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Neurocomputing 15 (1997) 91–106

NEUROCOMPUTING

A discrete version of the dynamic link network[1]

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Received 7 July 1995; accepted 8 November 1996

Abstract

Learning in a dynamic link network (DLN) is a composition of two dynamics: neural dynamics inside layers and link dynamics between layers. Based upon a rigorous analysis of the neural dynamics, we find an algorithm for selecting the parameters of the DLN in such a way that the neural dynamics preferentially converges to any chosen attractor. This control is important because the attractors of the neural dynamics determine the link dynamics which is the main tool for pattern retrieval. Thus in terms of our constructive algorithm it is possible to explore the link dynamics using all kinds of attractors of the neural dynamics. In particular, we show how to get on-center activity patterns which have been extensively used in the application of the DLN to image recognition tasks as well as having an important role in the image processing of the retina. We propose also a Hopfield-like discretized version of the neural dynamics which converges to the attractors much faster than the original DLN.

Keywords: Dynamical link network (DLN); Short term memory; Long term memory; Saturated attractor; On-center activity pattern; Neural dynamics; Link dynamics

1. Introduction

With the purpose of solving certain conceptual problems of conventional artificial neural networks, von der Malsburg proposed in 1981 a new neural information processing system, the two layered dynamic link network (DLN) (see [7, 9] and references therein). Since then, many successful applications in some difficult problems

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[1] Partially supported by the CNR of Italy. J.F. was also partially supported by the A. von Humboldt Foundation of Germany.

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PII S0925-2312(96)00055-0

have been carried out demonstrating the ability of the DLN. These applications have resulted in a massively parallel and fault-tolerant technique as well as giving new insights into brain function. The prominent features of the DLN are correlated neural activity and the self-organization of the dynamic links. There are two time scales in the DLN which makes it different from conventional artificial neural networks. For the fast varying time scale, the neurons in the same layer group themselves into an organized ensemble; then on the slowly varying time scale, the synaptic connections between the two layers develop according to a version of the Hebb learning rule evaluated on the neural activities of the organized attractor reached by the fast neural dynamics. Thus the evolution of the synaptic connections between the two layers essentially depends on the activities of the neural configuration reached by the neural dynamics on each single layer. Blob configurations i.e. on-center activity patterns [7] have been extensively used for this aim. But until now there has been no algorithm for selecting five key parameters of the neural dynamics to ensure the convergence of the neural dynamics to a blob configuration. In this research we present such an algorithm by showing how to choose the five parameters $(\beta, \gamma, s, I^X, I^Y)$ of the DLN in order to get any particular attractor per se. Furthermore an explicit method of finding the five parameters so that the neural dynamics reaches the blob configurations and, at the same time, avoids the trivial attractors which have all the neurons at maximum (or minimum) activity values is constructively given. We find also a Hopfield-like dynamics which acts only on the subspace of all the possible attractors. This discretized dynamics is faster than the original DLN dynamics which acts on the larger space of all possible neural activities. Our algorithm can be extended also to other important neural dynamics like the Hopfield model and Linsker's model employing the more general sigmoid response function: this analysis is done in another paper [3].

2. A brief description of the DLN

The dynamic link network is essentially a two layer network, say layer X and layer Y, with connections both inside the same layer and between different layers. Suppose that there are $n_X(n_Y)$ neurons in layer $X(Y)$ and all the neurons in layer $X(Y)$ are arranged in a two dimensional torus (i.e with periodic boundary conditions) as shown in Fig. 1.

The periodic boundary conditions are introduced here only for avoiding the boundary effects. We choose a coordinate system so that the first neuron sits at the origin. For $i = (r_1, r_2), j = (r_3, r_4), i, j = 1, \ldots, n_X$ or $i, j = 1, \ldots, n_Y$, the distance between i, j is given by

$$\|i - j\| = \max(|r_1 - r_3|, |r_2 - r_4|)$$

according to the boundary conditions.

There are two time scales in the DLN, a slow varying one $\tau = 1, 2, \ldots$, and a fast varying one $t \in \mathbb{R}^+$. For fixed τ, let $X_t(i, \tau)$ denote the activity of the ith neuron at time t in the layer X and $Y_t(i, \tau)$ be the activity of the ith neuron in the layer Y at time t. $X_i(t, \tau)$, $i = 1, \ldots, n_X$ is obtained by a weighted linear combination of the activities of

