^r\right] \mathbf{y}, \frac{1-\delta}{\delta} \cdot \left[R_s^T - R_j^T\right] \mathbf{x} \right\} \\ &\leq \max\left\{\delta, \frac{1-\delta}{\delta}\right\} \end{aligned}$$

WHY?

- Same reasoning as in previous case.
- NOTE: Not so tight analysis as before!!!

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Extension to Normalized Games (III)

Theorem

For any normalized bimatrix game there is a polynomial time computable $(\sqrt{11}/2 - 1)-$*SuppNE.*

WHY?

- Shift $\langle A, B\rangle$ to the $[-\frac{1}{2}, \frac{1}{2}]-$bimatrix game $\langle R, C\rangle$.
- Let $Z = -(R + C)$ and $0 < \delta < 1$.
- Any element $(R, C)_{i,j} \in [\frac{1}{2} - \zeta, \frac{1}{2}] \times [\frac{1}{2} - \zeta, \frac{1}{2}]$ would indicate a $\zeta-$SuppNE of the game.

$\Rightarrow$ Each element of $\langle R, C\rangle$ has $R_{ij} < \frac{1}{2} - \zeta \vee C_{ij} < \frac{1}{2} - \zeta$.

$\Rightarrow$ $Z \in (-1 + \zeta, 1]^{m\times n}$.

- Any NE of $\langle R + \delta Z, -(R + \delta Z)\rangle$ is an $\varepsilon(\delta)-$SuppNE of $\langle R, C\rangle$.
- Fine Tuning: For $\zeta^* = \frac{\sqrt{11}}{2} - 1$ we get a $\zeta-$SuppNE for $\langle R, C\rangle$.

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Random Bimatrix Games

- **Random normalized games:**
  - The entries of the bimatrix are independent (not necessarily identically distributed) random variables.
  - The sums of the elements of each row of A are sharply concentrated around the same value.
  - The sums of the elements of each column of B are sharply concentrated around the same value.

  $\Rightarrow$ The uniform full mix is $\mathrm{O}\left(\sqrt{\frac{\log m}{m}}\right)-$SuppNE of $\langle A, B\rangle$, **whp** .

- **Random Win Lose Games:**
  - All the probability mass is split among elements of $\{(0,0),(0,1),(1,0)\}$. All these elements have positive probability.

  $\Rightarrow$ There is either a PNE, or a polynomial time constructible $\frac{1}{2}-$SuppNE, **whp** .

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What We Have Seen

	Graph Theoretic		**LP Based**	**Random**
Win Lose	1-2/g		0.5	∃ PNE **OR** 2-MP, (whp)
	λ-sparse with large girth	$O(\lambda/g)=o(1)$		
Normalized	1-1/g		$\frac{\sqrt{11}}{2}-1$	Uniform Full Mix is (whp) $\sqrt{\frac{\log m}{m}}$ -SuppNE
	λ-sparse win lose **image** with large girth	$\frac{1+o(1)}{2}$		

Open Issues

- Is there a PTAS for ApproxNE?
- Is there a polynomial time algorithm for ε–SuppNE, for some constant $\frac{\sqrt{11}}{2} - 1 > \varepsilon > 0$?
- Is there a PTAS for SuppNE?
- What is the relation of $\mathcal{PPAD}$ with other complexity classes (eg, $\mathcal{PLS}$)?

Thank you
for your attention!