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Metastability of exponentially perturbed Markov chains*

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Abstract A family of irreducible Markov chains on a finite state space is considered as an exponential perturbation of a reducible Markov chain. This is a generalization of the Freidlin-Wentzell theory, motivated by studies of stochastic Ising models, neural network and simulated annealing. It is shown that the metastability is a universal feature for this wide class of Markov chains. The metastable states are simply those recurrent states of the reducible Markov chain. Higher level attractors, related attractive basins and their pyramidal structure are analysed. The logarithmic asymptotics of the hitting time of various sets are estimated. The hitting time over its mean converges in law to the unit exponential distribution.

Keywords: exponential perturbation, metastability, higher level attractive basin, pyramidal structure.

The problems discussed in this paper are originated from the following sources: i) the metastable behavior observed in stochastic Ising models at very low temperature[1-3]; ii) breaking down of ergodicity of neural networks[4-6]; iii) simulated annealing techniques[7,8]. They seem to be quite different, but their underlying mathematical structures are closely related to each other. In this article, we try to explore them in a unified way, inspired by the theory of Freidlin-Wentzell[9].

Let the state space $S=\{\xi_1,\cdots,\xi_N\}$ be finite. The family of Markov chains on S with transition matrices $\{\boldsymbol{P}^\beta;\beta\in[0,\infty]\}$ are assumed to satisfy the following assumptions.

$$\lim_{\beta\to\infty} p^\beta(\xi,\eta)=p^\infty(\xi,\eta),\ \ \forall\xi,\eta\in S. \tag{0.1}$$

Assume that the following limit exists:

$$C_{\xi\eta}=\lim_{\beta\to\infty}-\frac{1}{\beta}\log p^\beta(\xi,\eta) \tag{0.2}$$

with the convention that $\log 0=-\infty$. Assume that

$$C_{\xi\eta}>0 \text{ if } p^\infty(\xi,\eta)=0. \tag{0.3}$$

In other words, the family of Markov chains with transition probability matrices $\{\boldsymbol{P}^\beta;\beta\in[0,\infty)\}$ is an exponential perturbation of the Markov chain with transition probability matrix $\boldsymbol{P}^\infty$. This is just a generalization of stochastic perturbations of ref. [9]. For the

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Glauber dynamics of stochastic Ising model[3] and the Little's model of neural network[4], (0.1)—(0.3) hold and furthermore

$$\lim_{\beta\to\infty}\frac{p(\xi,\eta)}{e^{-\beta C_{\xi\eta}}}\quad \text{exists for } \forall \xi,\eta\in S. \tag{0.4}$$

We only need (0.4) for Theorem 6 of §3. It is assumed in ref. [9] that $\boldsymbol{P}^{\infty}$ is degenerated in the sense that $p^{\infty}(\xi,\eta)=0$ or 1. We shall relax this restriction here.

For a subset $K\subset S$, the exit time of Markov chain $\{X_n\}$ is denoted by

$$\tau(K)=\inf\{n;\, X_n\notin K\}, \tag{0.5}$$

and the hitting time by

$$\sigma(K)=\inf\{n;\, X_n\in K\}.$$

Let $\{A_1,\cdots,A_s\}$ be the set of all recurrent classes of $\boldsymbol{P}^{\infty}$. Without loss of generality we assume that $\boldsymbol{P}^{\infty}_{A_i}$'s, the transition probability matrices restricted to each recurrent class A_i, are all nonperiodic. Define for each A_i (which corresponds to an attractor of dynamical systems) the attractive basin B_i to be

$$B_i=\{\xi\,|\,P^{\infty}(\sigma(A_i)<\infty\,|\,X_0=\xi)>0\}. \tag{0.6}$$

Certainly $\bigcup\limits_i B_i=S$. For the dynamical systems considered in ref. [9], $B_i\bigcap B_j=\varnothing$ for $i\neq j$. However, it is well possible that $B_i\bigcap B_j\neq\varnothing$ for the stochastic version we are considering now. We shall study the behavior of Markov chain with very large β, e.g. the exit time, exit distribution and ergodicity, etc. Much information is given by $C_{\xi\eta}$'s of (0.2). Define

$$T(B_i)=\min\left\{\sum_{k=1}^{l}C_{\xi_{k-1}\xi_k}\Big|\, l\geqslant 1,\ \xi_0\in A_i,\ \xi_1,\ \xi_2,\cdots,\ \xi_{l-1}\in B_i,\ \xi_l\notin B_i\right\} \tag{0.7}$$

and for $\xi\in B_i,\eta\notin B_i$, $T_{\xi\eta}(B_i)$ is a similar quantity (see Definition 2.1). The following theorems are precise statements.

Theorem 0.1. *Suppose that* $\{X_n\}$ *is a Markov chain starting from* $\xi\in B_i$.

i) *Exit time*

$$\lim_{\beta\to\infty}\frac{1}{\beta}\log E^{\beta}_{\xi}\tau(B_i)=T(B_i). \tag{0.8}$$

ii) *Exit distribution: For* $\eta\notin B_i$,

$$\lim_{\beta\to\infty}\frac{-1}{\beta}\log P^{\beta}(X_{\tau(B_i)}=\eta\,|\,X_0=\xi)=-T(B_i)+T_{\xi\eta}(B_i), \tag{0.9}$$

$$\lim_{\beta\to\infty}P^{\beta}_{\xi}(X_{\tau(B_i)}\in\{\eta\notin B_i\,|\,T_{\xi\eta}(B_i)=T(B_i)\})=1. \tag{0.10}$$

Theorem 0.2. *If the initial state* $\xi\in B_i\backslash\bigcup\limits_{j\neq i}B_j$, *then* $\tau(B_i)/E^{\beta}_{\xi}\tau(B_i)$ *converges in distribution to the exponential random variable with mean* 1, *as* $\beta\to\infty$. *In particular, for* $\delta>0$ *small* *enough, we have*

$$\lim_{\beta\to\infty} P_\xi^\beta\left(e^{(T(B_i)-\delta)\beta}<\tau(B_i)<e^{(T(B_i)+\delta)\beta}\right)=1, \tag{0.11}$$

$$\lim_{\beta\to\infty} \frac{-1}{\beta}\log P_\xi^\beta(\sigma(\zeta)>e^{\delta\beta})\geqslant\delta,\ \forall\zeta\in A_i. \tag{0.12}$$

Theorem 0.3. *Suppose that* $\{v_i(\xi)\}$ *is the invariant measure of* $P_{A_i}^{\infty}$. *For* $\xi\in A_i$, $\xi\in B_i\setminus\bigcup_{j\neq i}B_j$, $0<\delta<T(B_i)$

$$\lim_{\beta\to\infty} E_\xi^\beta\left[\frac{1}{N_\beta}\sum_{k=1}^{N_\beta}I_{\{\zeta\}}(X_k)-v_i(\zeta)\right]^2=0,$$

where I *is the indicator function and* N_β *is the integral part of* $e^{(T(B_i)-\delta)\beta}$.

The phenomena characterized by the theorems are collectively called metastability. Our second part is to demonstrate that this kind of metastability is also observed at higher levels. If we do not consider the actual state but only the set B_i in which X_n lies, then transition from one B_i to another B_j also exhibits the metastability. Some B_i's are grouped together to form level 2 attractive basins B_j^2, $j=1, 2,\cdots s_2$. X_n stays in B_k^2 for an asymptotically expontial random time, and then enters into other B_j^2, $j\neq k$ with a definite hitting probability. Repeating this argument for higher levels, one gets a pyramidal of classes of states, with $\{B_1,\cdots B_s\}$ renamed as $\{B_1^1,\cdots B_s^1\}$ at the bottom, and $B_1^r=S$ on the top, as shown below.

$$\begin{array}{ccccccc}
 & & & B_1^r & & & \\
 & & B_1^{r-1} & \cdots & B_{s_{r-1}}^{r-1} & & \\
 & \cdots & & \cdots & & \cdots & \\
B_1^2 & B_2^2 & & \cdots & \cdots & & B_{s_2}^2 \\
B_1^1\cdots B_{n_1}^1 & B_{n_1+1}^1\cdots B_{n_2}^1 & \cdots & \cdots & \cdots & \cdots & B_s^1.
\end{array}$$

Roughly speaking, B_i^k is the union of some B_j^{k-1}'s. The precise definitions and rigorous mathematical statements (Theorems 3.1, 3.2 and 3.3 concerning higher level attractive basins B_i^k's require much preparation). We shall discuss the details in section 3.

As an example, let us consider the d-dimensional stochastic Ising model with the Glauber dynamics. At the top level $B_1^r=S$. The configuration with all spins up $(+\underline{1})$ is the unique attractor of the highest level. On the second top level there are two attractors, $+\underline{1}$ and the configuration with all spins down $(-\underline{1})$. The corresponding attractive basin of $-\underline{1}$ is exactly the collection of subcritical configurations. From the physical point of view, it is important to find the so-called critical droplet from the interacting potential which determines $C_{\xi\eta}$ in (0.2). By the theorems of this paper, the problem is essentially reduced to finding a path connecting $-\underline{1}$ to $+\underline{1}$ with minimum barrier. The details are discussed in references [10, 11].

1 Preliminaries

We now discuss a Markov chain on the finite state space S. The results are very similar to Lemmas 3.3 and 3.4 of Chapter 6 in ref. [9], but are derived directly by an elementary method of determinants. They are not only the stepping stone for our main results, but also of independent interest.