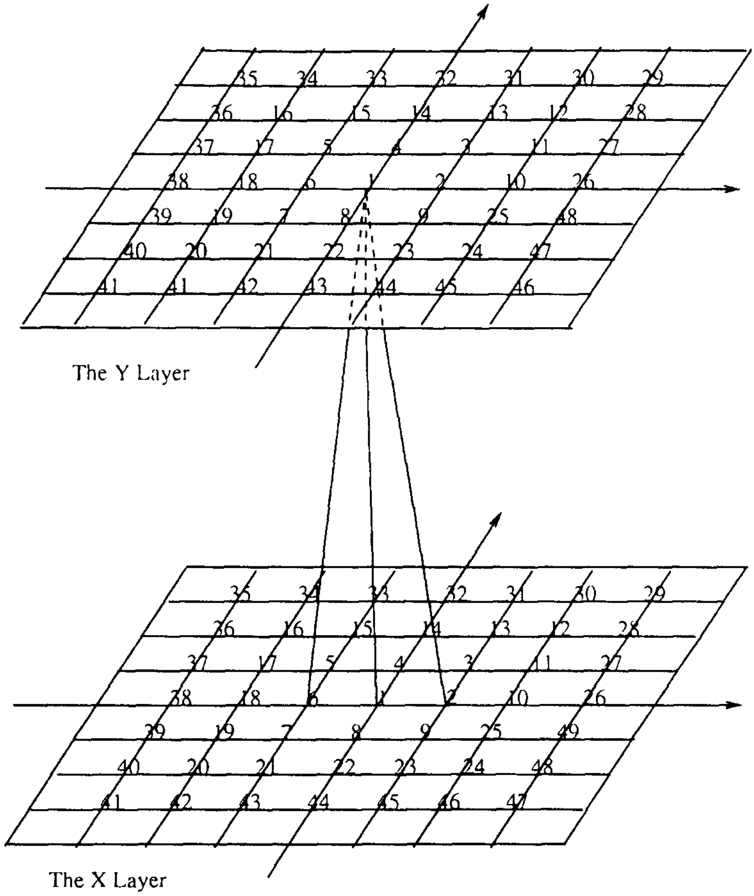

Fig. 1. An explanation of the structure of the dynamic link network. Neurons are numbered as in the figure. Only some dynamic links $J_{1i}(\tau)$ between the first neuron in the X layer and i neuron in the Y layer are showed in the figure.

the other neurons in the same layer and then by an application of the sigmoid transformation σ. More precisely,

$$X_t(i,\tau) = \sigma(x_t(i,\tau)),$$

$$\dot{x}_t(i,\tau) = -\alpha x_t(i,\tau) + \sum_{j=1}^{n_X} k(i,j)X_t(j,\tau) + I_t^X(i,\tau), \tag{1}$$

$$X_0(i,\tau) = 0,$$

where $i = 1, \ldots, n_X$, $\alpha > 0$ is a parameter of the dynamics,

$$k(i,j) = \gamma \exp\left(-\frac{\|i-j\|^2}{2s^2}\right) - \beta = \gamma p(i,j) - \beta \tag{2}$$

is the weight (interaction) function inside the layer X and γ, β the intensities of the excitatory and inhibitory connection, are all positive parameters, $I_t^X(i,\tau)$ with $\langle I_t^X(i,\tau)\, I_t^X(j,\tau)\rangle = \delta_{ij}, i,j = 1, \ldots, n_X$ is the input signal presented at the neuron i of the layer X. Note that the interaction kernel $k(i,j)$ consists of short-range excitatory connections with range s and global inhibitory connections of relative strength β.

The activities of the Y layer evolve with the same dynamics as the X layer except that the input signal $I_t^Y(i,\tau)$ is different, i.e. for $\tau = 1,2, \ldots, i = 1, \ldots, n_Y$

$$\begin{aligned} &Y_t(i,\tau) = \sigma(y_t(i,\tau)),\\ &\dot{y}_t(i,\tau) = -\alpha y_t(i,\tau) + \sum_{j=1}^{n_Y} k(i,j) Y_t(j,\tau) + I_t^Y(i,\tau), \\ &Y_0(i,\tau) = 0, \end{aligned} \tag{3}$$

where $I_t^Y(i,\tau)$, the input signal in the layer Y, will be specified in Eq. (6).

We refer to dynamics (1) and (3) as *neural dynamics.*

Let the matrix $T(i,j)$ be defined according to the matching algorithm i.e. it is equal to one if the feature presented at the neuron i of the X layer is similar to the feature associated with neuron j of the Y layer and equals zero otherwise. Then the strength $J_{ij}(\tau), \tau = 1, 2, \ldots, i = 1, \ldots, n_X, j = 1, \ldots, n_Y$ of the connections between the ith neuron in the layer X and the jth neuron in the layer Y are changed using the normalized Hebb learning rule

$$J_{ij}(\tau+1) = \frac{J_{ij}(\tau) + \varepsilon J_{ij}(\tau) T(i,j) Y(j,\tau) X(i,\tau)}{\sum_{j=1}^{n_X} [J_{ij}(\tau) + \varepsilon J_{ij}(\tau) T(i,j) Y(j,\tau) X(i,\tau)]}, \tag{4}$$

where

$$Y(i,\tau) = \lim_{t\to\infty} Y_t(i,\tau), \qquad X(i,\tau) = \lim_{t\to\infty} X_t(i,\tau) \tag{5}$$

are the activity values of the attractors reached on the layer Y and the layer X respectively. The formula (5) shows that the structure of the activity patterns of the neural attractors is crucial for the evolution of the synaptic weights among the X and Y layers and in this model the asymptotic values of the entries of the synaptic matrix give the matching between the stored pattern and the input pattern. The initial condition on the J_{ij} is

$$J_{ij}(0) = \frac{Tij}{\sum_{i=1}^{n_X} T_{ij}}, \quad i = 1, \ldots, n_X,\ j = 1, \ldots, n_Y.$$

Dynamics (4) is referred to as *link dynamics.*

Note that the existence of the limit in Eq. (5) is ensured by the existence of the Lyapunov function corresponding to the dynamics (1) and the dynamics (3) [7].

