An Efficient Cost Sharing Mechanism for the Prize-Collecting Steiner Forest Problem

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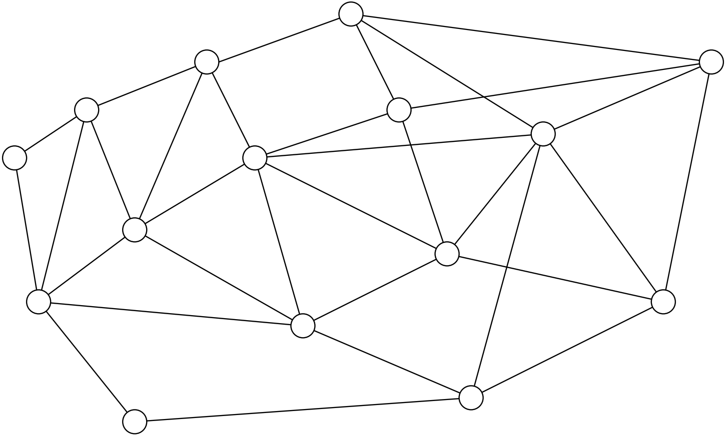
Outline

- Part I: Cost Sharing Mechanisms
  - cost sharing model, definitions, objectives
  - state of affairs, new trade-offs
  - tricks of the trade
- Part II: Prize-Collecting Steiner Forest
  - primal-dual algorithm PCSF
  - cross-monotonicity and budget balance
  - general reduction technique
- Conclusions and Open Problems

Motivation



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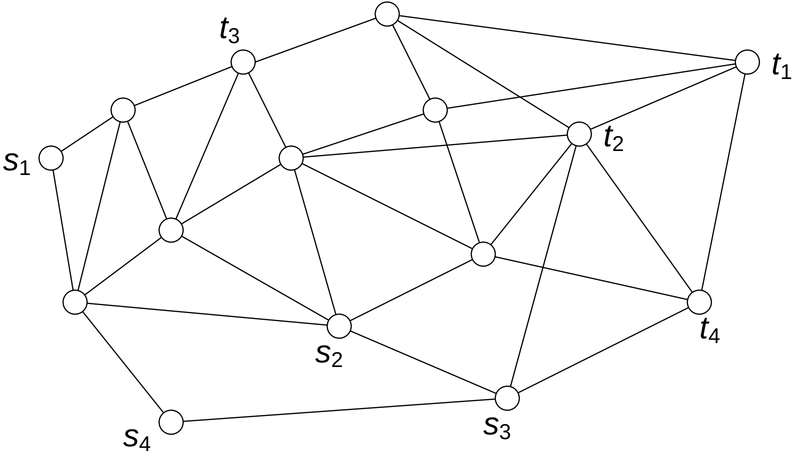

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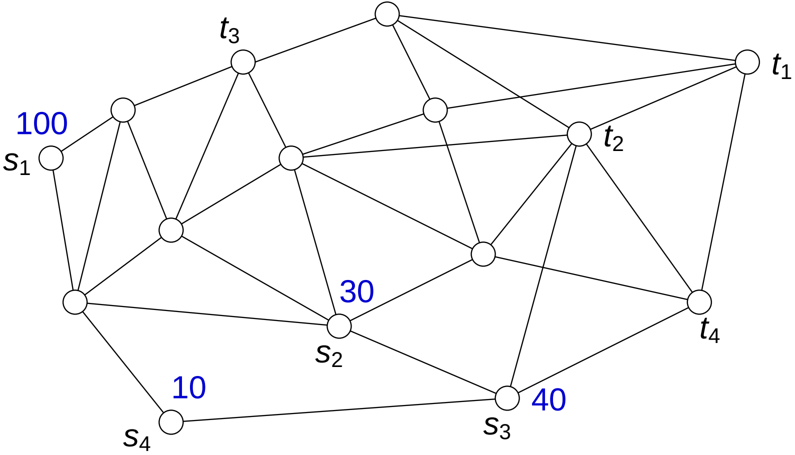