Throughout this section we will identify the state space S with $\{1, 2, \cdots, N\}$.

$$\boldsymbol{P}=\{p(i, j): 1 \leqslant i, j \leqslant N\}$$

is a stochastic matrix. This means that every $p(i, j) \geqslant 0$ and $\sum_{j=1}^{N} p(i, j)=1$ for $1 \leqslant i \leqslant N$. Let K be a proper subset of S.

Definition 1.1. $G(K)$ is the set of maps $g: K \rightarrow S$ with the property that g maps no non-empty subset of K into itself.

Let $p \in K$ and $q \in S \backslash K$. We say that $g \in G(K)$ leads from p to q if there is a sequence $x_1, \cdots, x_k$ of distinct elements in K such that

$$g(p)=x_1, \ g(x_k)=q, \text{ and } g(x_j)=x_{j+1}, \ 1 \leqslant j \leqslant k-1.$$

Definition 1.2. $G_{pq}(K)$ is the subset of $g \in G(K)$ that leads from p to q.

Definition 1.3. For $g: K \rightarrow S$, define

$$\pi(g)=\prod_{x \in K} p(x, g(x)). \tag{1.1}$$

In particular $\pi(g)$ is defined by (1.1) for $g \in G(K)$.

Definition 1.4. Let $K=\{i_1, \cdots, i_s\}$ with $1 \leqslant i_1<\cdots<i_s \leqslant N$. Then

$$\boldsymbol{P}_K=\{p(i_k, i_l); 1 \leqslant k, \ l \leqslant s\}.$$

By relabelling the elements of S we can always arrange matters so that $K=\{1, \cdots, s\}$ with $1 \leqslant s \leqslant N-1$. From now on we take it for granted that we have done this. In the expression $\boldsymbol{I}-\boldsymbol{P}_K$, the symbol $\boldsymbol{I}$ is understood to represent the identity matrix with the same dimensions as $\boldsymbol{P}_K$.

Definition 1.5. For $p \in K$ and $q \in S \backslash K$ define the matrix $D_{pq}(K)$ by deleting the p-th column of $\boldsymbol{I}-\boldsymbol{P}_K$ and adjoining at the extreme right the column

$$\begin{pmatrix} -p(1,q) \\ -p(2,q) \\ \vdots \\ -p(s,q) \end{pmatrix}.$$

Also we can arrange matters so that $q=s+1$ and we take this for granted below.

We now state three lemmas. Their proof appears in the Appendix.

Lemma 1.1.

$$\det(\boldsymbol{I}-\boldsymbol{P}_K)=\sum_{g\in G(K)}\pi(g). \tag{1.2}$$

For $p\in K$, and $q\in S\backslash K$,

$$\det D_{pq}(K)=(-1)^{p+q}\sum_{g\in G_{pq}(K)}\pi(g). \tag{1.3}$$

Lemma 1.2 (Formula for the Hitting Probability). *For $p\in K$ and $q\in S\backslash K$,*

$$P(X_{\tau(K)}=q|X_0=p)=\frac{(-1)^{p+q}\det D_{pq}(K)}{\det(\boldsymbol{I}-\boldsymbol{P}_K)}=\frac{\sum\limits_{g\in G_{pq}(K)}\pi(g)}{\sum\limits_{g\in G(K)}\pi(g)}\ . \tag{1.4}$$

Lemma 1.3 (Formula for Mean Exit Time).

$$E_p\tau(K)=\frac{\sum\limits_{g\in G(K\backslash\{p\})}\pi(g)+\sum\limits_{\substack{j=1\\ j\neq p}}^{s}\ \sum\limits_{g\in G_{pj}(K\backslash\{j\})}\pi(g)}{\sum\limits_{g\in G(K)}\pi(g)}\ . \tag{1.5}$$

2 The first level metastability

Definition 2.1. For $g\in G(k)$ define

$$\Pi(g)=\lim_{\beta\to\infty}\frac{-1}{\beta}\log\pi^{\beta}(g), \tag{2.1}$$

and

$$T(K)=\min_{g\in G(K)}\Pi(g),\ \ T_{\xi\eta}(K)=\min_{g\in G_{\xi\eta}(K)}\Pi(g). \tag{2.2}$$

Under assumption (0.2), we have

$$\Pi(g)=\sum_{\xi\in K}C_{\xi\,g(\xi)},$$

and

$$T(K)=\min_{g\in G(K)}\sum_{\xi\in K}C_{\xi,g(\xi)},\ \ T_{\xi\eta}(K)=\min_{g\in G_{\xi\eta}(K)}\sum_{\xi\in K}C_{\xi\,g(\xi)}.$$

Lemma 2.1. *Assume $K\subset S$, $\xi\in K$, $\eta\notin K$.*

$$\lim_{\beta\to\infty}\ \frac{-1}{\beta}\log P(X_{\tau(K)}=\eta|X_0=\xi)=T_{\xi\eta}(K)-T(K), \tag{2.3}$$

$$\lim_{\beta\to\infty}\ \frac{1}{\beta}\log E_{\xi}\tau(K)=-[T(K\backslash\{\xi\})\wedge\min_{\substack{\zeta\in K\\ \zeta\neq\xi}}\ T_{\xi\zeta}(K\backslash\{\zeta\})]+T(K). \tag{2.4}$$

Proof. By Lemma 1.2,

$$\begin{aligned}
&\lim_{\beta\to\infty}\frac{-1}{\beta}\log P(X_{\tau(K)}=\eta|X_0=\xi)\\
&=\lim_{\beta\to\infty}\frac{-1}{\beta}\left[\log\sum_{g\in G_{\xi\eta}(K)}\pi(g)-\log\sum_{g\in G(K)}\pi(g)\right]\\
&=-\lim_{\beta\to\infty}\frac{1}{\beta}\log\sum_{g\in G_{\xi,\eta}(K)}\mathrm{e}^{-\beta\Pi(g)}+\lim_{\beta\to\infty}\frac{1}{\beta}\log\sum_{g\in G(K)}\mathrm{e}^{-\beta\Pi(g)}\\
&=\min_{g\in G_{\xi\eta}(K)}\Pi(g)-\min_{g\in G(K)}\Pi(g)\\
&=T_{\xi\eta}(K)-T(K).
\end{aligned}$$

Similarly, by Lemma 1.3, we have

$$\begin{aligned}
&\lim_{\beta\to\infty}\frac{1}{\beta}\log E_\xi\tau(K)\\
&=\lim_{\beta\to\infty}\frac{1}{\beta}\log\left[\sum_{g\in K\setminus\{\xi\}}\pi(g)+\sum_{\substack{\zeta\in K\\ \zeta\neq\xi}}\sum_{g\in G_{\xi\zeta}(K\setminus\{\zeta\})}\pi(g)\right]-\lim_{\beta\to\infty}\frac{1}{\beta}\log\left[\sum_{g\in G(K)}\pi(g)\right]\\
&=\lim_{\beta\to\infty}\frac{1}{\beta}\log\left[\sum_{g\in K\setminus\{\xi\}}\mathrm{e}^{-\beta\Pi(g)}+\sum_{\substack{\zeta\in K\\ \zeta\neq\xi}}\sum_{g\in G_{\xi\zeta}(K\setminus\{\zeta\})}\mathrm{e}^{-\beta\Pi(g)}\right]-\lim_{\beta\to\infty}\frac{1}{\beta}\log\left[\sum_{g\in G(K)}\mathrm{e}^{-\beta\Pi(g)}\right]\\
&=-\left[\min_{g\in K\setminus\{\xi\}}\Pi(g)\wedge\min_{\substack{\zeta\in K\\ \zeta\neq\xi}}\min_{g\in G_{\xi\zeta}(K\setminus\{\zeta\})}\Pi(g)\right]+\min_{g\in G(K)}\Pi(g)\\
&=-\left[T(K\setminus\{\xi\})\wedge\min_{\substack{\zeta\in K\\ \zeta\neq\xi}}T_{\xi\zeta}(K\setminus\{\zeta\})\right]+T(K).\qquad\text{Q.E.D.}
\end{aligned}$$

Lemma 2.2. *Take a recurrent class A_i with its attractive basin B_i. Assume $\zeta\in A_i,\zeta\in B_i$. Then $T_{\xi\zeta}(B_i\setminus\{\xi\})=0$, and*

$$\min_{\eta\notin B_i}T_{\xi\eta}(B_i)=\min_{\eta\notin B_i}\left\{\sum_{K=1}^{l}C_{\xi_{K-1}\xi_K}\middle|\, l\geqslant 1,\ \xi_0=\zeta,\ \xi_1,\ \xi_2,\cdots,\ \xi_{l-1}\in B_i,\ \xi_l=\eta\right\}$$

which is independent of ξ.

Because of this lemma, $T(B_i)$ given by (0.7) is consistent with Definition 2.1.

Proof. The proof is a direct verification of Definition 2.1. For any $\xi\in B_i$, $\zeta\in A_i$ there exists a sequence $\{\xi_k,\ k=0,\ 1,\cdots,n\}$ leading from ξ to $\zeta\in A_i$ such that $\prod_k p^\infty(\xi_k,\xi_{k+1})>0$. Therefore $\sum_k C_{\xi_k\xi_{k+1}}=0$. Define $g(\xi_k)=\xi_{k+1}$. If $\xi'\notin B_i\setminus\{\xi_0,\xi_1,\cdots\xi_n\}$, we can again find a sequence $\{\xi'_i,\ i=0,1,\cdots,n'\}\subset B_i$ leading from ξ' to ζ. Suppose $j=\min\{i:\xi'_i\in\{\xi_0,\xi_1,\cdots,\xi_n\}\}$. Then define $g(\xi_k')=\xi_{k+1}'$ for $k=0,1,\cdots,j-1$. In this way we extend g to a larger domain of definition such that (1) $\Pi(g)=0$ and (2) g leads ξ to ζ. Since B_i is finite, after a finite number of extensions we get $g\in G_{\xi\zeta}(B_i\backslash\zeta)$ with the two properties preserved. So the first assertion follows.

