Pushdown Automata

Last Time
- Chomsky Normal Form
- A Pumping Lemma for CFLs

Today
- Decision procedures for CFLs
- Pushdown Automata
- A Kleene Theorem for CFLs
- Determinism

Decision Procedures for CFGs

Recall what a decision procedure is: an algorithm for answering a yes/no question.

A several yes/no questions involving CFGs have decision procedures.

1. Given CFG $G$, is $\varepsilon \in L(G)$?
2. Given CFG $M$ and $x \in \Sigma^*$, is $x \in L(M)$?
3. Given FA $G$, is $L(M) = \emptyset$?

Answering these questions is harder in the case of CFGs than FAs, but all are decidable.
Deciding Whether $\varepsilon \in \mathcal{L}(G)$

... use “nullability algorithm”!

- Let $G = (V, \Sigma, S, P)$.
- Compute $N(G) \subseteq V$, the set of nullable variables (i.e. $A \in N(G)$ if and only if $A \Rightarrow^*_G \varepsilon$).
- Then $\varepsilon \in \mathcal{L}(G)$ if and only if $S \in N(G)$! (Why?)

Deciding Whether $x \in \mathcal{L}(G)$

What we want is an algorithm that, given a CFG $G$ and word $x$, determines whether or not $x$ can be generated from $G$.

Our approach will rely on Chomsky Normal Form!

- We’ll generate a CNF grammar $G_4$ from $G$.
- We’ll then use the special properties of CNF grammars to answer the question.

How Does CNF Help?

Consider CFG $G$ given by: $S \rightarrow \varepsilon \mid 0S1$. Our CNF equivalent $G_4$ is:

- $S \rightarrow X_0X_1 \mid X_0Y$
- $Y \rightarrow SX_1$
- $X_0 \rightarrow 0$
- $Y_1 \rightarrow 1$

To determine if $S \Rightarrow_{G_4} 001$, do we need to consider derivations beginning

$S \Rightarrow_{G_4} X_0Y \Rightarrow_{G_4} X_0SX_1 \Rightarrow_{G_4} X_0X_0YX_1 \Rightarrow_{G_4} \cdots$?

No! $|X_0X_0YX_1| = 4 > 3 = |001|$, and in a CNF grammar only words of length $\geq 4$ can be generated from $X_0X_0YX_1$.

A Decision Procedure for $x \in \mathcal{L}(G)$

1. If $x = \varepsilon$, apply previous decision procedure.
2. Otherwise, do following.
   (a) Convert $G$ into CNF, yielding $G_4$
   (b) Generate all possible derivation sequences whose final configuration has length $|x|$.
   (c) If one derivation sequence leads to $x$, return “true”, else return “false”.

Why does this work? Because for CNF grammars, only finitely many appropriate derivation sequences are possible!

Note There are much better algorithms....
Machines for CFLs

Recall our study of regular languages.
- They were defined in terms of regular expressions (syntax).
- We then showed that FAs provide the computational power needed to process them.

We would like to mimic this line of development for CFLs.
- We have a “syntactic” definition of CFLs in terms of CFGs.
- What kind of computing power is needed to “process” (i.e. recognize) CFLs?

Do FAs suffice?

The problem with FAs is that a given FA only has a finite amount of memory.
- States allow you to “store” information about the input seen so far.
- Finite states = finite memory!

However, some CFLs require an unbounded amount of “memory”!

E.g. \( L = \{ 0^n1^n \mid n \geq 0 \} \). To determine if a word is in \( L \), you need to be able to “count” arbitrarily high in order to keep track of the number of initial 0’s. This implies a need for an unbounded number of bits of memory. (Why?)

Consequently, we need to have some form of “unbounded memory” in the machines for CFLs.

It turns out that an unbounded stack, or pushdown, will do!

Pushdown Automata

... are (N)FAs with an auxiliary stack.

- State transitions can read inputs and top stack symbol.
- When a transition “fires”, new symbols can be pushed onto stack.
Formalizing PDAs

What does a PDA specification need to contain?