Now we can give the definition of the input signal $I_t^Y(j,\tau)$, $j = 1, \ldots, n_Y$, $\tau = 1, 2, \ldots$, for the layer Y,

$$I_t^Y(j,\tau) = \sum_{i=1}^{n_X} J_{ij}(\tau) X_t(i,\tau) T(i,j). \tag{6}$$

Hence in the dynamic link network, the time scale t is explained as a kind of 'short term memory' and τ is a kind of 'long term memory'. First the neurons in the layers X and Y evolve according to the self-organization mechanism (1) and (3). Then the learning procedure for $J_{ij}(s)$ is switched on using the limit configurations (5) in the formula (4).

However, one disadvantage of the above algorithm is the cost in computing time. In order to speed up the process, an algorithm called the fast dynamic link network (FDLN) is proposed in [7]. They replaced the sigmoid function by the step function and some analytical conclusions corroborated by the numerical simulations on the parameter family $(\beta, \gamma, s, I^X, I^Y)$ are obtained. In [7] the neural dynamics is substituted by the search of the blob which minimizes the Lyapunov potential of the DLN dynamics. The blobs are special configurations of the neurons such that all the neurons inside a circle have maximum positive activity while the neuron outside have minimum negative activity, such configurations are typical of the structure of the receptive fields of the retina and appear whenever any image reaches the retina. In the DLN they are used for optimizing the learning procedure of the synaptic weights. This special version of the DLN is consistent and moreover the DLN and the FDLN are compatible if the blobs really appear as fixed points of the neural dynamics and this in turn can be shown only by applying the results of our paper which are shown below. Moreover since the introduction of the blobs as a basic structure of the neural dynamics has purely biological motivations one can explore alternative learning procedures using other attractors. This search can be done only if one has some available procedure for constructing other attractors; the reasons alluded to above are the main motivations for the introduction of our algorithm. In the present paper we approximate the sigmoid function σ with the clipping function f defined by

$$f(x) = \begin{cases} 1 & \text{if } kx > 1 \\ kx & \text{if } k|x| \le 1 \\ -1 & \text{if } kx < -1 \end{cases} \quad (k, \text{ a positive number}). \tag{7}$$

Note that it is easy to transform all our results in the next two sections to the setting in which the clipping function is replaced by the function g given by

$$g(x) = \begin{cases} 1 & \text{if } k_1x > 1 \\ k_1x & \text{if } 0 \le k_1x \le 1 \\ 0 & \text{if } k_1x < 0 \end{cases} \quad (k_1, \text{ a positive number}). \tag{8}$$

The networks with a dynamics generated by the function f have been widely used in the field of neural network [8]. We have shown in another paper [3] that the dynamics with the function σ and the dynamics with the function f will have a similar behavior, at least qualitatively. In the next section, explicit relationships among the parameters $(\beta, \gamma, s, I^X, I^Y)$ are obtained from which we get a full comprehension of the role of each of them in the construction of a given attractor. Our analysis supplies a guidance for the numerical realization of the DLN and further theoretical analysis of it.

3. Relationship among (β, γ, s, I^X, I^Y)

Since all the conclusions below are true for both the X and Y layers, let us use the symbols ξ, I, n to represent either X, I^X, n_X or X, I^Y, n_Y. For the sake of simplicity we take the parameter $\alpha = 1$ and note that this parameter also does not appear in the FDLN. So now the dynamics (1) and (3) read

$$\begin{aligned} &\xi_t(i,\tau) = f(x_t(i,\tau)),\\ &\dot{x}_t(i,\tau) = -x_t(i,\tau) + \sum_{j=1}^{n} k(i,j)\xi_t(j,\tau) + I_t(i,\tau), \qquad (9)\\ &\xi_0(i,\tau) = 0. \end{aligned}$$

Discretizing (9) with time step h, without loss of generality, let $h = 1$ and we have

$$\begin{aligned} &x_{t+1}(i,\tau) = \sum_{j=1}^{n} k(i,j) f(x_t(j,\tau)) + I_t(i,\tau),\\ &x_0(i,\tau) = 0. \end{aligned} \qquad (10)$$

for $t \in N$. We can transform the dynamical system generated by the solutions $x_t(i,\tau)$ of (10) in a more suitable system by making the transformation $\eta_t(i,\tau) = f(x_t(i,\tau))$

$$\begin{aligned} &\eta_{t+1}(i,\tau) = f\left(\sum_{j=1}^{n} k(i,j)\,\eta_t(j,\tau) + I_t(i,\tau)\right),\\ &\eta_0(i,\tau) = 0, \end{aligned} \qquad (11)$$

which is the dynamics we will focus on. The trajectory of the original system is easily recovered using (10):

$$x_{t+1}(i,\tau) = \sum_{j=1}^{n} k(i,j)\eta_t(j,\tau) + I_t(i,\tau).$$

It is evident from these formulas that the attractors of the dynamical system $\eta_t(i,\tau)$ are in one-to-one correspondence with the attractors of the dynamical system $x_t(i,\tau)$.

We suppose now that $I_t(i,\tau)$ is independent of i and t and denote it as $I(\tau)$. The case of the dependence of I on i and t is discussed after Theorem 6.

As in previous papers [4, 5, 8], we only consider those fixed points which have activity values ± 1 for the dynamics (11). We call such attractors saturated according to the next definition.

Definition 1. A fixed point $w \in \{-1, 1\}^n$ of the dynamics (11) is said to be a saturated fixed point if

$$I(\tau) + \sum_{j=1}^{n} \bar{k}(i,j)w(j) \neq 0, \quad \forall i = 1, \dots, n,$$

where $\bar{k}(i,j) = \gamma p(i,j) - \beta - \delta_{ij}$.