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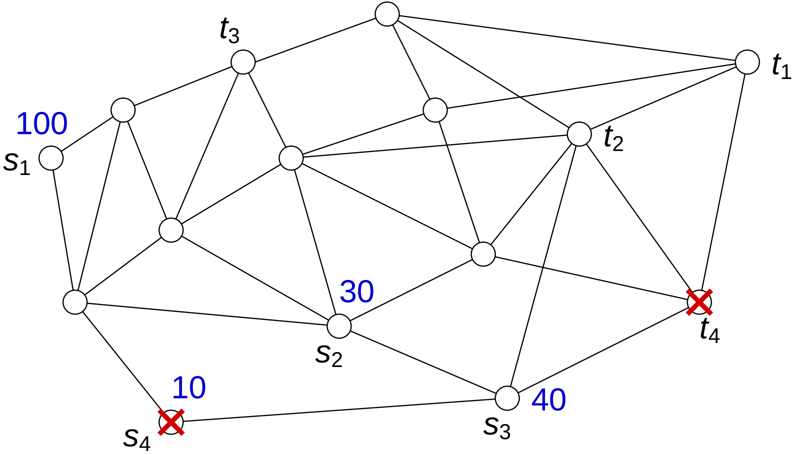

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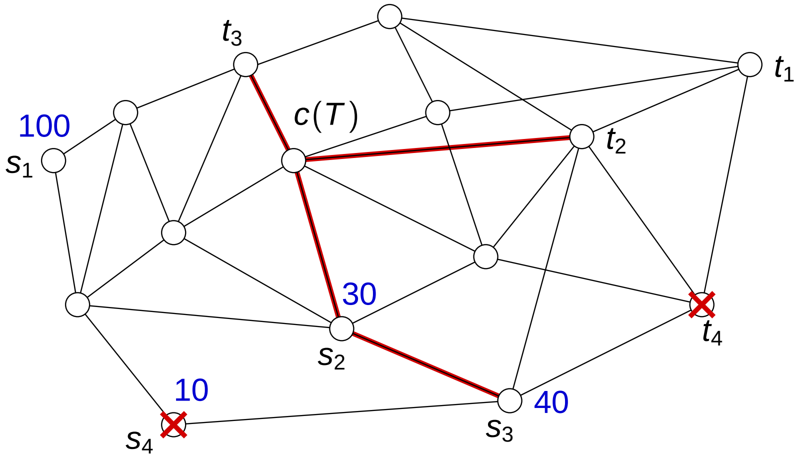

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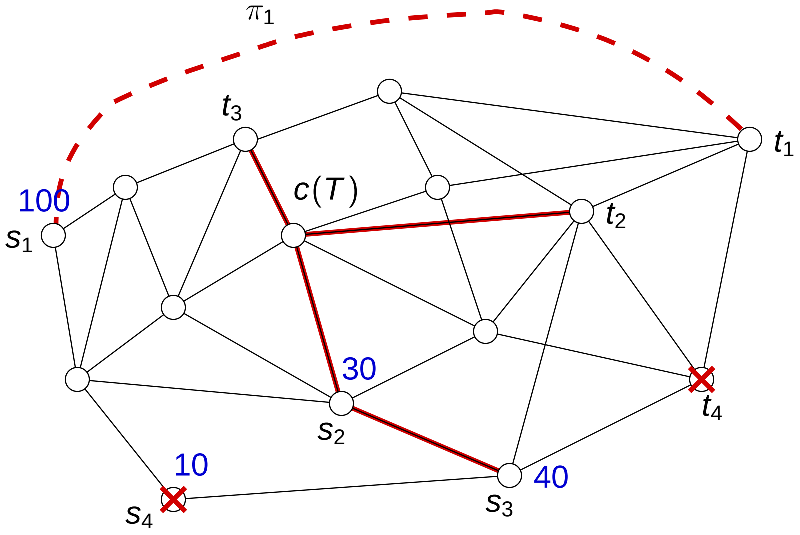

Prize-Collecting Steiner Forest Problem (PCSF)

Given:

- network $N = (V, E, c)$ with edge costs $c : E \to \mathbb{R}^+$
- set of n terminal pairs $R = \{(s_1, t_1), \ldots, (s_n, t_n)\} \subseteq V \times V$
- penalty $\pi_i \geq 0$ for every pair $(s_i, t_i) \in R$.