- The things we found in FAs: states, input alphabet, start state, transitions, accepting states ...
- ... plus the stack alphabet and initial stack symbol.
- Also, transitions need to be able manipulate the stack.

Defining PDAs

**Definition** A pushdown automaton (PDA) is a septuple \((Q, \Sigma, \Gamma, q_0, Z_0, \delta, A)\) where:

- \(Q\) is a finite set of states;
- \(\Sigma\) and \(\Gamma\) are the input and stack alphabets, respectively;
- \(q_0 \in Q\) is the start state;
- \(Z_0 \in \Gamma\) is the initial stack symbol;
- \(\delta: Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow 2^{Q \times (\Gamma^*)}\) is the transition function; and
- \(A \subseteq Q\) is the set of accepting states.

The Transition Function PDA \(M = (Q, \Sigma, \Gamma, q_0, Z_0, \delta, A)\)

\(\delta\) has type \(Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow 2^{Q \times (\Gamma^*)}\).

**Inputs** triples \((q, a, X)\).

- \(q\) is the source state of the transition.
- \(a\) can either be an input symbol (element of \(\Sigma\)) or \(\varepsilon\), in which case the transition consumes no input when it fires.
- \(X\) is the symbol currently on top of the stack.

**Outputs** sets of pairs \((q', \gamma)\). (Why sets? PDAs can be nondeterministic!).

- \(q'\) is the target state of the transition.
- \(\gamma \in \Gamma^*\) is a sequence of symbols pushed onto the stack in place of \(X\).
PDA Transitions (cont.)

So
if
\[ (q', \gamma) \in \delta(q, a, X) \]
and the current state is \( q \)
and \( a \in \Sigma \) and the current input symbol is \( a \)
and the current top symbol on the stack is \( X \)
then
the input symbol is consumed
and the state changes to \( q' \)
and \( X \) is popped from the stack, with \( \gamma \) then pushed.

(Case for \( a = \varepsilon \) is the same, except that input stream is not disturbed.)

Example: PDA for \( \{ 0^m1^n \mid m \leq n \leq 2m \} \)

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The Language of a PDA

A PDA \( M \) should accept a word \( w \) if, starting with the initial stack, \( M \) "processes" \( w \) and winds up in an accepting state.

What do we need to keep track of to determine if this holds?
- PDA’s current state
- Current stack contents
- Remaining input

If we have this information, then we can determine which transitions can "fire" and what the new stack contents and input stream are when a transition takes place!

Formalizing Acceptance in a PDA

We need to define the notions of:
- Configuration of a PDA (i.e. a “snapshot” of an executing PDA)
- A one-step configuration transition relation, \( \vdash^1_M \)

We’ll then use these to define the language accepted by a PDA.

In what follows fix PDA \( M = \langle Q, \Sigma, \Gamma, q_0, Z_0, \delta, A \rangle \).
Formalizing the Language of a PDA

**Definition** A *configuration* of $M$ is a triple $(q, x, \alpha) \in Q \times (\Sigma^*) \times (\Gamma^*)$.

$q$ is current state, $x$ is remaining input, $\alpha$ is stack, with top element first.

**Definition** The *configuration transition relation* $\vdash_M$, is defined by:

$$\langle q, x, \alpha \rangle \vdash_M \langle q', x', \alpha' \rangle$$

if there exist $a \in \Sigma \cup \{\varepsilon\}$, $X \in \Gamma$, and $\beta \in \Gamma^*$ such that:

- $(q', \gamma) \in \delta(q, a, X)$.
- $x = a \cdot x'$
- $\alpha = X \cdot \beta$
- $\alpha' = \gamma \cdot \beta$

Example

Configuration: $\langle B, 01, 0Z_0 \rangle$

Sequence of configuration transitions:

$$\langle A, 001, Z_0 \rangle \vdash_M \langle B, 01, 0Z_0 \rangle \vdash_M \langle B, 1, 00Z_0 \rangle \vdash_M \langle C, \varepsilon, 0Z_0 \rangle$$

Formalizing the Language of a PDA (cont.)