The reason for the introduction of the $\bar{k}(i,j)$ is that the system of equations (11) coincides exactly with the evolution equations of Linsker's model [8] if we write them in the form

$$\eta_{t+1}(i,\tau) = f(\eta_t(i,\tau) + \sum_{j=1}^{n} \bar{k}(i,j)\eta_t(j,\tau) + I(i,\tau)).$$

One may think that the definition of the saturated fixed points should include those fixed points such that there exists an $i, i = 1, \ldots, n$ with the equality

$$I(\tau) + \sum_{j=1}^{n} \bar{k}(i,j)w(j) = 0. \tag{12}$$

However, if we look at the parameter space of $(I(\tau), \beta)$ (see Lemma 2 and Fig. 3) the Lebesgue measure of the set of parameters $(I(\tau), \beta)$ satisfying Eq. (12) is zero since it is the union of a finite number of lines. Hence there is no loss of generality if we consider only the saturated fixed points of Definition 1.

For $w \in \{-1, 1\}^n$, let

$$J^+(w) = \{i, w(i) = 1\}, \qquad J^-(w) = \{i, w(i) = -1\}$$

be the set of index set of neurons with excitatory and inhibitory activities. We define

$$c(w) = |J^+(w)| - |J^-(w)|$$

as the *slope function* and

$$d_1(w) := \max_{i \in J^+(w)} \left[\sum_{j \in J^-(w)} p(i,j) - \sum_{j \in J^+(w)} p(i,j) \right],$$

$$d_2(w) := \min_{i \in J^-(w)} \left[\sum_{j \in J^-(w)} p(i,j) - \sum_{j \in J^+(w)} p(i,j) \right]$$

as the two *intercept functions*. The reason why we call them slope and intercept function will be clear after the next lemma. A fixed point is called an attractor if and only if it is a stable fixed point. For a proof of the next lemma, we refer the reader to [5, Theorem 2, p. 1008].

Lemma 2. *A state $w \in \{-1, 1\}^n$ is a saturated attractor of the dynamics* (9) *if and only if*

$$\gamma d_1(w) + 1 < I(\tau) - c(w)\beta < \gamma d_2(w) - 1.$$

Thus the region of the parameter space $(I(\tau), \beta)$ which ensures that ω is an attractor of the dynamics (9) is a band between the two parallel lines

$$\gamma d_1(w) + 1 = I(\tau) - c(w)\beta$$

and

$$I(\tau) - c(w)\beta = \gamma d_2(w) - 1.$$

These two lines have an intercept with the x-axis given by $d_2(w)$ and $d_1(w)$ respectively, and the tangent of the angle with the x-axis given by $c(w)$. This is the reason why we call $c(w)$ the slope function and $d_1(w)$, $d_2(w)$ the intercept functions (Fig. 3).

Assume that $c = \sum_{j=1}^{n} p(i,j)$, we could easily calculate the function $d_2(w)$ and $d_1(w)$ if w is an on-center activity pattern with radius r. Without loss of generality, we suppose that $w(i) = 1$, if $|i| \leqslant r$ and $w(i) = -1$ if $|i| > r$, where r is an integer, the radius of the excitatory neural activities. In this setting we know from the special form of $p(i,j)$ that

$$d_2(w) = [c - 2\max_{i \in J^-(w)} \sum_{j \in J^+(w)} p(i,j)]$$

$$= [c - 2 \sum_{j \in J^+(w)} p(i^*,j)], \tag{13}$$

where $i^* = (r+1, 0)$ and similarly

$$d_1(w) = [c - 2\min_{i \in J^+(w)} \sum_{j \in J^+(w)} p(i,j)]$$

$$= [c - 2 \sum_{j \in J^+(w)} p(i_*,j)], \tag{14}$$

where $i_* = (r, r)$ (see Fig. 2).

Theorem 3. $\forall w \in \{-1, 1\}^n$, $w \neq (1, \ldots, 1), (-1, \ldots, -1)$, *a necessary and sufficient condition ensuring that there exists a nonempty set of* (β, γ, s, I) *in which* w *is a saturated attractor of the dynamics* (11) *is*

$$d_2(w) > d_1(w)$$

and

$$\gamma > \gamma_0 := \frac{2}{d_2(w) - d_1(w)} .$$

Furthermore the larger the γ, *the bigger the parameter region ensuring that* w *is a saturated attractor of the dynamics* (11).

Proof. We know from Lemma 2 that there exists a set of parameters such that w is a saturated attractor of the dynamics (11) if and only if

$$\gamma d_2(w) - 1 > \gamma d_1(w) + 1$$

which implies the first conclusion of the theorem. Furthermore, since the bigger the γ, the wider the band between the lines