Feasible solution: forest F and subset $Q \subseteq R$ such that for all $(s_i, t_i) \in R$: either s_i, t_i are connected in F, or $(s_i, t_i) \in Q$

Objective: compute feasible solution (F, Q) such that $c(F) + \pi(Q)$ is minimized

Previous and Our Results

Approximation algorithms:

- 2.54-approximate algorithm (LP rounding)
- 3-approximate combinatorial algorithm (primal-dual)

[Hajiaghayi and Jain ’06]

This talk:

- simple 3-approximate primal-dual combinatorial algorithm that additionally achieves several desirable game-theoretic objectives

Cost Sharing Model

Setting:

- service provider offers some service
- set U of n potential users, interested in service
- every user $i \in U$:
  - has a (private) utility $u_i \geq 0$ for receiving the service
  - announces bid $b_i \geq 0$, the maximum amount he is willing to pay for the service
- cost function $C : 2^U \to \mathbb{R}^+$
  $C(S) =$ cost to serve user-set $S \subseteq U$
  (here: $C(S) =$ optimal cost of PCSF for S)

Cost Sharing Mechanism

Cost sharing mechanism M:

- collects all bids $\{b_i\}_{i \in U}$ from users
- decides a set $S^M \subseteq U$ of users that receive service
- determines a payment $p_i \geq 0$ for every user $i \in S^M$

Benefit: user i receives benefit $u_i - p_i$ if served, zero otherwise

Strategic behaviour: every user $i \in U$ acts selfishly and attempts to maximize his benefit (using his bid)

Objectives

1. β-budget balance: approximate total cost

$$C(S^M) \leq p(S^M) \leq \beta \cdot C(S^M), \quad \beta \geq 1$$

2. Group-strategyproofness: bidding truthfully $b_i = u_i$ is a dominant strategy for every user $i \in U$, even if users cooperate

3. α-efficiency: approximate maximum social welfare

$$u(S^M) - c(S^M) \geq \frac{1}{\alpha} \cdot \max_{S \subseteq U}[u(S) - C(S)], \quad \alpha \geq 1$$

No mechanism can achieve (approximate) budget balance, truthfullness and efficiency [Feigenbaum et al. '03]

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Previous Results

Authors	Problem	β
[Moulin, Shenker '01]	submodular cost	1
[Jain, Vazirani '01]	MST	1
	Steiner tree and TSP	2
[Devanur, Mihail, Vazirani '03]	set cover	$\log n$
(strategyproof only)	facility location	1.61
[Pal, Tardos '03]	facility location	3
	SRoB	15
[Leonardi, Schäfer '03], [Gupta et al. '03]	SRoB	4
[Leonardi, Schäfer '03]	CFL	30
[Könemann, Leonardi, Schäfer '05]	Steiner forest	2
Lower bounds		
[Immorlica, Mahdian, Mirrokni '05]	edge cover	2
	facility location	3
	vertex cover	$n^{1/3}$
	set cover	n
[Könemann, Leonardi, Schäfer, van Zwam '05]	Steiner tree	2

Objectives

1. β-budget balance: approximate total cost

$$C(S^M) \leq p(S^M) \leq \beta \cdot C(S^M), \quad \beta \geq 1$$

2. Group-strategyproofness: bidding truthfully $b_i = u_i$ is a dominant strategy for every user $i \in U$, even if users cooperate

3. α-approximate: approximate minimum social cost

$$\Pi(S^M) \leq \alpha \cdot \min_{S \subseteq U} \Pi(S), \quad \alpha \geq 1$$

where $\Pi(S) := u(U \setminus S) + C(S)$

[Roughgarden and Sundararajan '06]

Previous/Recent Work

Authors	**Problem**	β	α
[Roughgarden, Sundararajan '06]	submodular cost	1	$\Theta(\log n)$
	Steiner tree	2	$\Theta(\log^2 n)$
[Chawla, Roughgarden, Sundararajan '06]	Steiner forest	2	$\Theta(\log^2 n)$
[Roughgarden, Sundararajan]	facility location	3	$\Theta(\log n)$
	SRoB	4	$\Theta(\log^2 n)$
[Gupta et al. '07]	prize-collecting Steiner forest	3	$\Theta(\log^2 n)$

Tricks of the Trade...

Cost sharing method: function $\xi : U \times 2^U \to \mathbb{R}^+$
$\xi(i, S) =$ cost share of user i with respect to set $S \subseteq U$

β-budget balance:

$$C(S) \leq \sum_{i \in S} \xi(i, S) \leq \beta \cdot C(S) \quad \forall S \subseteq U$$

Cross-monotonicity: cost share of user i does not increase as additional users join the game:

$$\forall S' \subseteq S,\ \forall i \in S' : \quad \xi(i, S') \geq \xi(i, S)$$

Moulin Mechanism

Given: cross-monotonic and β-budget balanced cost sharing method ξ

Thm: Moulin mechanism $M(\xi)$ is a group-strategyproof cost sharing mechanism that is β-budget balanced

[Moulin, Shenker '01]
[Jain, Vazirani '01]

Moulin mechanism $M(\xi)$:

1: Initialize: $S^M \leftarrow U$
2: If for each user $i \in S^M$: $\xi(i, S^M) \leq b_i$ then STOP
3: Otherwise, remove from S^M all users with $\xi(i, S^M) > b_i$ and repeat

Summability

Given: arbitrary order σ on users in U

Order subset $S \subseteq U$ according to σ:

$$S := \{i_1, \dots, i_{|S|}\}$$

Let $S_j :=$ first j users of S

α-summability: ξ is α-summable if

$$\forall \sigma,\ \forall S \subseteq U : \quad \sum_{j=1}^{|S|} \xi(i_j, S_j) \leq \alpha \cdot C(S)$$

Approximability

Given: cross-monotonic and β-budget balanced cost sharing method ξ that satisfies α-summability

Thm: Moulin mechanism $M(\xi)$ is a group-strategyproof cost sharing mechanism that is β-budget balanced and $(\alpha + \beta)$-approximate

[Roughgarden, Sundararajan '06]

Our Results

- cost sharing method ξ that is cross-monotonic and 3-budget balanced for PCSF
  (byproduct: simple primal-dual 3-approximate algorithm)
- reduction technique that shows that Moulin mechanism $M(\xi)$ is $\Theta(\log^2 n)$-approximate
  (technique applicable to other prize-collecting problems)
- simple proof of $O(\log^3 n)$-summability for Steiner forest cost sharing method

Goal and Main Idea

Goal: develop an algorithm that for each set $S \subseteq U$ of users (terminal pairs) defines a cost share $\xi(i, S)$ for each user $i \in S$ such that cost shares are

- 3-budget balanced and
- cross-monotonic

Main idea: develop 3-approximate primal-dual algorithm for PCSF and share dual growth among terminal pairs

- budget balance follows from approximation guarantee
- cross-monotonicity requires new ideas!!

LP Formulation

$$\begin{aligned}
\texttt{min} \quad & \sum_{e \in E} c_e \cdot x_e + \sum_{(u,\bar{u}) \in R} \pi(u,\bar{u}) \cdot x_{u\bar{u}} \\
\text{s.t.} \quad & \sum_{e \in \delta(S)} x_e + x_{u\bar{u}} \geq 1 \quad \forall S \in \mathcal{S},\ \forall (u,\bar{u}) \odot S \\
& x_e \geq 0 \quad \forall e \in E \\
& x_{u\bar{u}} \geq 0 \quad \forall (u,\bar{u}) \in R
\end{aligned}$$

$\mathcal{S}$ = set of all Steiner cuts (separate at least one pair)

$\delta(S)$ = edges that cross cut defined by S

$(u,\bar{u}) \odot S$ = terminal pair $(u,\bar{u})$ separated by S

Dual LP — Simplified

$$\max \sum_{S \in \mathcal{S}} y_S$$

$$\text{s.t.} \quad \sum_{S: e \in \delta(S)} y_S \leq c_e \quad \forall e \in E$$

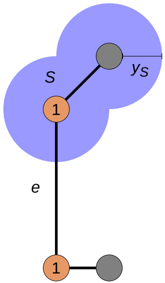
$$\xi_{u\bar{u}} \leq \pi(u, \bar{u}) \quad \forall (u, \bar{u}) \in R$$

$$\xi_{S, u\bar{u}} \geq 0 \quad \forall S \in \mathcal{S}, \ \forall (u, \bar{u}) \odot S$$

$$\xi_{u\bar{u}} := \sum_{S:(u,\bar{u}) \odot S} \xi_{S, u\bar{u}} \quad \text{(total cost share of } (u, \bar{u}))$$

$$y_S := \sum_{(u,\bar{u}) \odot S} \xi_{S, u\bar{u}} \quad \text{(total dual of Steiner cut } S)$$

Visualizing the Dual

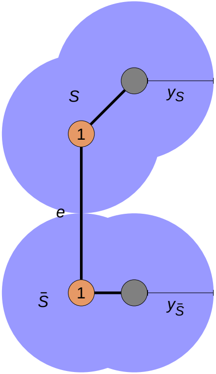


- dual y_S of Steiner cut S is visualized as moat around S of radius y_S
- edge e is tight if

$$\sum_{S:e\in\delta(S)} y_S = c_e$$

- growth of moat corresponds to an increase in the dual value

Visualizing the Dual



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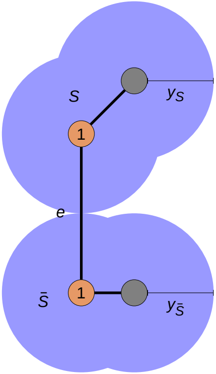