**Definition** $M$ *accepts* $x \in \Sigma^*$ if there are configurations $c_0, c_1, ..., c_n$, for some $n \geq 0$ such that:

- $c_0 = \langle q_0, x, Z_0 \rangle$;
- $c_i = \langle q, \varepsilon, \alpha \rangle$ for some $q \in A, \alpha \in \Gamma^*$; and
- $c_i \vdash_M c_{i+1}$ all $i$ such that $0 \leq i < n$.

The language $M$ is $L(M) = \{ x \in \Sigma^* \mid M \text{ accepts } x \}$.

Example

$M$ accepts 0011, since

$$\langle A, 0011, Z_0 \rangle \vdash_M \langle B, 011, 0Z_0 \rangle \vdash_M \langle B, 11, 00Z_0 \rangle \vdash_M \langle C, \varepsilon, 0Z_0 \rangle$$

$M$ does not accept 010, since only possible configuration sequence is:

$$\langle A, 010, Z_0 \rangle \vdash_M \langle B, 10, 0Z_0 \rangle \vdash_M \langle C, 0, Z_0 \rangle \vdash_M \langle D, 0, Z_0 \rangle$$
A Kleene Theorem for CFLs

Recall the statement of the Kleene Theorem for regular languages.

A language is regular iff it is accepted by some finite automaton.

This result says that FAs provide the requisite computing power for “processing” regular languages.

We proved one direction by showing how to convert any regular expression into a language-equivalent NFA.

We want to prove a similar connection between CFLs and PDAs:

A language is context-free if and only if it is recognized by some PDA.

How Do We Prove This?

It suffices to prove two things.

1. For every CFG $G$ there is a PDA $M$ with $L(G) = L(M)$.
2. For every PDA $M$ there is a CFG $G$ with $L(M) = L(G)$.

Why does this suffice?

We will only prove the first direction.

Building PDAs from CFGs

Theorem

For any CFG $G = (V, \Sigma, S, P)$ there is a PDA $M$ with $L(G) = L(M)$.

How do we build $M$? CFGs and PDAs seem very different (one generates words, the other accepts them). But ...

- CFGs generate words using derivation sequences ($\Rightarrow_G$).
- Any variable can be “expanded” at any time: e.g. in $AaB$ either $A$ or $B$ can have a production applied. But expanding one variable does not affect the potential expansions for the others.
- Once a terminal appears, it remains.

Idea

Build a PDA that simulates “appropriate” derivations in $G$. 
**Leftmost Derivations**

In a “leftmost” derivation, the leftmost variable in a sequence of terminals and nonterminals is always worked on “first”.

**Example**  Consider $G$ given by:

- $S \rightarrow AC$
- $A \rightarrow aAb \mid \varepsilon$
- $C \rightarrow cC \mid \varepsilon$

Here is a leftmost derivation of $abc$:

\[
S \Rightarrow G AC \Rightarrow G aAbC \Rightarrow G abcC \Rightarrow G abc
\]

Here is a derivation that is not leftmost:

\[
S \Rightarrow G AC \Rightarrow G AcC \Rightarrow G aAbcC \Rightarrow G abcC \Rightarrow G abc
\]

**Formalizing Leftmost Derivations**

**Definition**  Let $G = (V, \Sigma, S, P)$ be a CFG, with $\alpha, \beta \in (V \cup \Sigma)^*$. Then $\alpha \Rightarrow_G^* \beta$ if there exist $x \in \Sigma^*$, $\alpha', \gamma \in (V \cup \Sigma)^*$, and $A \in V$ such that:

- $\alpha = xA\alpha'$;
- $\beta = x\gamma\alpha'$; and
- $A \rightarrow \gamma$ is a production in $P$.

We write $\alpha \Rightarrow_G^* \beta$ if $\alpha$ can be rewritten to $\beta$ via a sequence of $\Rightarrow_G$ steps.

**Lemma**  For any CFG $G = (V, \Sigma, S, P)$ $w \in L(G)$ iff $S \Rightarrow_G^* w$.

In other words: even though $\Rightarrow_G$ is more restricted than $\Rightarrow_G$, you can still generate the same words!