$$I(\tau) - c(w)\beta = \gamma d_1(w) + 1$$

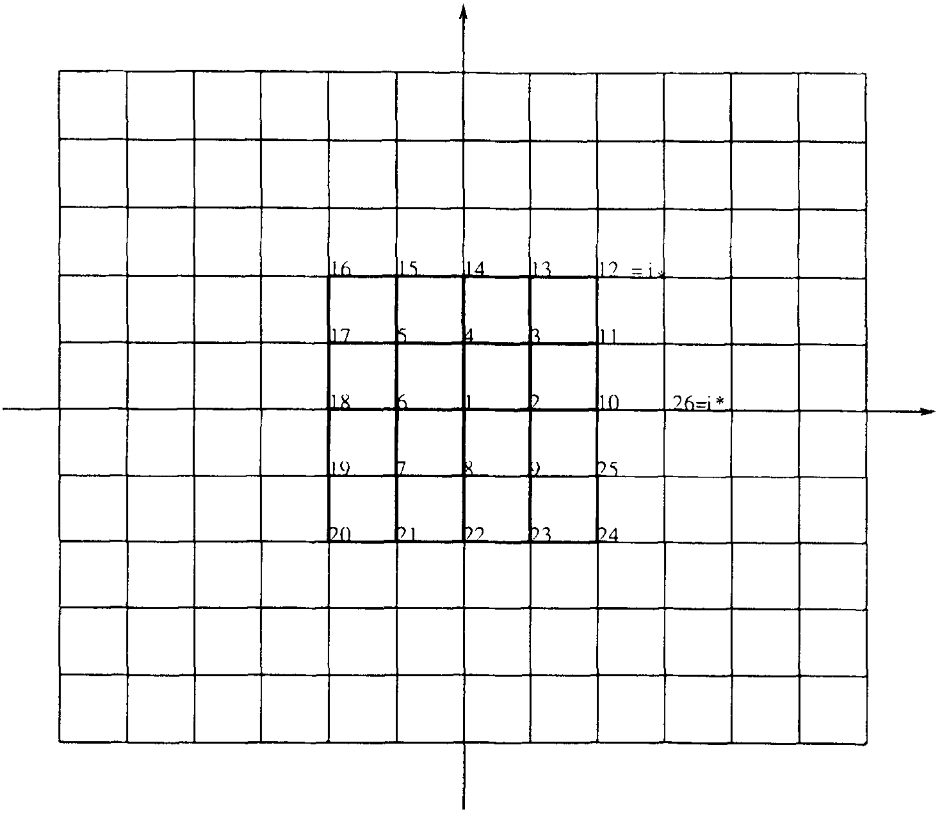

Fig. 2. A typical on-center neuron activity pattern. In Fig. 2, $w(i) = 1, i = 1, \ldots, 25$ and $w(i) = -1, i > 25$. $i_* = (2, 2) = 12$ and $i^* = (3, 0) = 26$.

and

$$I(\tau) - c(w)\beta = \gamma d_2(w) - 1,$$

we arrive at the second conclusion. ☐

For an explanation of Theorem 3, we refer the reader to Fig. 3.

Theorem 4. $\forall w \in \{-1, 1\}^n$, $w \neq (1, \ldots, 1)$, $(-1, \ldots, -1)$ *there exists a positive number* β_0 *such that when* β *is in the set*

$$\{\beta, \beta \geqslant \beta_0\} \cap \{\beta, \gamma d_1(w) + 1 < I(\tau) - c(w)\beta < \gamma d_2(w) - 1\},$$

w *is a saturated attractor of the dynamics* (11) *and* $(1, \ldots, 1)$, $(-1, \ldots, -1)$ *will no longer be attractors of the dynamics* (11).