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- growth of moat corresponds to an increase in the dual value

Activity Notion

Death time: let $d_G(u, \bar{u})$ be distance between $u, \bar{u}$ in G

$$\mathrm{d}(u, \bar{u}) := \frac{1}{2} d_G(u, \bar{u})$$

Activity: terminal $u \in R$ is active at time τ iff

$$\xi^{\tau}_{u\bar{u}} < \pi(u, \bar{u}) \quad \text{and} \quad \tau \leq \mathrm{d}(u, \bar{u}).$$

Call a moat active if it contains at least one active terminal

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Primal-dual Algorithm

- process over time
- at every time τ: grow all active moats uniformly
- share dual growth of a moat evenly among active terminals contained in it
- if two active moats collide: add all new tight edges on path between them to the forest F
- if a terminal pair $(u, \bar{u})$ becomes inactive since its cost share reaches its penalty, add $(u, \bar{u})$ to the set Q
- terminate if all moats are inactive

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Primal-dual Algorithm

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- terminate if all moats are inactive

Primal-dual Algorithm

- process over time
- at every time τ: grow all active moats uniformly
- share dual growth of a moat evenly among active terminals contained in it
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- terminate if all moats are inactive

Primal-dual Algorithm

- process over time
- at every time τ: grow all active moats uniformly
- share dual growth of a moat evenly among active terminals contained in it
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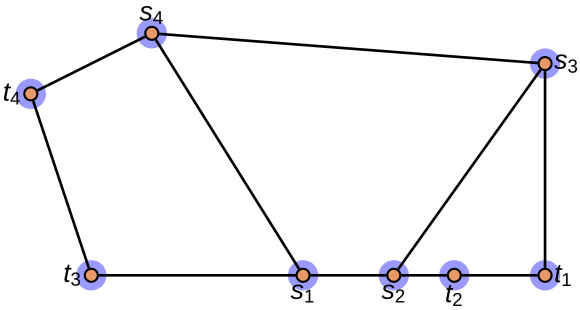
- terminate if all moats are inactive

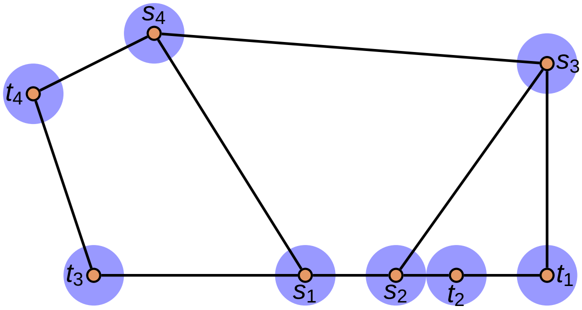
Illustration



$\tau = 0.5$

	(s_1, t_1)	(s_2, t_2)	(s_3, t_3)	(s_4, t_4)
$d(\cdot)$	4	1	22	3
$\pi(\cdot)$	5	5	∞	2
ξ^τ	1	1	1	1

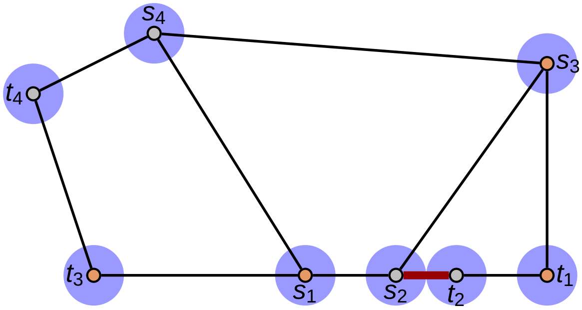
Illustration



$\tau = 1$

	(s_1, t_1)	(s_2, t_2)	(s_3, t_3)	(s_4, t_4)
$d(\cdot)$	4	1	22	3
$\pi(\cdot)$	5	5	∞	2
ξ^τ	2	2	2	2