Turn to the second equality. Take a sequence $\{\xi_j, j=0,1,\cdots,n\}$ leading from ζ to $\eta\notin B_i$ which minimizes $\sum c_{\xi_j\xi_{j+1}}$, define $g(\xi_j)=\xi_{j+1}$ and extend it to B_i such that

$$\sum_{\xi\in B_i} C_{\xi,g(\xi)}=\sum_{j=0}^{n-1} C_{\xi_j\xi_{j+1}} \text{ and } g\in G_{\xi\eta}(B_i).$$

Hence

$$\min_{\eta\notin B_i} T_{\xi\eta}(B_i)\leqslant\min_{\eta\notin B_i}\left\{ \sum_{k=1}^{l} C_{\xi_{k-1}\xi_k}\middle| l\geqslant 1,\ \xi_0=\zeta,\ \xi_1, \xi_2\cdots,\ \xi_{l-1}\in B_i,\ \xi_l=\eta\right\};$$

on the other hand, if $g\in G_{\xi\eta}(B_i)$, then $\{\zeta=g^{(0)}(\zeta),\ g^{(1)}(\zeta)=g(\zeta),\ g^{(2)}(\zeta)=g(g(\zeta)),\cdots,\cdots\}$ is a sequence leading from ζ to some $\eta'\notin B_i$, and

$$\Pi(g)\geqslant\sum_k C_{g^{(k)}(\xi),g^{(k+1)}(\zeta)}.$$

So the opposite inequality holds, and the equality follows. Q.E.D.

Proof of Theorem 0.1. By Lemmas 2.1 and 2.2, we obtain (0.8) and (0.9) immediately. (0.10) is an easy consequence of (0.9). Q.E.D.

Proof of Theorem 0.2. We prove (0.12) first. The assumption $\xi\in B_i\backslash U_{j\neq i}B_j$ implies that $P_\xi^\infty(\sigma(A_i)<\infty)=1$. Hence

$$\min_{\eta\notin B_i} T_{\xi\eta}(B_i\backslash\{\zeta\})>0 \quad \forall_\zeta\in A_i.$$

By Lemma 1.2 and Lemma 2.2

$$\begin{aligned} P_\xi(\sigma(\zeta)>e^{\delta\beta}) &\leqslant P_\xi(\sigma(\zeta)\neq\tau(B_i\backslash\{\zeta\}))+P_\xi(\tau(B_i\backslash\{\zeta\})>e^{\delta\beta}) \\ &\leqslant P_\xi(X_{\tau(B_i\backslash\{\zeta\})}\neq\zeta)+e^{-\delta\beta}E_\xi\tau(B_i\backslash\{\zeta\}) \\ &\leqslant e^{-\beta\min_{\eta\notin B_i} T_{\xi\eta}(B_i\backslash\{\zeta\})}+e^{-\delta\beta}E_\xi\tau(B_i\backslash\{\zeta\})\leqslant e^{-\delta_1\beta}. \end{aligned}$$

This proves (0.12).

Take $h_\beta=E_\zeta^\beta\tau(B_i)$ for a fixed $\zeta\in A_i$. For any $\xi\in B_i$, it follows from the proof of (0.12) that

$$\lim_{\beta\to\infty} P_\xi^\beta(\sigma(\zeta)<\tau(B_i))\geqslant\lim_{\beta\to\infty} P_\xi^\beta(\sigma(\zeta)=\tau(B_i\backslash\{\zeta\})<\tau(B_i))=1,$$

$$\begin{aligned} \lim_{\beta\to\infty}\frac{E_\xi^\beta\tau(B_i)}{E_\zeta^\beta\tau(B_i)} &=\lim_{\beta\to\infty}\frac{E_\xi^\beta\ (\tau(B_i),\ \sigma(\zeta)<\tau(B_i))}{E_\zeta^\beta\tau(B_i)}+\lim_{\beta\to\infty}\frac{E_\xi^\beta(\tau(B_i),\ \sigma(\zeta)\geqslant\tau(B_i))}{E_\zeta^\beta\tau(B_i)} \\ &=\lim_{\beta\to\infty}\frac{E_\xi^\beta(\tau(B_i)-\sigma(\zeta)+\sigma(\zeta),\ \sigma(\zeta)<\tau(B_i))}{E_\zeta^\beta\tau(B_i)} \end{aligned}$$

$$=\lim_{\beta\to\infty}\frac{E^{\beta}_{\zeta}\tau(B_i)+E^{\beta}_{\xi}\sigma(\zeta)}{E^{\beta}_{\zeta}\tau(B_i)}\,P^{\beta}_{\xi}(\sigma(\zeta)<\tau(B_i))\ \text{(strong Markov!)}$$

$$=\lim_{\beta\to\infty}P^{\beta}_{\xi}(\sigma(\zeta)<\tau(B_i))=1. \tag{2.5}$$

Step 1. Let $\mu^{\beta}_{\xi}(\cdot)$ be the probability distribution of $\tau(B_i)/h_{\beta}$. Then

$$\mu^{\beta}_{\xi}\left(\left[0,\ \frac{2}{\varepsilon}\ \right]\right)=1-P^{\beta}_{\xi}\left(\frac{\tau(B_i)}{h_{\beta}}\ >\frac{2}{\varepsilon}\ \right)\geqslant 1-\frac{\varepsilon}{2}\,E^{\beta}_{\xi}\frac{\tau(B_i)}{h_{\beta}}\ .$$

By (2.5) there exists β_0 such that as $\beta\geqslant\beta_0$, $E_{\xi}\tau(B_i)/h_{\beta}\leqslant 2$. Hence $\mu^{\xi}_{\beta}([0, 2/\varepsilon])\geqslant 1-\varepsilon$ and $\{\mu^{\beta}_{\xi}(\cdot);\beta_0\leqslant\beta<+\infty\}$ is tight. Consequently we can choose a sequence $\{\beta_n\}$ such that $\mu^{\beta}_{\xi}(\cdot)$ converges weakly to a distribution $\mu_{\xi}(\cdot)$ for $\forall\xi\in B_i$, as $n\to+\infty$.

Step 2. We show now that μ_{ξ} is independent of ξ. By the Chebyshev inequality and (0.12), $\forall\delta>0$, as $\beta\to\infty$,

$$P(\tau(B_i)\geqslant \mathrm{e}^{(T(B_i)+\delta)\beta})\leqslant\frac{E\tau(B_i)}{\mathrm{e}^{(T(B_i)+\delta)\beta}}\ \to\ 0, \tag{2.6}$$

$$P(\sigma(\zeta)\geqslant \mathrm{e}^{(T(B_i)-\delta)\beta})\leqslant\frac{E\sigma(\zeta)}{\mathrm{e}^{(T(B_i)-\delta)\beta}}\ \to\ 0. \tag{2.7}$$

Using the strong Markovian property and the fact $X_{\sigma(\zeta)}=\zeta$,

$$\begin{aligned}\mu_{\xi}([x,\ +\infty))&=\lim_{n\to\infty}\mu^{\beta_n}_{\xi}([x,\ +\infty))=\lim_{n\to\infty}P^{\beta_n}_{\xi}(\tau(B_i)\geqslant h_{\beta_n}x)\\&=\lim_{n\to\infty}P^{\beta_n}_{\xi}(\tau(B_i)\geqslant h_{\beta_n}x>\mathrm{e}^{(T(B_i)-\delta)\beta_n}\geqslant\sigma(\zeta))\\&=\lim_{n\to\infty}P^{\beta_n}_{\xi}(\sigma(\zeta)\leqslant\mathrm{e}^{(T(B_i)-\delta)\beta_n},\ P^{\beta_n}_{\zeta}(\tau(B_i)\geqslant h_{\beta_n}x))\\&=\mu_{\zeta}([x,\ +\infty)).\end{aligned}$$

Step 3. Now simply write $\mu_{\xi}([x,\ +\infty))=\mu(x)$. Let

$$F=\{z\geqslant 0|\exists\,\beta_n,\ \text{and}\ \xi\in B_i\ \text{such that}\ \mu^{\beta_n}_{\xi}(\{z\})>0\}.$$

For $x,\ y,\ x+y\notin F$,

$$\begin{aligned}\mu(x+y)&=\lim_{n\to\infty}P^{\beta_n}_{\xi}(\tau(B_i)\geqslant h^{\beta_n}(x+y))\\&=\lim_{n\to\infty}E^{\beta_n}_{\xi}(\tau(B_i)\geqslant h^{\beta_n}x,\ P^{\beta_n}_{\xi_{[h^{\beta_n}x]}}(\tau(B_i)\geqslant h^{\beta_n}y))=\mu(x)\mu(y).\end{aligned} \tag{2.8}$$

To show that (2.8) holds for any $x,\ y>0$, we notice that F is at most a countable set. Take sequences $x_n\nearrow x, y_n\nearrow y$ such that $x_n,y_n,x_n+y_n\notin F$ and (2.8) holds for all x_n, y_n. Since $\mu(\cdot)$ is left continuous we obtain

$$\mu(x+y)=\operatorname*{im}_{n\to\infty}\mu(x_n+y_n)=\lim_{n\to\infty}\mu(x_n)\,\mu(y_n)=\mu(x)\mu(y).$$

It then follows that $\mu(x) = C\mathrm{e}^{-ax}$. $\mu(0)=1$ and $\int_0^\infty x\mu(\mathrm{d}x)=1$ imply $C=1$ and $a=1$, respectively.