**We Can Build a PDA for Leftmost Derivations!**

- Stack contains (part of) current sequence of terminals and nonterminals in derivation, with topmost variable in stack being leftmost variable.
- Variables at top of stack are popped and replaced by right-hand sides of productions.
- Terminals at top of stack are matched against input and popped.

**Example**

- Stack contains (part of) current sequence of terminals and nonterminals in derivation, with topmost variable in stack being leftmost variable.
- Variables at top of stack are popped and replaced by right-hand sides of productions.
- Terminals at top of stack are matched against input and popped.

Note correspondence:

\[
S \Rightarrow G 0S1 \quad \Rightarrow M (q_0, 0, 1Z_0) \quad \Rightarrow M (q_1, 01, 0S1Z_0) \quad \Rightarrow M (q_1, 01, 0S1Z_0) \\
S \Rightarrow G 1 \quad \Rightarrow M (q_1, 1Z_0) \quad \Rightarrow M (q_1, 1Z_0) \quad \Rightarrow M (q_1, 1Z_0) \quad \Rightarrow M (q_1, 1Z_0)
\]
Formalizing the Construction

Let \( G = \langle V, \Sigma, S, P \rangle \) be a CFG. We can construct PDA \( M_G \) as follows.

- \( Q_G = \{ q_0, q_1, q_2 \} \)
- \( \Gamma = V \cup \Sigma \cup \{ Z_0 \} \) where \( Z_0 \notin V \cup \Sigma \) is a new symbol.
- \( \delta_G \) is defined as follows.
  \[
  \delta_G(q_0, \varepsilon, Z_0) = \{ (q_1, SZ_0) \} \\
  \delta_G(q_1, a, a) = \{ (q_1, \varepsilon) \} \text{ if } a \in \Sigma \\
  \delta_G(q_1, \varepsilon, A) = \{ (q_1, \alpha) \mid A \rightarrow \alpha \in P \} \text{ if } A \in V \\
  \delta_G(q_1, \varepsilon, Z_0) = \{ (q_2, Z_0) \} \\
  \]
- \( A_G = \{ q_2 \} \)

Why Does \( M_G \) Accept the Language of \( G \)?

Recall \( w \in L(M_G) \) iff \( \langle q_0, w, Z_0 \rangle \xrightarrow{\ast}_{M_G} \langle q_2, \varepsilon, \alpha \rangle \) some \( \alpha \).

Claims

1. For any \( q' \in Q_G, w, w' \in \Sigma^*, \alpha \in \Gamma^*_G \), \( \text{ if } \langle q_0, w, Z_0 \rangle \xrightarrow{\ast}_{M_G} \langle q', w', \alpha \rangle \text{ then } \alpha = \beta \cdot Z_0 \) for some \( \beta \in (V \cup \Sigma)^* \).
2. For any \( w, w' \in \Sigma^*, \alpha \in \Gamma^*, \text{ if } \langle q_0, w, Z_0 \rangle \xrightarrow{\ast}_{M_G} \langle q_2, w', \alpha \rangle \text{ then } \alpha = Z_0. \)
3. For any \( w, w' \in \Sigma^*, \alpha \in (V \cup \Sigma)^*, \langle q_0, w, Z_0 \rangle \xrightarrow{\ast}_{M_G} \langle q_1, w', \alpha \cdot Z_0 \rangle \) iff \( S \xrightarrow{x} x \cdot \alpha \), where \( x \in \Sigma^* \) is such that \( w = x \cdot w' \).

On the basis of these claims, we can prove that \( L(G) = L(M_G) \).