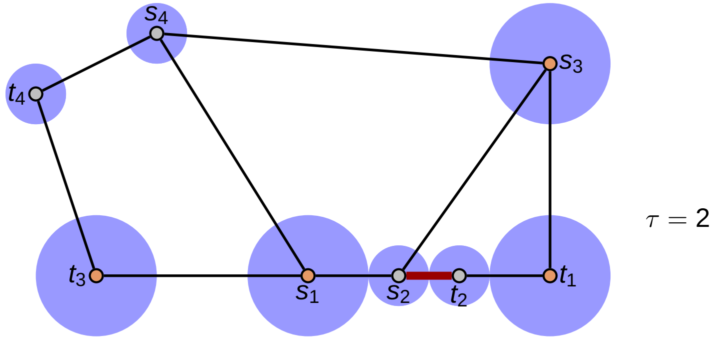
Illustration



$\tau = 1$

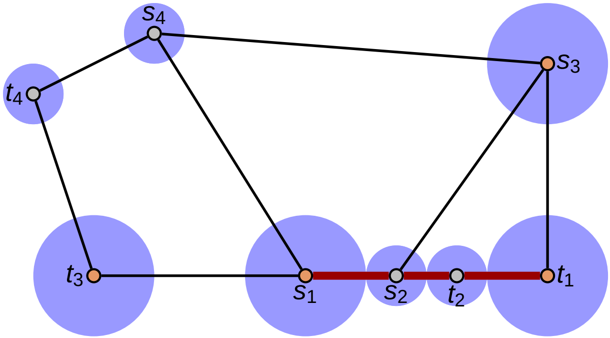
	(s_1, t_1)	(s_2, t_2)	(s_3, t_3)	(s_4, t_4)
$d(\cdot)$	4	1	22	3
$\pi(\cdot)$	5	5	∞	2
ξ^τ	2	2	2	2

Illustration



	(s_1, t_1)	(s_2, t_2)	(s_3, t_3)	(s_4, t_4)
$d(\cdot)$	4	1	22	3
$\pi(\cdot)$	5	5	∞	2
ξ^τ	4	2	4	2

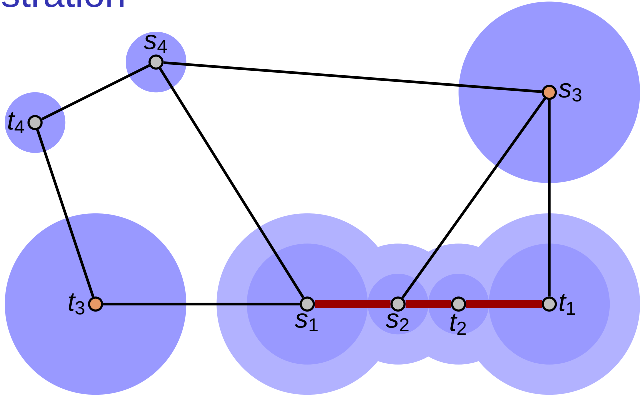
Illustration



$\tau = 2$

	(s_1, t_1)	(s_2, t_2)	(s_3, t_3)	(s_4, t_4)
$\mathrm{d}(\cdot)$	4	1	22	3
$\pi(\cdot)$	5	5	∞	2
ξ^τ	4	2	4	2