Step 4. Finally, by the uniqueness of the weak limit of $\{\mu_\xi^{\beta_n}(\cdot)\}$, $\tau(B_i)/h_\beta$ and $\tau(B_i)/E_\xi^\beta\tau(B_i)$ converge in law to the unit exponential random variable.

Finally, rewrite the left-hand side of (0.11) as

$$\lim_{\beta\to\infty} P_\xi^\beta\left(\mathrm{e}^{-\delta\beta} < \frac{\tau(B_i)}{\mathrm{e}^{T(B_i)\beta}} < \mathrm{e}^{\delta\beta}\right) = \lim_{\beta\to\infty}\int_{\mathrm{e}^{-\delta\beta}}^{\mathrm{e}^{\delta\beta}} \mathrm{e}^{-x}\mathrm{d}x = 1.$$

Q.E.D

In the following two lemmas and the proof of Theorem 0.3, let A be one of A_i's, $B=B_i\setminus\bigcup_{j\neq i}B_j$. We omit the subscript for simplicity.

Lemma 2.3. *Suppose that* $\lambda_1^\beta, \lambda_2^\beta, \cdots\lambda_{|B|}^\beta$ *are eigenvalues of* $\boldsymbol{I}-\boldsymbol{P}_B^\beta$ *and* λ_1^β *is the first eigenvalue in the sense that* $|\lambda_1^\beta|<|\lambda_j^\beta|, j=2,3,\cdots|B|$. *Then*

$$\lim_{\beta\to\infty}\frac{-1}{\beta}\log\lambda_1^\beta = T(B),\ \lim_{\beta\to\infty}\frac{-1}{\beta}\log|\lambda_j^\beta| = 0,\ j\geqslant 2.$$

Proof. Notice that the first eigenvalue of $\boldsymbol{I}-\boldsymbol{P}_B^\infty$ is 0 and all other eigenvalues have positive norms if $\boldsymbol{P}_B^\infty$ is an irreducible stochastic matrix. So the second part is immediate. By the expansion of the eigenpolynomial, we have

$$\det(\boldsymbol{I}-\boldsymbol{P}_B)=\prod_{k=1}^{|B|}\lambda_k^\beta,\ \sum_{\xi\in B}\det(\boldsymbol{I}-\boldsymbol{P}_{B\setminus\{\xi\}})=\sum_{j=1}^{|B|}\prod_{k\neq j}\lambda_k^\beta. \tag{2.9}$$

Therefore,

$$\begin{aligned}\lim_{\beta\to\infty}\frac{-1}{\beta}\log\lambda_1^\beta &= \lim_{\beta\to\infty}\frac{-1}{\beta}\log\frac{\det(\boldsymbol{I}-\boldsymbol{P}_B^\beta)}{\sum_{\xi\in B}\det(\boldsymbol{I}-\boldsymbol{P}_{B\setminus\{\xi\}})} \qquad (2.10)\\ &= -\lim_{\beta\to\infty}\frac{1}{\beta}\log\sum_{g\in G(B)}\mathrm{e}^{-\beta\Pi(g)}+\lim_{\beta\to\infty}\log\sum_{\xi\in B}\sum_{g'\in G(B\setminus\{\xi\})}\mathrm{e}^{-\beta\Pi(g')}\\ &= \min_{g\in G(B)}\prod(g)-\min_{\xi\in B}\min_{g'\in G(B\setminus\{\xi\})}\prod(g')=T(B).\end{aligned}$$

Q.E.D.

Lemma 2.4. *Let* $\varphi^\beta, \psi^\beta$ *be the unique positive right eigenvector and left eigen vector of* $\boldsymbol{P}_B^\beta$ *for the largest eigenvalue* $1-\lambda_1^\beta$ *such that the maximum entry of* φ^β *is* $\mathbf{1}$, *and* $\psi^\beta\cdot\varphi^\beta=1$. *Then*

$$\varphi^\beta\to\mathbf{1},\ \psi^\beta\to\boldsymbol{V}=(v(\zeta_1),\cdots,v(\zeta_m),0,\cdots,0),$$

where $A=\{\zeta_1,\cdots,\zeta_m)\}$ *and* $\{v(\zeta_1),\cdots,v(\zeta_m)\}$ *is the invariant measure of* $\boldsymbol{P}_A^\infty$.

Proof. Take a subsequence $\{\beta_n\}$ such that $\lim_{n\to\infty}\varphi^{\beta_n}$ exists, say φ. Take the limit of both sides of $\boldsymbol{P}_B^\beta\varphi^\beta=(1-\lambda_1^\beta)\varphi^\beta$ along the subsequence $\{\beta_n\}$. We get $\boldsymbol{P}_B^\infty\varphi=\varphi$. It follows that $\varphi=\mathbf{1}$ is the unique solution. Hence $\lim\limits_{\beta\to\infty}\varphi^\beta=\mathbf{1}$. The same argument works for ψ^β. Q.E.D.

Proof of Theorem 3. Let $\boldsymbol{P}_B^\beta$ have the following Jordon decomposition.

$$\boldsymbol{P}_B^\beta=\Gamma_\beta\begin{pmatrix}1-\lambda_1^\beta & & 0\\ & \boldsymbol{E}_2^\beta\ddots & \\ 0 & & \boldsymbol{E}_j^\beta\end{pmatrix}\Gamma_\beta^{-1},$$

where the first column of Γ_β and first row of Γ_β^{-1} can be chosen as φ^β and ψ^β.

$$\boldsymbol{E}_i^\beta=\begin{pmatrix}1-\lambda_i^\beta & 1 & \\ & \ddots & \ddots & 1\\ 0 & & \ddots & 1-\lambda_i^\beta\end{pmatrix}_{r_i\times r_i},$$

$$(\boldsymbol{E}_i^\beta)^k=\left[(1-\lambda_i^\beta)\boldsymbol{I}+\begin{pmatrix}0 & 1 & 0\\ & \ddots & \ddots\\ & & \ddots & 1\\ 0 & & & 0\end{pmatrix}_{r_i\times r_i}\right]^k=(1-\lambda_i^\beta)^k\boldsymbol{I}+k(1-\lambda_i^\beta)^{k-1}\begin{pmatrix}0 & 1 & 0\\ & \ddots & \ddots\\ & & \ddots & 1\\ 0 & & & 0\end{pmatrix}$$

$$+\frac{k(k-1)}{2}(1-\lambda_i^\beta)^{k-2}\begin{pmatrix}0 & 0 & 1 & 0\\ & \ddots & \ddots & \ddots\\ & & \ddots & \ddots & 1\\ & & & \ddots & 0\\ 0 & & & & 0\end{pmatrix}+\cdots+\binom{k}{r_i}(1-\lambda_i^\beta)^{k-r_i}\begin{pmatrix}0 & \cdots & 1\\ 0 & \ddots & 0\\ 0 & \cdots & 0\end{pmatrix}.$$

Fix $\delta>0$. For any l such that $\mathrm{e}^{\delta\beta}<l<\mathrm{e}^{(T(B_1)-\delta)\beta}-\mathrm{e}^{\delta\beta}$, as $\beta\to\infty$,

$$\lambda_1^\beta l\to 0,\ \lambda_i^\beta l\to\infty,\ i\geqslant 2,$$

$$\frac{1}{l}\sum_{k=0}^{l-1}(1-\lambda_1^\beta)^k=\frac{1-(1-\lambda_1^\beta)^l}{l\lambda_1^\beta}\to 1,$$

$$\frac{1}{l}\sum_{k=0}^{l-1}(1-\lambda_i^\beta)^k\leqslant\frac{1}{l\lambda_i^\beta}\to 0,$$

$$\frac{1}{l}\sum_{k=0}^{l-1}\frac{k!}{(k-r)!}(1-\lambda_i^\beta)^k\leqslant\frac{r!}{l(\lambda_i^\beta)^{r+1}}\to 0,$$

$$\frac{1}{l}\sum_{k=0}^{l-1}(\boldsymbol{P}_B^\beta)^k=\Gamma_\beta\begin{pmatrix}\frac{1}{l}\sum\limits_{k=0}^{l-1}(1-\lambda_1^\beta)^k & & \\ & \frac{1}{l}\sum\limits_{k=0}^{l-1}\boldsymbol{E}_1^k & \\ & & \ddots\end{pmatrix}\Gamma_\beta^{-1}\xrightarrow{\beta\to+\infty}\begin{pmatrix}1 & \\ \vdots & *\\ 1 & \end{pmatrix}\begin{pmatrix}1 & \\ & 0\end{pmatrix}\begin{pmatrix}V\\ \\ *\end{pmatrix}=\mathbf{1}\cdot V.\quad(2.11)$$

Namely,

$$\frac{1}{l}\sum_{k=0}^{l-1} P_\xi^\beta(X_k=\xi) \to \begin{cases} V(\zeta), & \text{if } \zeta\in A, \\ 0, & \text{otherwise.}\end{cases}$$