Recall the definition of a PDA

**Definition** A **pushdown automaton** (PDA) is a septuple \( \langle Q, \Sigma, \Gamma, q_0, Z_0, \delta, A \rangle \) where:

- \( Q \) is a finite set of **states**;
- \( \Sigma \) and \( \Gamma \) are the **input** and **stack** alphabets, respectively;
- \( q_0 \in Q \) is the **start state**;
- \( Z_0 \in \Gamma \) is the **initial stack symbol**;
- \( \delta : Q \times (\Sigma \cup \{ \varepsilon \}) \times \Gamma \rightarrow 2^Q \times (\Gamma^*) \) is the **transition function**; and
- \( A \subseteq Q \) is the set of **accepting states**.
**A PDA for** \( L = \{ wcw^R \mid w \in \{a,b\}^* \} \)

- It will have two states that correspond to “have not seen the c” and “have seen the c”. The former will be the starting state, and the latter will be the final state.
- When in state “have not seen the c”, it will push the symbols that it reads onto the stack.
- When it encounters the c it switches states without changing the stack.
- In the state “have seen the c”, it compares the current input symbol to the symbol on the top of the stack and advances past both if they match.
- Only valid strings, that is, ones that have matching a’s and b’s and contain a c in the middle will cause acceptance. Any other string will reach a situation where there is no transition to take.

**Definition**

Let \( M = \langle \{s,f\}, \{a,b,c\}, \{a,b,Z_0\}, s, Z_0, \delta, \{f\} \rangle \), where

1. \( \delta(s, a, \gamma) = (s, a\gamma) \)
2. \( \delta(s, b, \gamma) = (s, b\gamma) \)
3. \( \delta(s, c, \gamma) = (f, \gamma) \)
4. \( \delta(f, a, a) = (f, \varepsilon) \)
5. \( \delta(f, b, b) = (f, \varepsilon) \)

**Sample accepting computation:** \( w = abacaba \)

<table>
<thead>
<tr>
<th>State</th>
<th>Unread input</th>
<th>Stack</th>
<th>Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s )</td>
<td>( abacaba )</td>
<td>( \varepsilon )</td>
<td>—</td>
</tr>
<tr>
<td>( s )</td>
<td>( bacaba )</td>
<td>( a )</td>
<td>1</td>
</tr>
<tr>
<td>( s )</td>
<td>( acaba )</td>
<td>( ba )</td>
<td>2</td>
</tr>
<tr>
<td>( s )</td>
<td>( caba )</td>
<td>( aba )</td>
<td>1</td>
</tr>
<tr>
<td>( f )</td>
<td>( aba )</td>
<td>( aba )</td>
<td>3</td>
</tr>
<tr>
<td>( f )</td>
<td>( ba )</td>
<td>( ba )</td>
<td>4</td>
</tr>
<tr>
<td>( f )</td>
<td>( a )</td>
<td>( a )</td>
<td>5</td>
</tr>
<tr>
<td>( f )</td>
<td>( \varepsilon )</td>
<td>( \varepsilon )</td>
<td>4</td>
</tr>
</tbody>
</table>

**Sample rejecting computation:** \( w = aaaa \)

<table>
<thead>
<tr>
<th>State</th>
<th>Unread input</th>
<th>Stack</th>
<th>Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s )</td>
<td>( aaaa )</td>
<td>( \varepsilon )</td>
<td>—</td>
</tr>
<tr>
<td>( s )</td>
<td>( aaa )</td>
<td>( a )</td>
<td>1</td>
</tr>
<tr>
<td>( s )</td>
<td>( aa )</td>
<td>( aa )</td>
<td>1</td>
</tr>
<tr>
<td>( s )</td>
<td>( a )</td>
<td>( aaaa )</td>
<td>1</td>
</tr>
<tr>
<td>( s )</td>
<td>( \varepsilon )</td>
<td>( aaaa )</td>
<td>1</td>
</tr>
</tbody>
</table>
Now consider the language \( L = \{ ww^R \mid w \in \{a, b\}^* \} \).

There is no center marker \( c \) to tell us when to switch from the state that pushes input onto the stack into the state that reads input while popping characters off the stack.

We will have to use nondeterminism to “guess” when to make the switch.