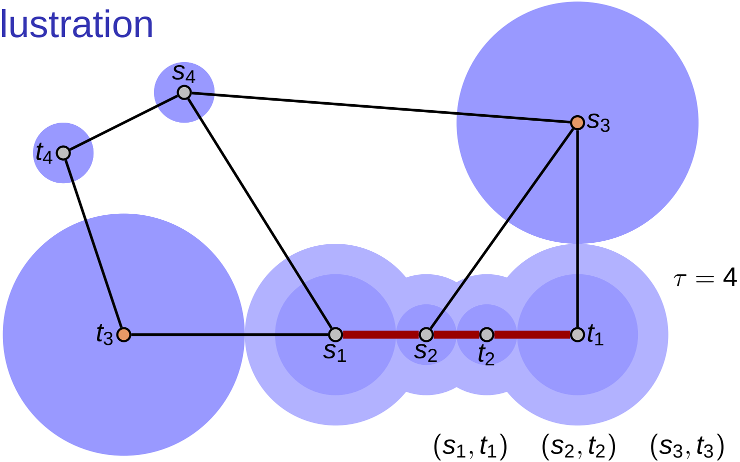
Illustration



$\tau = 3$

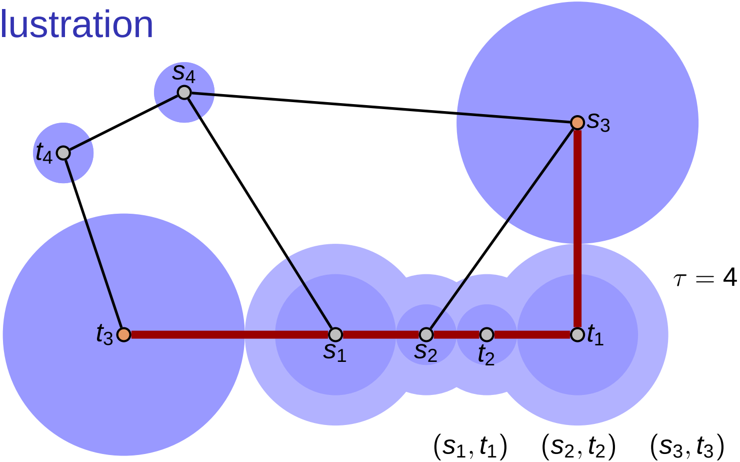
	(s_1, t_1)	(s_2, t_2)	(s_3, t_3)	(s_4, t_4)
$d(\cdot)$	4	1	22	3
$\pi(\cdot)$	5	5	∞	2
ξ^{τ}	5	2	6	2

Illustration



	(s_1, t_1)	(s_2, t_2)	(s_3, t_3)	(s_4, t_4)
$d(\cdot)$	4	1	22	3
$\pi(\cdot)$	5	5	∞	2
ξ^τ	5	2	8	2

Illustration



	(s_1, t_1)	(s_2, t_2)	(s_3, t_3)	(s_4, t_4)
$d(\cdot)$	4	1	22	3
$\pi(\cdot)$	5	5	∞	2
ξ^τ	5	2	8	2

Two Quick Proofs

Lem: ξ is cross-monotonic

Proof (idea): at every time τ and for any $S \subseteq S'$

- moat system wrt. S is a refinement of moat system wrt. S'
- cost share of u wrt. S is at least cost share of u wrt. S'

Lem: ξ is 3-budget balanced

Proof (idea):

- cost of solution is at most $2 \sum y_S$ for Steiner forest and $\sum \xi_{u\bar{u}}$ for total penalty
- need to prove that $\sum y_S = \sum_{(u,\bar{u}) \in R} \xi_{u,\bar{u}} \leq C(R)$

Proving budget balance

Lemma: $\sum_{(u,\overline{u})\in R} \xi_{u,\overline{u}} \leq C(R)$

Proof:

- Let $C(R) = c(F^*) + \pi(Q^*)$, with (F^*, Q^*) denoting the optimal solution.
- We have

$$\sum_{(u,\bar{u})\in Q^*} \xi_{u\bar{u}} \leq \pi(Q^*).$$

- It remains to be shown:

$$\sum_{(u,\bar{u})\in R/Q^*} \xi_{u\bar{u}} \leq c(F^*)$$

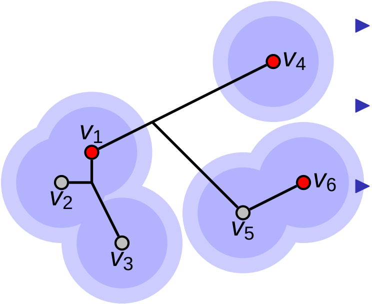
Proving $\sum_{(u,\bar{u})\in R} \xi_{u,\bar{u}} \leq C(R)$

- For each connected component $T \in F^*$, let $R(T)$ be the set of terminal pairs that are connected by T.
- We prove a slightly weaker result:

$$\sum_{(u,\bar{u})\in R(T)} \xi_{u\bar{u}} \leq \frac{3}{2} c(T). \qquad (1)$$

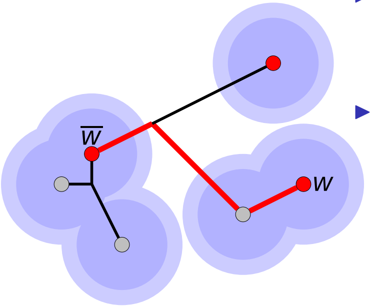
- $\mathcal{M}^{\tau}(T)$: set of moats at time τ that contain at least one active terminal of $R(T)$.
- Let let $(w, \bar{w}) \in R(T)$, be the pair that is active longest.
- Need to show that the total growth of $\mathcal{M}^{\tau}(T)$ for all $\tau \in [0, \mathrm{d}(w, \bar{w})]$ is at most $\frac{3}{2}c(T)$.