For $\forall e^{\delta\beta} \leqslant l \leqslant e^{(T(B_i)-\delta)\beta} - e^{\delta\beta}$,

$$\lim_{\beta\to\infty} \frac{1}{N_\beta}\sum_{\substack{k=0\\k\neq l}}^{N_\beta-1} (\boldsymbol{P}_{B_i}^\beta)^{|k-l|} = \lim_{\beta\to\infty} \frac{1}{N_\beta}\,\frac{1}{l}\sum_{k=1}^{l}(\boldsymbol{P}_{B_i}^\beta)^k + \lim_{\beta\to\infty} \frac{N_\beta-l-1}{N_\beta}\,\frac{1}{N_\beta-l-1}\sum_{k=1}^{N_\beta-l-1} (\boldsymbol{P}_{B_i}^\beta)^k$$

$$= \lim_{\beta\to\infty}\frac{1}{N_\beta}\,\mathbf{1}\cdot \boldsymbol{V} + \lim_{\beta\to\infty}\frac{N_\beta-l-1}{N_\beta}\,\mathbf{1}\cdot\boldsymbol{V} = \mathbf{1}\cdot\boldsymbol{V}. \tag{2.12}$$

For $l \leqslant e^{\delta\beta}$

$$\lim_{\beta\to\infty} l/N\beta = 0.$$

For $l > e^{(T(B_i)-\delta)\beta} - e^{\delta\beta}$

$$\lim_{\beta\to\infty} (N_\beta - l)/N_\beta = 0.$$

Equality (2.12) still holds. Finally the proof is completed by

$$E_\xi^\beta\left|\frac{1}{N_\beta}\sum_{k=0}^{N_\beta-1} I_{\{\zeta\}}(X_k) - v(\zeta)\right|^2 = E_\xi^\beta\left[\frac{1}{N_\beta}\sum_{k=0}^{N_\beta-1} I_{\{\xi\}}(X_k)\right]^2 - (v(\zeta))^2$$

$$= \frac{1}{N_\beta}\sum_{l=0}^{N_\beta-1} P(X_0=\xi,\ X_l=\zeta)\,\frac{1}{N_\beta}\sum_{\substack{k=0\\k\neq l}}^{N_\beta-1} P(X_l=\zeta,\ X_k=\eta) - (v(\zeta))^2 \to (v(\zeta))^2 - (v(\zeta))^2 = 0. \quad \text{Q.E.D.}$$

3 The higher level metastability

We now define iteratively the higher level attractors. The definition seems to be inherently cumbersome. To motivate and to unify the definition, we first reformulate the recurrent classes and their attractive basins by new terminology. From now on the recurrent classes $\{A_1, A_2, \cdots, A_s\}$ are called level 1 attractors and are denoted as $\{A_1^1, A_2^1, \cdots, A_s^1\}$. In the meantime, attractive basins $B_1, \cdots B_s$ are called level 1 attractive basins and are denoted as $B_1^1, \cdots, B_s^1$.

Definition 3.1.

(i) We say $\xi \overset{(0)}{\Rightarrow} \eta$ if $C_{\xi\eta}=0$;

(ii) we say $\xi \overset{(0)}{\to} \eta$ if there is a sequence $\{\xi_i, i=0, 1, \cdots, k\}$ such that

$$\xi=\xi_0,\ \eta=\xi_k,\ \xi_i \overset{(0)}{\Rightarrow} \xi_{i+1},\ i=0,1,\cdots,k-1;$$

(iii) we say $\xi \overset{(0)}{\longleftrightarrow} \eta$ if $\xi \overset{(0)}{\to} \eta$ and $\eta \overset{(0)}{\to} \xi$.

Definition 3.2. A level 1 attractor A_i^1 is a subset of S such that

(i) $\forall \xi, \eta \in A_i^1, \xi \overset{(0)}{\longleftrightarrow} \eta$, and

(ii) if $\xi \in A_i^1, \xi \overset{(0)}{\to} \eta$, then $\eta \in A_i^1$.

The set of all level 1 attractors is denoted by $\mathscr{A}_1$.

Definition 3.3. The attractive basin B_i^1 of a level 1 attractor A_i^1 is defined as

$$B_i^1 = \{\xi \mid \xi \overset{(0)}{\to} \zeta \text{ for some } \zeta \in A_i^1\}.$$

One can verify that Definition 3.2 coincides with the definition of the recurrent class, and that of Definition 3.3 is consistent with (0.6).

Definition 3.4. Define $T_1 = \min\limits_{1 \leqslant j \leqslant s} T(B_j^1)$, $V_i^1 = T(B_i^1)$.

We now define inductively higher level attractors and attractive basins by the same procedure as follows. Suppose that we have defined level k attractors, level k attractive basins, and T_k already.

Definition 3.5.

(i) We say $A_i^k \overset{(k)}{\Longrightarrow} A_j^k$ if there exist $\xi \in A_i^k, \eta \in B_j^k \setminus B_j^k$ such that

$$T_{\xi\eta}(B_i^k) - \min_{\zeta \in B_i^k} T(B_i^k \setminus \{\zeta\}) = V_i^k = T_k;$$

(ii) we say $A_i^k \overset{(k)}{\to} A_j^k$ if there is a sequence $A_i^k = A_{n_0}^k, A_{n_1}^k, \cdots, A_{n_m}^k = A_j^k$ such that $A_{n_l}^k \overset{(k)}{\Longrightarrow} A_{n_{l+1}}^k$ for $l = 0, \cdots, m-1$;

(iii) we say $A_i^k \overset{(k)}{\longleftrightarrow} A_j^k$ if $A_i^k \overset{(k)}{\longleftrightarrow} A_j^k$ and $A_j^k \overset{(k)}{\to} A_i^k$.

Definition 3.6. A level $(k+1)$ attractor A_i^{k+1} is a subset of $\mathscr{A}_k$ such that

(i) $\forall A_m^k, A_j^k \in A_i^{k+1}, A_m^k \overset{(k)}{\longleftrightarrow} A_j^k$, and

(ii) if $A_m^k \in A_i^{k+1}, A_m^k \overset{(k)}{\to} A_j^k$, then $A_j^k \in A_i^{k+1}$.

The set of all level $(k+1)$ attractors is denoted by $\mathscr{A}_{k+1}$.

Definition 3.7. The attractive basin B_i^{k+1} of a level $(k+1)$ attractor A_i^{k+1} is defined as

$$B_i^{k+1} = \bigcup \{B_j^k \mid A_j^k \overset{(k)}{\to} A_m^k \text{ for some } A_m^k \in A_i^{k+1}\}.$$

Definition 3.8. Define

$$V_i^{k+1} = T(B_i^{k+1}) - \min_{\zeta \in B_i^{k+1}} T(B_i^{k+1} \setminus \{\zeta\}),$$

$$T_{k+1} = \min_{1 \leqslant j \leqslant s_k+1} V_j^{k+1}.$$

It is well possible that $B_i^k \cap B_j^k \neq \varnothing$ for $i \neq j$. By Definition 3.7 it is evident that there is a pyramidal structure among the attractive basin B_i^k's, as shown below.

$$\begin{array}{ccccccccc}
 & & & & B_1^r & & & & \\
 & & & B_1^{r-1} & \cdots & B_{s_{r-1}}^{r-1} & & & \\
 & & \cdots & & \cdots & & \cdots & & \\
B_1^2 & B_2^2 & & & \cdots & \cdots & & B_{s_2}^2 & \\
B_1^1 \cdots B_{n_1}^1 & B_{n_{1+1}}^1 \cdots B_{n_2}^1 & \cdots & & \cdots & \cdots & \cdots & & B_s^1.
\end{array}$$

Theorem 3.1. *Suppose that* $\xi \in B_i^k$ *and* $\eta \notin B_i^k$.

$$\lim_{\beta \to \infty} \frac{-1}{\beta} \log P_\xi^\beta (X_{\tau(B_i^k)} = \eta) = -T(B_i^k) + T_{\xi\eta}(B_i^k), \tag{3.1}$$

$$\lim_{\beta \to \infty} P_\xi^\beta (X_{\tau(B_i^k)} \in \{\eta \notin B_i^k \mid T(B_i^k) = T_{\xi\eta}(B_i^k)\}) = 1, \tag{3.2}$$

$$\lim_{\beta \to \infty} \frac{1}{\beta} \log E_\xi \tau(B_i^k) = V_i^k. \tag{3.3}$$

(3.1) and (3.3) are immediate from Lemma 2.1 and (3.2) is in turn a consequence of (3.1). To see that the exit distribution is independent of the initial state $\xi \in B_i^k \setminus \bigcup_{j \neq i} B_j^k$, we have to study $T(B_i^k)$'s further.