1. \( \delta(s, a, \gamma) = (s, a\gamma) \)
2. \( \delta(s, b, \gamma) = (s, b\gamma) \)
3. \( \delta(s, \varepsilon, \gamma) = (f, \gamma) \)
4. \( \delta(f, a, a) = (f, \varepsilon) \)
5. \( \delta(f, b, b) = (f, \varepsilon) \)

Sample accepting computation: \( w = abba \)

<table>
<thead>
<tr>
<th>State</th>
<th>Unread input</th>
<th>Stack</th>
<th>Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>abba</td>
<td>( \varepsilon )</td>
<td>—</td>
</tr>
<tr>
<td>s</td>
<td>bba</td>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>s</td>
<td>ba</td>
<td>ba</td>
<td>2</td>
</tr>
<tr>
<td>f</td>
<td>ba</td>
<td>ba</td>
<td>3</td>
</tr>
<tr>
<td>f</td>
<td>a</td>
<td>a</td>
<td>5</td>
</tr>
<tr>
<td>f</td>
<td>( \varepsilon )</td>
<td>( \varepsilon )</td>
<td>4</td>
</tr>
</tbody>
</table>

If there is no way to “guess” correctly, then the string will not be accepted, for example with \( w = babaa \)

Recall the configuration transition relation

**Definition** A configuration of \( M \) is a triple \( (q, x, \alpha) \in Q \times (\Sigma^*) \times (\Gamma^*) \).

(q is current state, \( x \) is remaining input, \( \alpha \) is stack, with top element first.)

**Definition** The configuration transition relation, \( \Gamma_M \), is defined by:

\( (q, x, \alpha) \in \Gamma_M (q', x', \alpha') \) if there exist \( a \in \Sigma \cup \{ \varepsilon \} \), \( X \in \Gamma \), and \( \beta, \gamma \in \Gamma^* \) such that:

1. \( (q', \gamma) \in \delta(q, a, X) \).
2. \( x = a \cdot x' \)
3. \( \alpha = X \cdot \beta \)
4. \( \alpha' = \gamma \cdot \beta \)

Determinism

A pushdown automaton is **deterministic** if for each configuration there is at most one configuration that can succeed it in a computation by \( M \).

**Question** Can we always find an equivalent deterministic pushdown automaton for a given context-free language?

**Answer:** Unfortunately not.

There are some context-free languages that cannot be accepted by deterministic pushdown automata.

This is a dire result, especially if we actually want to produce a parser for the context-free language.

**Some good news:** For most programming languages one can construct deterministic pushdown automata that accept all syntactically correct programs.
Determinism and complementation

**Theorem** The class of deterministic context-free languages is closed under complementation.

This is trickier than it was for DFAs, because of the stack. **Sticking point**: a PDA may reject because it never finished reading the input.

This can happen in the following two circumstances:

- $M$ reaches a configuration that has no following configuration;
- $M$ enters a configuration from which it can apply an infinite sequence of configurations that do not consume any input.

Proof idea: add in explicit transitions for these cases, then negate the accepting states.

Consider the language $L = \{a^m b^m c^p \mid m, n, p \geq 0, \text{ and } m \neq n \text{ or } m \neq p\}$.

Suppose that $L$ is deterministic. Then $\overline{L}$ is deterministic context-free, and thus, context-free.

So $\overline{L} \cap a^* b^* c^*$ would be context-free since intersection of CFL and RL is a CFL.

But $\overline{L} \cap a^* b^* c^* = \{a^n b^n c^n \mid n \geq 0\}$, a language that is not context-free.

Thus, $L$ cannot be deterministic.

**Corollary** The class of deterministic context-free languages is properly contained in the class of context-free languages.

*End result*: For pushdown automata, non-determinism is more powerful than determinism.

Ambiguity and Determinism

An ambiguous CFG is one which has more than one derivation for the same string. Note that ambiguity is a property of a grammar, not a language.

An inherently ambiguous CFL is a CFL that is not expressible using an unambiguous CFG. For example $\{a^m b^m c^p \mid m = n \text{ or } m = p\}$ is an inherently ambiguous CFL.

Every deterministic PDA language is expressible in an unambiguous CFG. The converse is not true. For example, $S \rightarrow 0S0 \mid 1S1 \mid \varepsilon$ is unambiguous but is accepted by no deterministic PDA.


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