Proving $\sum_{(u,\bar{u})\in R} \xi_{u,\bar{u}} \leq C(R)$



- The moats of $\mathcal{M}^{\tau}(T)$ are disjoint at any time τ.
- If there are at least two active moats in $\mathcal{M}^{\tau}(T)$, they all intersect a different part of the edges of T.
- Let $\tau_0 \leq \mathrm{d}(w, \bar{w})$ be the first time such that $\mathcal{M}^{\tau_0}(T)$ does not load T.
- The total growth of moats in $\mathcal{M}^{\tau}(T)$ for all $\tau \leq \tau_0$ is at most $c(T)$.
- We are left with bounding the growth of the single moat $\mathcal{M}^{\tau_0}(T) = \{M^{\tau_0}\}$ for each $\tau \in [\tau_o, \mathrm{d}(w, \bar{w})]$.

Proving $\sum_{(u,\bar{u})\in R} \xi_{u,\bar{u}} \leq C(R)$



- Growth of M^τ for all times $\tau \in [\tau_0, \mathrm{d}(w,\bar{w})]$ is at most $\mathrm{d}(w,\bar{w}) - \tau_0$.
- Since w and $\bar{w}$ are connected by T, this additional growth is at most $\mathrm{d}(w,\bar{w}) \leq c(T)/2$.
- The $\frac{3}{2}c(T)$ upper bound on the total cost shares of pairs in $R(T)$ then follows.

Approximate social cost

α-approximate minimum social cost

$$\Pi(S^M) \leq \alpha \cdot \min_{S \subseteq U} \Pi(S), \quad \alpha \geq 1$$

where $\Pi(S) := u(U \setminus S) + C(S)$

Given: cross-monotonic and β-budget balanced cost sharing method ξ that satisfies α-summability

Thm: Moulin mechanism $M(\xi)$ is a group-strategyproof cost sharing mechanism that is β-budget balanced and $(\alpha + \beta)$-approximate

[Roughgarden, Sundararajan '06]

Approximate social cost

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Partitioning Lemma

Given: cross-monotonic cost sharing method ξ on U that is β-budget balanced for C

Lem: If there is a partition $U = U_1 \dot\cup U_2$ such that the Moulin mechanism $M(\xi)$ is α_i-approximate on U_i for all $i \in \{1, 2\}$, then $M(\xi)$ is $(\alpha_1 + \alpha_2)\beta$-approximate on U

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High-Utility Users

U_1 = set of all users i with $u_i \geq \pi_i$

Lem: (High-Utility Lemma): $M(\xi)$ is 1-approximate on U_1.

Proof: By construction, $\xi(i, S) \leq \pi_i \leq u_i$ for all i, for all $S \subseteq U_1$.
Thus, set S^M output by Moulin mechanism $M(\xi)$ is U.
Moreover, U minimizes social cost.

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Low-Utility Users

$U_2 =$ set of all users i with $u_i < \pi_i$

$\xi' =$ cross-monotonic cost sharing method for Steiner forest problem

Similarity Property: For every $S \subseteq U_2$: If there is a user $i \in S$ with $\xi(i, S) > u_i$ or $\xi'(i, S) > u_i$ then there exists a user $j \in S$ with $\xi(j, S) > u_j$ and $\xi'(j, S) > u_j$.

Lem: When starting with a low-utility set $S \subseteq U_2$, the final user sets produced by $M(\xi)$ and $M(\xi')$ are the same

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Lem: (Low-Utility Lemma): $M(\xi)$ is α-approximate on U_2 if $M(\xi')$ is α-approximate on U_2

Proof: Solution for set with minimum social cost never pays a penalty, as $u_i < \pi_i$. Thus, optimal social cost for PCSF and SF are the same. Furthermore, $C(S) \leq C'(S)$ for all $S \subseteq U_2$. Due to the similarity property, both mechanisms output the same set S.

$$\Pi(S) = u(U \backslash S) + C(S) \leq u(U \backslash S) + C'(S) = \Pi'(S) \leq \alpha \Pi'^{*} = \alpha \Pi^{*}$$

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Putting the Pieces together...

We showed:

- $M(\xi)$ is 1-approximate on high-utility users
- $M(\xi)$ is $\Theta(\log^2 n)$-approximate on low-utility users

Thm: $M(\xi)$ is a group-strategyproof cost sharing mechanism for PCSF that is 3-budget balanced and $\Theta(\log^2 n)$-approximate

Remark: technique extends to other prize-collecting problems, e.g., prize-collecting facility location

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Conclusions and Open Problems

- developed a group-strategyproof cost sharing mechanism for PCSF that is 3-budget balanced and $\Theta(\log^2(n))$-approximate

- **open problem:** find an LP formulation for our PCSF primal-dual algorithm
- **open problem:** give a combinatorial $(3 - \epsilon)$-approximate algorithm for PCSF