Lemma 3.1. *For any* $\zeta \in A_i^k$, $\xi \in B_i^k$

$$T(B_i^k \setminus \{\xi\}) \geqslant T(B_i^k \setminus \{\zeta\}) = T_{\xi\zeta}(B_i^k \setminus \{\zeta\}). \tag{3.4}$$

Proof. Suppose that

$$B_i^k = B_{i_1}^{k-1} \cup \cdots \cup B_{i_r}^{k-1}, \quad \zeta \in B_{i_1}^{k-1}.$$

For any $g \in G(B_i^k \setminus \{\zeta\})$,

$$g|_{B_{i_1}^{k-1} \setminus \{\zeta\}} \in G(B_{i_1}^{k-1} \setminus \{\zeta\}), \ g|_{B_{i_m}^{k-1}} \in G(B_{i_m}^{k-1}), \quad m = 2, 3, \cdots, r,$$

and

$$\prod(g) = \sum_{\eta \in B_i^k \setminus \{\zeta\}} C_{\eta, g(\eta)} = \sum_{\eta \in B_{i_1}^{k-1} \setminus \{\zeta\}} C_{\eta, g(\eta)} + \sum_{m=2}^{r} \sum_{\eta \in B_{i_m}^{k-1}} C_{\eta, g(\eta)}. \tag{3.5}$$

In the last step we have to add to the sum some $C_{\xi\eta}$'s $(=0)$ if $\xi \in B_i^{k-1} \cap B_j^{k-1}$ so that $C_{\xi g(\xi)}$ is counted in one B_i^{k-1} and $C_{\xi\eta}$ is counted in the other B_j^{k-1}. From (3.5) we get

$$T(B_i^k \setminus \{\zeta\}) \geqslant T(B_{i_1}^{k-1} \setminus \{\zeta\}) + \sum_{l=2}^{r} T(B_{i_l}^{k-1}). \tag{3.6}$$

We shall construct $g \in G_{\xi\zeta}(B_i^k \setminus \{\zeta\})$ such that

$$\prod(g)=T(B_{i_1}^{k-1}\setminus\{\zeta\})+\sum_{l=2}^{r}T(B_{i_1}^{k-1}). \tag{3.7}$$

But

$$T(B_i^k\setminus\{\zeta\})\leqslant T_{\xi\zeta}(B_i^k\setminus\{\zeta\})\leqslant \Pi(g). \tag{3.8}$$

(3.6), (3.7) and (3.8) together imply that the second equality of (3.4) holds.

The construction of g satisfying (3.7) is similiar to what we did in proving Lemma 3.2. Using induction we suppose that (3.4) is true for level $k-1$. First take $g_1\in G_{\xi\zeta}(B_{i_1}^{k-1}\setminus\{\zeta\})$ such that

$$\prod(g_1)=T(B_{i_i}^{k-1}\setminus\{\zeta\})=T_{\xi\zeta}(B_{i_1}^{k-1}\setminus\{\zeta\}),\ \forall\xi\in B_{i_1}^{k-1}(\zeta\}.$$

To extend the domain of definition of g_1 choose $B_{i_m}^{k-1}$ such that $A_{i_m}^{k-1}\overset{(k-1)}{\Rightarrow}A_{i_1}^{k-1}$. Without loss of generality we may assume that $A_{i_2}^{k-1}\overset{(k-1)}{\Rightarrow}A_{i_1}^{k-1}$. By definition there exist $\xi\in B_{i_1}^{k-1}$, $\eta\in B_{i_2}^{k-1}\setminus B_{i_1}^{k-1}$ and $g_2\in G_{\xi\eta}(B_{i_2}^{k-1})$ such that $\Pi(g_2)=T(B_{i_2}^{k-1})$. Define on $B_{i_1}^{k-1}\bigcup B_{i_2}^{k-1}$

$$g(\xi)=\begin{cases} g_1(x), & \text{if } \xi\in B_{i_1}^{k-1}\setminus\{\zeta\}, \\ g_2(x), & \text{if } \xi\in B_{i_2}^{k-1}\setminus B_{i_1}^{k-1}. \end{cases}$$

Continue the extension by taking $B_{i_m}^{k-1}$ such that $A_{i_m}^{k-1}\overset{(k-1)}{\Rightarrow}A_{i_1}^{k-1}$ or $A_{i_m}^{k-1}\overset{(k-1)}{\Rightarrow}A_{i_2}^{k-1}$, and by repeating the same argument, and so on. After a finite number of extensions we get a desired g satisfying (3.7). This finishes the proof of the second equality in (3.4).

Suppose that $\xi\in B_{i_m}^{k-1}$, like (3.6) we also have

$$T(B_i^k\setminus\{\xi\})\geqslant T(B_{i_m}^{k-1}\setminus\{\xi\})+\sum_{l=1,l\neq m}^{r}T(B_{i_l}^{k-1}). \tag*{(3.6)$'$}$$

If $m=1$, by induction hypothesis,

$$T(B_{i_1}^{k-1}\setminus\{\xi\})\geqslant T(B_{i_1}^{k-1}\setminus\{\zeta\}). \tag{3.9}$$

(3.7), (3.6)$'$ and (3.9) combine to give the first inequality in (3.4).

If $m\neq 1$, by the induction hypothesis,

$$\begin{aligned} & T(B_{i_1}^{k-1})-T(B_{i_1}^{k-1}\setminus\{\zeta\}) \\ & =T(B_{i_1}^{k-1})-\min_{\eta}T(B_{i_1}^{k-1}\setminus\{\eta\}) \\ & =V_{i_1}^{k-1}\geqslant T_{k-1}=V_{i_m}^{k-1} \\ & =T(B_{i_m}^{k-1})-\min_{\eta}T(B_{i_m}^{k-1}\setminus\{\eta\})\geqslant T(B_{i_m}^{k-1})-T(B_{i_1}^{k-1}\setminus\{\xi\}). \end{aligned} \tag{3.10}$$

Again the first inequality in (3.4) holds by (3.7), (3.6$'$) and (4.10). This completes the proof. Q.E.D.

Definition 3.9. Let

$$U_i^k = \min_{\zeta \in B_i^k} T(B_i^k \setminus \{\zeta\}) - \min_{\substack{\eta \neq \xi \\ \eta, \xi \in B_i^k}} T(B_i^k \setminus \{\xi, \eta\}). \tag{3.11}$$

Lemma 3.3

$$U_i^k \leqslant T_{k-1} < T_k \leqslant V_i^k. \tag{3.12}$$

Proof. Choose $\zeta, \xi, \eta \in B_i^k$ such that

$$T(B_i^k \setminus \{\zeta\}) = \min_{\zeta' \in B_i^k} T(B_i^k \setminus \{\zeta'\}),$$

$$T(B_i^k \setminus \{\eta, \xi\}) = \min_{\eta', \xi' \in B_i^k} T(B_i^k \setminus \{\eta', \xi'\}).$$

Suppose that

$$B_i^k = B_{i_1}^{k-1} \cup \cdots \cup B_{i_r}^{k-1}, \ \zeta \in B_{i_1}^{k-1},$$

and

$$T(B_i^k \setminus \{\zeta\}) = \sum_{l=2}^{r} T(B_{i_l}^{k-1}) + T(B_{i_1}^{k-1} \setminus \{\zeta\}).$$

If $\xi \in B_{i_2}^{k-1}$, $\eta \in B_{i_3}^{k-1}$, then $T(B_i^k \setminus \{\eta, \xi\})$ is not minimum because

$$T(B_{i_3}^{k-1}) - \min_{\xi} T(B_{i_3}^{k-1} \setminus \{\xi\}) = T_{k-1} \leqslant T(B_{i_1}^{k-1}) - \min_{\zeta} T(B_{i_1}^{k-1} \setminus \{\zeta\}).$$

Similarly,

$$T(B_i^k \setminus \{\xi, \eta\}) = \begin{cases} \sum_{l=2}^{r} T(B_{i_l}^{k-1}) + T(B_{i_1}^{k-1} \setminus \{\xi, \eta\}), \text{ if } \xi, \eta \in B_{i_1}^{k-1}. \\ \sum_{l=3}^{r} T(B_{i_l}^{k-1}) + T(B_{i_1}^{k-1} \setminus \{\xi\}) + T(B_{i_2}^{k-1} \setminus \{\eta\}), \text{ if } \xi \in B_{i_1}^{k-1}, \ \eta \in B_{i_2}^{k-1}. \end{cases}$$

If $\xi, \eta \in B_{i_1}^{k-1}$, by induction hypothesis

$$\begin{aligned} U_i^k &= T(B_{i_1}^{k-1} \setminus \{\zeta\}) - T(B_{i_1}^{k-1} \setminus \{\xi, \eta\}) \\ &\leqslant \min_{\xi \in B_{i_1}^{k-1}} T(B_{i_1}^{k-1} \setminus \{\xi\}) - \min_{\xi, \eta \in B_{i_1}^{k-1}} T(B_{i_1}^{k-1} \setminus \{\xi, \eta\}) \\ &= U_{i_1}^{k-1} \leqslant T_{k-2} < T_{k-1}. \end{aligned}$$

If $\xi \in B_{i_1}^{k-1}$, $\eta \in B_{i_2}^{k-1}$,

$$\begin{aligned} U_i^k &= T(B_i^k \setminus \{\zeta\}) - T(B_i^k \setminus \{\xi, \eta\}) \\ &= T(B_i^k \setminus \{\zeta\}) + T(B_{i_2}^{k-1}) - T(B_{i_1}^{k-1} \setminus \{\xi\}) - T(B_{i_2}^{k-1} \setminus \{\eta\}) \\ &\leqslant T(B_{i_2}^{k-1}) - T(B_{i_2}^{k-1} \setminus \{\eta\}) \\ &\leqslant T(B_{i_2}^{k-1}) - \min_{\eta \in B_{i_2}^{k-1}} T(B_{i_2}^{k-1} \setminus \{\eta\}) = V_{i_2}^{k-1} = T_{k-1}. \end{aligned}$$

Here we use the fact that $B_{i_2}^{k-1} \subset B_i^k$, so $A_{i_2}^{k-1} \overset{(k-1)}{\rightarrow} A_{i_1}^{k-1}$. In both cases we conclude that $U_i^k \leqslant T_{k-1}$. This completes the proof of the first inequality.