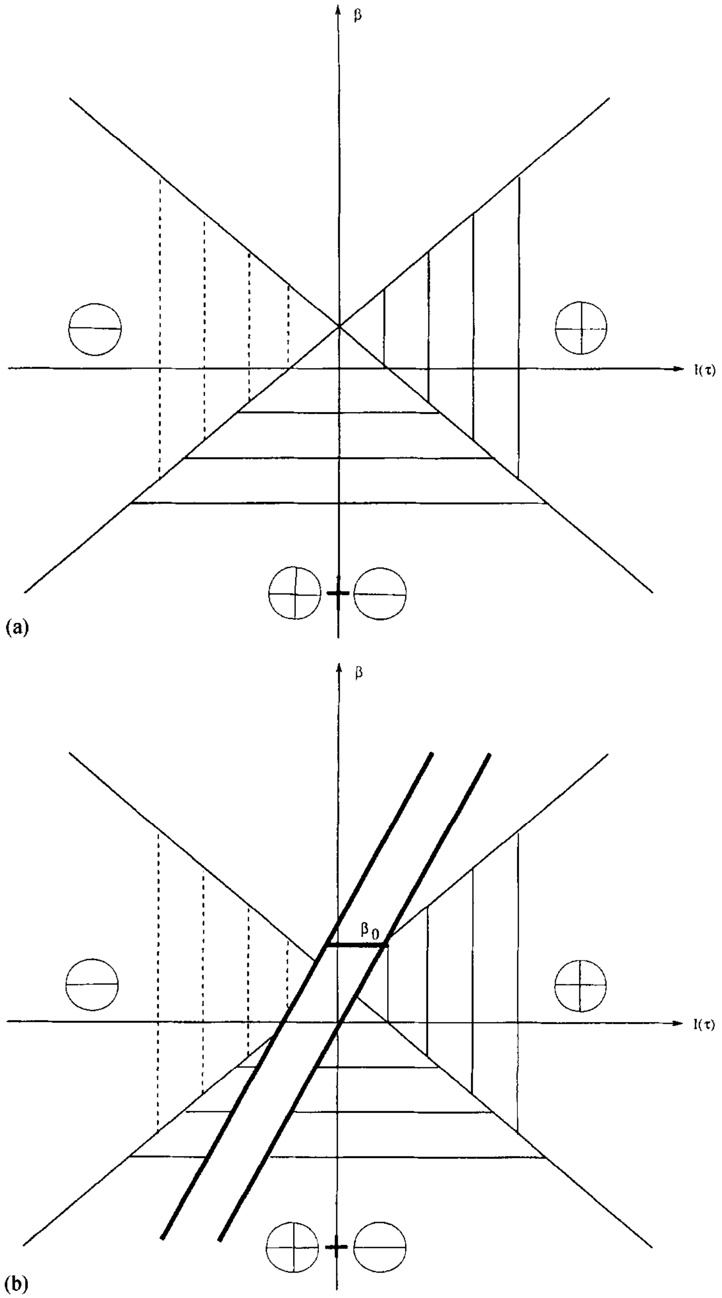

Fig. 3. (a) The whole space of $(I(\tau), \beta)$ is divided into four parts. ⊕ is the parameter region with all excitatory neuron activities; ⊖ is the parameter region with all inhibitory neuron activities; ⊕ + ⊖ the parameter region with both all excitatory and all inhibitory neuron activities. (b) The parameter region, which ensures that $w \neq (1, \ldots, 1), (-1, \ldots, -1)$ is an attractor of the dynamics (11), is a band between two parallel lines (thick lines). β_0 is showed in the figure as $c(w) < 0$. Note that as $\beta > \beta_0$ and $\gamma d_1(w) + 1 < I(\tau) - c(w)\beta < \gamma d_2(w) - 1$, $(1, \ldots, 1)$ and $(-1, \ldots, -1)$ will no longer be attractors of the dynamics (11). The width of the band between the thick lines is $\gamma(d_2(w) - d_1(w)) - 2$.

Proof. For $w_1 = (1, \ldots, 1)$, we have that $c(w_1) = n$ and $d_1(w_1) = -c$. Thus in terms of Lemma 2, we deduce that in the region

$$I(\tau) - n\beta > -\gamma c + 1,$$

the configuration $w_1 = (1, \ldots, 1)$ is an attractor of the dynamics (11). Similarly we also have that in the region

$$I(\tau) + n\beta < \gamma c - 1,$$

the configuration $w_2 = (-1, \ldots, -1)$ is an attractor of the dynamics (11). For fixed γ, w, denote $\beta_1, \beta_2, \beta_3$ and β_4 as the solutions of the following four equations:

$$I(\tau) - n\beta = -\gamma c + 1,$$

$$I(\tau) - c(w)\beta = \gamma d_1(w) + 1,$$

$$I(\tau) - n\beta = -\gamma c + 1,$$

$$I(\tau) - c(w)\beta = \gamma d_2(w) - 1,$$

$$I(\tau) + n\beta = \gamma c - 1,$$

$$I(\tau) - c(w)\beta = \gamma d_1(w) + 1,$$

$$I(\tau) + n\beta = \gamma c - 1,$$

$$I(\tau) - c(w)\beta = \gamma d_2(w) - 1,$$

respectively. Then we could choose $\beta_0 = \max(\beta_1, \beta_2, \beta_3, \beta_4)$, and one obtains the conclusion (see also Fig. 3). In fact, we have

$$\beta_0 = \max\left(\frac{\gamma c - \gamma d_1(w) - 2}{n + c(w)}, \frac{\gamma c + \gamma d_2(w) - 2}{n - c(w)}\right)$$

$$= \max\left(\frac{\gamma(\sum_{j=1}^{n} p(i,j) - d_1(w)) - 2}{n + c(w)}, \frac{\gamma(\sum_{j=1}^{n} p(i,j) + d_2(w)) - 2}{n - c(w)}\right). \qquad \square$$

We could define a Lyapunov function for the dynamics (11)

$$H(w) = -\frac{1}{2}\sum_{i,j}^{n} \bar{k}(i,j)\, w(i)\, w(j) + \sum_{i} \int_{0}^{w(i)} f^{-1}(z)\, dz - \sum_{i} I(\tau)\, w(i) \qquad (15)$$

as in [7, p. 14] or [2] if $\bar{k}(i,i) \geqslant 0$, $i = 1, \ldots, n$, $w \in [-1, 1]^n$. It is readily seen that $(1, \ldots, 1)$ is the global minimum of the dynamics if the input $I(\tau) > 0$ and $\beta \leqslant 0$, or $(-1, \ldots, -1)$ is the global minimum of the dynamics if the input $I(\tau) < 0$ and $\beta \leqslant 0$. So $(1, \ldots, 1)$ or $(-1, \ldots, -1)$ will dominate the behavior of the dynamics in the sense that if the input is contaminated by the noise, the neural configuration will converge to the global minima with large probability as in the simulated annealing. However, the lateral inhibition $\beta > \beta_0$($|\beta_0|$ is small usually) guarantees that some nontrivial activity patterns (not all neuron activities are excitatory or inhibitory) are attractors of the neural dynamics since now the parameters are outside of the parameter region where $(1, \ldots, 1)$ or $(-1, \ldots, -1)$ is the attractor of the dynamics (11) (Fig. 3).

Similar dynamics have been discussed in Linsker's network before. But only now we realize *the role that lateral inhibition plays in the neural model.* The lateral inhibition pulls the dynamics outside of the region dominated by the attractors $(1, \dots, 1)$ or $(-1, \dots, -1)$. These models are claimed to be a good approximation to the biological system. Here we supply an argument that explains why lateral inhibition is necessary in the natural biological network.

Theorem 5. (1) *If s is large enough so that $p(i,j)$, $i,j = 1, \dots, n$ are constants independent of i,j, then* $(1, \dots, 1)$ *and* $(-1, \dots, -1)$ *are the only possible saturated attractors of the dynamics* (11).

(2) *If s is small enough so that $p(i,j) = \delta_{ij}$ with $\gamma > 1$, then any state $w \in \{-1, 1\}^n$ is an attractor of the dynamics* (11).

Proof. (1) In this case, $d_2(w) = d_1(w)$, from Theorem 3 we know that any $w \in \{-1, 1\}^n$ will not be a saturated attractor of the dynamics (11) if $w \neq (1, \dots, 1)$, $(-1, \dots, -1)$. Moreover $(1, \dots, 1)$ is a saturated attractor of the dynamics (11) if

$$I(\tau) - n\beta > -\gamma c + 1,$$

and $(-1, \dots, -1)$ is a saturated attractor of the dynamics (11) if

$$I(\tau) + n\beta < \gamma c - 1,$$

(2) In this setting, $d_2(w) = \gamma - 1 > d_1(w) = -\gamma + 1$ for any $w \in \{-1, 1\}^n$ so we prove the conclusion by Lemma 2. □

Theorem 5 tells us that the range of the parameter s of the excitatory connection controls the correlation length of activity patterns. As s is small, the activity of each neuron could develop independently, so any activity pattern could be a saturated attractor of the dynamics. As $s \to \infty$, the activity of each neuron is highly correlated, only all excitatory and all inhibitory attractors are saturated attractors of the dynamics.

As a simple corollary of Lemma 2, we have the following theorem.

Theorem 6. *w is a saturated attractor of the dynamics* (11) *if and only if*

$$I(\tau) \in [\gamma d_1(w) + c(w)\beta + 1, \gamma d_2(w) + c(w)\beta - 1].$$

Theorem 6 gives an exact region for the fluctuation of the input signal. If $I(\tau)$ belongs to the region $[\gamma d_1(w) + c(w)\beta + 1, \gamma d_2(w) + c(w)\beta - 1]$, w will be an attractor of the dynamics (11) (Fig. 3). The interval in which $I(\tau)$ changes can be taken as an estimate for an effective interval in the case when the input signal is not translation invariant and depends on the time t of the neural dynamics.

From all the above arguments we now can give a useful way of choosing the four parameters β, γ, s, I^X. For any on-center activity pattern of radius r which is used in the FDLN and numerically founded in the simulation of the dynamics (1) and (3), the slope function

$$c(w) = 2(2r + 1)^2 - n$$

and the two intercept functions $d_2(w)$, $d_1(w)$ are obtained by (13) and (14). Let

$$\gamma > \gamma_0 = \frac{2}{d_2(w) - d_1(w)}$$

be fixed which is independent of the size n_X. For a given $s (s = 1, \ldots, 10)$, γ_0 as a function of r the radius of the on-center pattern, is given in Table 1. Then we easily find β_0 as in the proof of Theorem 3, i.e.

$$\beta_0 = \max\left(\frac{\gamma(\sum_{j=1}^{n} p(i,j) - d_1(w)) - 2}{n + c(w)}, \frac{\gamma(\sum_{j=1}^{n} p(i,j) + d_2(w)) - 2}{n - c(w)}\right)$$

Without loss of generality, set $\beta = \beta_0$. Now we have only one parameter I^X free, which is determined by the relation

$$\gamma d_1(w) + c(w)\beta + 1 < I^X(\tau) < \gamma d_2(w) + c(w)\beta - 1.$$

Note that based upon a similar principle, we could find the region of parameters $(\beta, \gamma, s, I^X, I^Y)$ in which a desired, constructed and biological oriented pattern is an attractor of the dynamics (11).

4. A discrete version of the DLN

Lemma 7. *Let Ω be the set of all saturated attractors of the dynamics* (11), *then*

$$\Omega = \left\{ w \in \{-1, 1\}^n, w(i)\,[I(\tau) + \sum_{j=1}^{n} \bar{k}(i,j)\, w(j)] > 0, \forall i = 1, \ldots, n \right\}.$$

The proof of the lemma is contained in [5, Theorem 1 p. 1006].

Table 1
$\gamma_0(r)$ is shown for $r = 1, \ldots, 10$

s	1	2	3	4	5	6	7	8	9	10
$\gamma_0(1)$	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
$\gamma_0(2)$	0.7	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
$\gamma_0(4)$	1.8	0.6	0.3	0.2	0.2	0.2	0.1	0.1	0.1	0.1
$\gamma_0(5)$	2.7	0.9	0.4	0.3	0.2	0.2	0.1	0.1	0.1	0.1
$\gamma_0(6)$	3.8	1.2	0.6	0.3	0.2	0.2	0.1	0.1	0.1	0.1
$\gamma_0(7)$	5.1	1.6	0.7	0.4	0.3	0.2	0.1	0.1	0.1	0.1
$\gamma_0(8)$	6.6	2.0	0.9	0.5	0.3	0.2	0.2	0.1	0.1	0.1
$\gamma_0(9)$	8.3	2.4	1.0	0.6	0.4	0.3	0.2	0.1	0.1	0.1
$\gamma_0(10)$	10.2	3.0	1.3	0.7	0.4	0.3	0.2	0.2	0.1	0.1

Note that the set of all saturated attractors of the dynamics coincides with that of the dynamics used in the Hopfield model [6], which is defined by

$$\xi_{t+1}(i,\tau) = \mathrm{sgn}(I_t(i,\tau) + \sum_{j=1}^{n} \bar{k}(i,j)\,\xi_t(j)), \quad i = 1, \ldots, n,\ t = 1, \ldots . \tag{16}$$

This implies that the self-organization dynamics (11) is a procedure of retrieving 'memory'. The 'memory' (i.e. a structured neuron activity pattern) is already stored in the matrix $K = (k(i,j), i, j = 1, \ldots, n)$.