To prove the second inequality we have to deal with two cases separately. If a level k attractor is also a level $(k-1)$ attractor, then

$$T(B_i^k) - T(B_i^k \setminus \{\zeta\}) \geqslant T(B_i^{k-1}) - T(B_i^{k-1} \setminus \{\zeta\}) > T_{k-1}.$$

If a level k attractor is a subset of $\mathscr{A}_{k-1}$ and contains at least two elements, then

$$T(B_i^k) > \sum_{l=1}^{r} T(B_{i_l}^{k-1}).$$

Hence

$$T(B_i^k) - T(B_i^k \setminus \{\zeta\}) > T(B_i^{k-1}) - T(B_i^{k-1} \setminus \{\zeta\}) \geqslant T_{k-1}.$$

In either case we have $T_k > T_{k-1}$.

The last inequality of (3.12) holds by definition. Q.E.D.

Theorem 3.2. *Suppose that* $\xi \in B_i^k \setminus \bigcup_{j \neq i} B_j^k$.

(i) *If* $\zeta \in A_i^k, U_i^k < \gamma < V_i^k$, *then*

$$\lim_{\beta \to \infty} P_\xi^\beta(\sigma(\zeta) \leqslant \mathrm{e}^{r\beta} \leqslant \tau(B_i^k) \leqslant \mathrm{e}^{(V_i^k + \delta)\beta}) = 1;$$

(ii) *as* $\beta \to \infty$, $\tau(B_i^k) / E_\xi \tau(B_i^k)$ *converges in law to a random variable having the exponential distribution with mean* 1.

Proof. The assumption that $\xi \in B_i^k \setminus \bigcup_{j \neq i} B_j^k$ implies that

$$\min_{\eta \notin B_i^k} T_{\xi\eta}(B_i^k \setminus \{\zeta\})) - T(B_i^k \setminus \{\zeta\}) > 0. \tag{3.13}$$

Applying Lemma 2.1 we have

$$\lim_{\beta \to \infty} \frac{1}{\beta} \log E_\xi \tau(B_i^k \setminus \{\zeta\}) = T(B_i^k \setminus \{\zeta\}) - \min_{\eta \in B_i^k} T(B_i^k \setminus \{\zeta, \eta\}) \leqslant U_i^k,$$

$$\begin{aligned} P_\xi(\sigma(\zeta) > \mathrm{e}^{\gamma\beta}) &= P_\xi(\sigma(\zeta) \neq \tau(B_i^k \setminus \{\zeta\})) + P_\xi(\tau(B_i^k \setminus \{\zeta\}) > \mathrm{e}^{\gamma\beta}) \\ &\leqslant \mathrm{e}^{-\beta(\min_{\eta \notin B_i^k} T_{\zeta\eta}(B_i^k \setminus \{\eta\}) - T(B_i^k \setminus \{\zeta\})} + \frac{E_\xi \tau(B_i^k \setminus \{\zeta\})}{\mathrm{e}^{\gamma\beta}} \\ &\leqslant \mathrm{e}^{-\delta'\beta} + \mathrm{e}^{-(r - U_i^k)\beta} \to 0. \end{aligned}$$

The remaining proof is identical with that of Theorem 0.2. Q.E.D.

Finally, let B_i^k be given. We write B instead of $B_i^k \setminus \bigcup_{j \neq i} B_j^k$ for simplicity. Assume $\xi \in B$. Let $M_{\xi\xi}^\beta$ be the algebraical cofactor of the ξ-row ξ-column entry of $\boldsymbol{I} - \boldsymbol{P}_B^\beta$. If (0.4) is true, namely, $\lim_{\beta \to \infty} p(\xi, \eta) / \mathrm{e}^{-\beta C_{\xi\eta}}$ exists $\forall \xi, \eta \in S$, then

$$v_i^k(\xi) = \lim_{\beta \to \infty} \frac{M_{\xi\xi}^\beta}{\sum_{\eta \in B} M_{\eta\eta}^\beta} \tag{3.14}$$

exists and

$$v_i^k(\xi) = 0 \text{ if } T(B \setminus \{\xi\}) > \min_{\zeta \in B} T(B \setminus \{\zeta\}).$$

Denote by $\boldsymbol{V}$ the row vector $\{v_i^k(\xi) | \xi \in B\}$. Suppose that $1 - \lambda_1^\beta, \cdots, 1 - \lambda_{|B|}^\beta$ are the eigenvalues of $\boldsymbol{P}_B$ with $|\lambda_1^\beta| < |\lambda_2^\beta| \leqslant \cdots \leqslant |\lambda_{|B|}^\beta|$. Let φ^β be the right eigenvector of $\boldsymbol{P}_B^\beta$ corresponding to λ_1^β

with $\max_\xi \varphi^\beta(\xi)=1$. Let ψ^β be the left eigenvector of $\boldsymbol{P}_B^\beta$ corresponding to λ_1^β such that $\psi^\beta \cdot \varphi^\beta=1$.

Lemma 3.4. *If* (0.4) *is true, then*

(i) $\lim\limits_{\beta\to\infty}\varphi^\beta=1,\ \lim\limits_{\beta\to\infty}\psi^\beta=\boldsymbol{V}$;

(ii) $\lim\limits_{\beta\to\infty}(\boldsymbol{P}_B^\beta)^{[\exp(\gamma\beta)]}=\boldsymbol{1}\cdot\boldsymbol{V}$ *for* $U_i^k<\gamma<V_i^k$.

Proof. Note that the eigenvector ψ^β is given by

$$\left\{\frac{M_{\xi\xi}^\beta}{\sum\limits_{\eta\in B} M_{\eta\eta}^\beta}\,|\,\xi\in B\right\}.$$

Under assumption (0.4),

$$\lim_{\beta\to\infty}\psi^\beta=\boldsymbol{V}.$$

Proof of $\lim\limits_{\beta\to\infty}\varphi^\beta=\boldsymbol{1}$ is the same as Lemma 2.4. Let $\boldsymbol{P}_B^\beta$ have the Jordon decomposition

$$\boldsymbol{P}_B^\beta=\Gamma_\beta\begin{pmatrix}1-\lambda_1^\beta & & & \\ & \boldsymbol{E}_1 & & \\ & & \ddots & \\ & & & \boldsymbol{E}_j\end{pmatrix}\Gamma_\beta^{-1}, \tag{3.15}$$

where

$$\Gamma_\beta=(\varphi^\beta\cdots),\ \Gamma_\beta^{-1}=\begin{pmatrix}\psi^\beta\\ \vdots\end{pmatrix},\quad \boldsymbol{E}_i=\begin{pmatrix}1-\lambda_i^\beta & 1 & 0\\ & \ddots & \ddots\ 1\\ 0 & \ddots & 1-\lambda_i^\beta\end{pmatrix}_{r_i\times r_i}$$

and $1-\lambda_i^\beta$ is an eigenvalue of $\boldsymbol{P}_B$ other than $1-\lambda_1^\beta$. λ_1^β is the first eigenvalue of $\boldsymbol{I}-\boldsymbol{P}_B$. By (2.10)

$$\lim_{\beta\to\infty}\ -\frac{1}{\beta}\log|\lambda_1^\beta|=T(B)-\min_{\zeta\in B}T(B\setminus\{\zeta\})=V_i^k, \tag{3.16}$$

$$\lim_{\beta\to\infty}\ -\frac{1}{\beta}\log|\lambda_i^\beta|\leqslant\min_{\xi\in B}\ T(B\setminus\{\zeta\})-\min_{\eta,\xi\in B}T(B\setminus\{\eta,\xi\})=U_i^k. \tag{3.17}$$

If $\gamma<V_i^k$, take δ such that $\gamma<V_i^k-\delta$ and for large β, by (3.16),

$$\lambda_1^\beta<\mathrm{e}^{-(V_i^k-\delta)\beta},\ \lim_{\beta\to\infty}\lambda_1^\beta[\mathrm{e}^{\gamma\beta}]=0,$$

and

$$\lim_{\beta\to\infty}(1-\lambda_1^\beta)^{[\mathrm{e}^{\gamma\beta}]}=\exp(-\lim\lambda_1^\beta[\mathrm{e}^{\gamma\beta}])=1.$$

For $\gamma>U_i^k$, take $\delta<\gamma-U_i^k$. Then $\lambda_i^\beta>\mathrm{e}^{-(U_i^k+\delta)\beta}$ for large β and

$$\begin{aligned}[\mathrm{e}^{\gamma\beta}]^r(1-\lambda_i^\beta)^{[\mathrm{e}^{\gamma\beta}]}&=\exp[r\gamma\beta+\mathrm{e}^{\gamma\beta}\log(1-\lambda_i^\beta)]\\ &\leqslant\exp[r\gamma\beta-\mathrm{e}^{\gamma\beta}\lambda_i^\beta]\\ &\leqslant\exp[r\gamma\beta-\mathrm{e}^{(\gamma-U_i^k-\delta)}\beta]\rightarrow 0 \text{ as } \beta\rightarrow+\infty.\end{aligned}$$

Then $E_i^{[\exp\gamma\beta]}\rightarrow 0$ as $\beta\rightarrow\infty$. Therefore the conclusion follows from (3.15). Q.E.D.

Theorem 3.3. *Assume* (0.4). *For* $U_i^k<\gamma<V_i^k$, ξ, $\zeta\in B_i^k\setminus\bigcup_{j\neq i}B_j^k$,

$$\lim_{\beta\rightarrow\infty}E_\xi\left(\frac{1}{\mathrm{e}^{\gamma\beta}}\sum_{k=1}^{\mathrm{e}^{[\gamma\beta]}}I_{\{\zeta\}}(X_k)-v_i^k(\zeta)\right)^2=0.$$

With Lemma 3.4 established, the proof of Theorem 3.3 is identical with that of Theorem 0.3.