Based on the above observation, the dynamics (1) and (3) could be now replaced by

$$\xi^X_{t+1}(i,\tau) = \mathrm{sgn}\left(I^X_t(i,\tau) + \sum_{j=1}^{n_X} \bar{k}(i,j)\,\xi^X_t(j,\tau)\right), \quad i = 1, \ldots, n_X \tag{17}$$

and

$$\xi^Y_{t+1}(i,\tau) = \mathrm{sgn}\left(I^Y_t(i,\tau) + \sum_{j=1}^{n_Y} \bar{k}(i,j)\,\xi^Y_t(j,\tau)\right), \quad i = 1, \ldots, n_Y, \tag{18}$$

respectively, for $t = 1, \ldots$. The advantages of the above dynamics are as follows:

(i) It is faster of course compared to the original dynamics (1) and (3) since dynamics (1) and (3) originates an evolution in all the cubes $[-1,1]^{n_X}$ or $[-1,1]^{n_Y}$ while our dynamics (17) and (18) jumps from one vertex of the cube to another until it reaches a fixed point.

(ii) Both the FDLN and the discrete version proposed here are very fast as shown in [7] and in our numerical simulation. In our numerical simulation of dynamics (16) the system starts from 20 randomly generated configurations. We find the same result as that reported in [1] for the Hopfield model: in spite of the fact that the number of neurons is very large, the state converges very quickly to an equilibrium configuration. These results are shown in the Table 2.

In the FDLN we have to artificially select a constant l first and then compute the energy $V(i)$, which is defined by

$$V(i) = -\sum_{j=1}^{n_Y} b(j-i) I^Y(\tau, j),$$

Table 2

Number gives the total number of neurons, i.e. number = $n \times n$. The time for computing the minimum of $V(i)$ is of the same order of magnitude as in the FDLN algorithm. AUT is the average updating time for the dynamics (16) to reach an equilibrium. It is given by the average number of steps performed during the evolution when starting from 20 random initial configurations. The parameters are $\gamma = 1.3$, $\beta = 0.1$, $s = 2.0$, $I = -0.2$

Number	121	169	225	289	361	441
AUT	199/20	334/20	331/20	330/20	471/20	508/20

where

$$b(i) = \begin{cases} 1 & \text{if } \|i\| < \frac{l}{2} \\ 0 & \text{else} \end{cases}$$

for $i = 1, \ldots, n_Y$ in order to find the minima of $V(i)$. The advantage of the discrete version of the DLN is that first, it has the same set of the fixed points as that of the corresponding dynamics in the DLN which is claimed to be biologically oriented [9]; second, as we all know, the dynamics (16) is easily realized on a parallel computer.

(iii) We could easily modify dynamics (16) so that it converges to the global minima of the energy. Since there are many attractors for dynamics (11) in general the possibility of choosing among them is clearly an advantage and this makes the Hopfield model interesting in optimization problems. In fact, in the past decades the Hopfield model has been widely used for this aim.

5. Concluding remarks

We have proposed a novel approach to analyze the DLN. With the use of this approach it is possible to choose the set of parameters in order to have the neural dynamics converge to a desired point attractor, which allows us to find attractors which make the synaptic dynamics on the J_{ij} to be optimal with respect to the pattern recognition problem as it follows from the considerations of this paper. Another consequence of the results of this paper is that the sole effect of every parameter in the dynamics is comprehensively understood. The lateral inhibitory parameter β pulls the dynamics out of the parameter region dominated by exclusively all excitatory or all inhibitory activities. γ plays the role of enlarging the parameter region. s reflects the correlation of activity pattern: the larger the s, the more homogeneous the activity patterns which develop. If w is an attractor of dynamics an exact fluctuation region of the input is given ensuring that w is still an attractor of dynamics. Based upon an observation in Section 3, we have developed a discrete version of the DLN.

There remain a number of interesting theoretical problems to be studied further. For example, there are extensive investigations on the number of attractors of the Hopfield model. Here, for our special matrix K, how many attractors exist for the dynamics (16)? The on-center neuron activity pattern appears between the second and the third layer of Linsker's network. If the oriented neuron activities pattern, observed in the further layer of Linsker's network and also in biological networks, appears at layer X (this could be easily realized if we replace the matrix K by the matrix Q given in [5, 8]), will the efficiency of the DLN be improved?

Acknowledgements

We would like to thank David Brown and the referees for useful comments on the manuscript.

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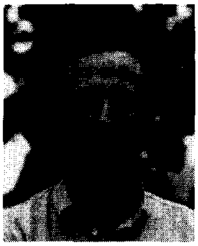

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