Appendix

For completeness we provide here the elementary proof of the three lemmas of §1. Readers who are familiar with the original proof of ref. [9] can skip this part. We often write p_{ij} for $p(i, j)$ in the sequel. We introduce now the convenient notation

$$Q_k=\sum_{j\neq k}p_{kj}=1-p_{kk},\quad 1\leqslant k\leqslant N.$$

Proof of Lemma 1.1. For $s=1$,

$$G(K)=G(\{1\})=\{g:g(1)=k,\ k=2,3,\cdots N\},$$

$$\det(\boldsymbol{I}-\boldsymbol{P}_K)=Q_1=\sum_{k=2}^N p_{1k}=\sum_{g\in G(K)}\pi(g)\tag{1.2}$$

proving (1.2) for $s=1$. In

$$\det D_{pq}(K)=(-1)^{p+q}\sum_{g\in G_{pq}(K)}\pi(g),\tag{1.3}$$

the only possible value for p is $p=1$ and $G_{1,2}(K)=\{g;g(1)=2\}$.

$$\det D_{12}(K)=-p_{12}=(-1)^{1+2}\sum_{g\in G_{12}(K)}\pi(g)$$

proving (1.3) for $s=1$.

Assume now that $s\geqslant 2$ and that we have proved conclusions (1.2) and (1.3) for $s-1$. Let $K_0=\{1,2,\cdots,s-1\}$. Then

$$\boldsymbol{I}-\boldsymbol{P}_K=\begin{pmatrix} Q_1 & -p_{12} & \cdots & -p_{1,s-1} & -p_{1s}\\ -p_{21} & Q_2 & \cdots & -p_{2,s-1} & -p_{2s}\\ -p_{31} & -p_{32} & \cdots & -p_{3,s-1} & -p_{3s}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ -p_{s-1,1} & -p_{s-1,2} & \cdots & Q_{s-1} & -p_{s-1,s}\\ -p_{s1} & -p_{s2} & \cdots & -p_{s,s-1} & Q_s\end{pmatrix}$$

$$= \begin{pmatrix} & & & -p_{1s} \\ & \boldsymbol{I}-\boldsymbol{P}_{K_0} & & \vdots \\ & & & -p_{s-1,s} \\ -p_{s1} & \cdots & -p_{s,s-1} & Q_s \end{pmatrix}$$

and expansion of the $\det(\boldsymbol{I}-\boldsymbol{P}_K)$ along the last row gives

$$\det(\boldsymbol{I}-\boldsymbol{P}_K)=Q_s\det(\boldsymbol{I}-\boldsymbol{P}_{K_0})-\sum_{k=1}^{s-1}(-1)^{s+k}p_{sk}\det D_{ks}(K_0),$$

which by the induction hypothesis

$$=\sum_{k=1}^{s-1}p_{sk}\left[\sum_{h\in G(K_0)}\pi(h)-\sum_{h\in G_{ks}(K_0)}\pi(h)\right]+\sum_{k=s+1}^{N}p_{sk}\sum_{h\in G(K_0)}\pi(h)$$

$$=\sum_{k=1}^{s-1}\left[\sum_{\substack{h:K\to S\\ h(s)=k\\ h|_{K_0}\in G(K_0)}}\pi(h)-\sum_{\substack{h:K\to S\\ h(s)=k\\ h|_{K_0}\in G_{ks}(K_0)}}\pi(h)\right]+\sum_{k=s+1}^{N}\sum_{\substack{h:K\to S\\ h(s)=k\\ h|_{K_0}\in G(K_0)}}\pi(h)$$

$$=\sum_{g\in G(K)}\pi(g).$$

In the last step we use the following fact. Extend $h_0\in G(K_0)$ to a map $h\colon K\to S$ by setting $h(s)=k$. If $k>s$ then automatically $h\in G(K)$ and if $1\leqslant k\leqslant s-1$ then $h\in G(K)$ if and only if $h_0\in G(K_0)\setminus G_{ks}(K_0)$. We have proved (1.2) for s.

Next we prove (1.3) for s and $p=s$.

$$D_{s,s+1}(K)=\begin{pmatrix} Q_1 & -p_{12} & \cdots & -p_{1,s-1} & -p_{1,s+1} \\ -p_{21} & Q_2 & \cdots & -p_{2,s-1} & -p_{2,s+1} \\ -p_{31} & -p_{32} & \cdots & -p_{3,s-1} & -p_{3,s+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -p_{s-1,1} & -p_{s-1,2} & \cdots & Q_{s-1} & -p_{s-1,s+1} \\ -p_{s1} & -p_{s2} & \cdots & -p_{s,s-1} & -p_{s,s+1} \end{pmatrix}$$

$$= \begin{pmatrix} & & & -p_{1,s+1} \\ & \boldsymbol{I}-\boldsymbol{P}_{K_0} & & \vdots \\ & & & -p_{s-1,s+1} \\ -p_{s1} & \cdots & -p_{s,s-1} & -p_{s,s+1} \end{pmatrix},$$

and expansion of the $\det D_{s,s+1}(K)$ along the last row gives

$$\det D_{s,s+1}(K)=-p_{s,s+1}\det(\boldsymbol{I}-\boldsymbol{P}_{K_0})-\sum_{k=1}^{s-1}(-1)^{s+k}p_{sk}\det D_{k,s+1}(K_0),$$

which by the induction hypothesis

$$= -p_{s,s+1} \sum_{h \in G(K_0)} \pi(h) - \sum_{k=1}^{s-1} p_{sk} \sum_{h \in G_{k,s+1}(K_0)} \pi(h)$$

$$= - \sum_{g \in G_{s,s+1}(K)} \pi(g).$$

Last equality holds because $g \in G(K)$ belongs to $G_{s,s+1}(K)$ if and only if either $g(s)=s+1$ and $g|_{K_0} \in G(K_0)$ or $g(s)=k$ for $1 \leqslant k \leqslant s-1$ and $g|_{K_0} \in G_{k,s+1}(K_0)$. We have proved (1.3) for s and $p=s$.

For $1 \leqslant p \leqslant s-1$, $D_{p,s+1}(K)=$

$$\begin{pmatrix} Q_1 & \cdots & -p_{1,p-1} & -p_{1,p+1} & \cdots & -p_{1,s-1} & -p_{1,s+1} \\ -p_{21} & \cdots & -p_{2,p-1} & -p_{2,p+1} & \cdots & -p_{2,s-1} & -p_{2,s+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & -p_{p,p-1} & p_{p,p+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -p_{s-1,1} & \cdots & -p_{s-1,p-1} & -p_{s-1,p+1} & \cdots & Q_{s-1} & -p_{s-1,s+1} \\ -p_{s1} & \cdots & -p_{s,p-1} & -p_{s,p+1} & \cdots & -p_{s,s-1} & -p_{s,s+1} \end{pmatrix}$$

and to make $D_{p,s+1}(K)$ exactly the same as $D_{s,s+1}(K)$ with $(p,\cdots,s)$ permuted into $(p+1,\cdots,s,p)$ we must perform the exact same permutation on the rows of $D_{p,s+1}(K)$; that is, we must move the p-th row to the s-th row and move the rows labeled by $p+1,\cdots,s$ up one. This amounts to $s-p-1$ transpositions of the rows of $D_{p,s+1}(K)$ and multiplies its determinant by $(-1)^{s-p-1}$. So by the result just proved for $p=s$,

$$\det D_{p,s+1}(K) = (-1)^{s-p-1}\left[- \sum_{g \in G_{p,s+1}(K)} \pi(g) \right] = (-1)^{s+p} \sum_{g \in G_{p,s+1}(K)} \pi(g).$$

This completes the induction argument and therefore the proof of the lemma. Q.E.D.

Proof of Lemma 1.2. First

$$P(X_{\tau(K)}=q|X_0=p) = \sum_{k=1}^{\infty} P(X_{\tau(K)}=q, \tau(K)=k|X_0=p)$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{s} p_K^k(p,j) p(j,q) = \sum_{j=1}^{s} (\boldsymbol{I}-\boldsymbol{P}_K)^{-1}_{p,j} p(j,q). \tag{A1}$$

It is obvious by cofactor expression of the inverse of matrix that

$$(\boldsymbol{I}-\boldsymbol{P}_K)^{-1}_{p,j} = \frac{(-1)^{p+j} \det M_{j,p}}{\det(\boldsymbol{I}-\boldsymbol{P}_K)}, \tag{A2}$$

where $M_{j,p}$ is obtained from $\boldsymbol{I}-\boldsymbol{P}_K$ by deleting the j-th row and p-th column, $1 \leqslant j \leqslant s$.

If $j<p$,

$$\det M_{jp} = (-1)^{s-1-j} \det D_{pj}(K \setminus \{j\}), \tag{A3}$$

and since p is the $(p-1)$-st member of $K \setminus \{j\}$,

$$\det D_{pj}(K \setminus \{j\}) = (-1)^{p-1+s} \sum_{g \in G_{pj}(K \setminus \{j\})} \pi(g), \tag{A4}$$

then (A2), (A3) and (A4) combine to give

$$(\boldsymbol{I} - \boldsymbol{P}_K)^{-1}_{pj} = \frac{\sum_{g \in G_{pj}(K \setminus \{j\})} \pi(g)}{\det(\boldsymbol{I} - \boldsymbol{P}_K)}. \tag{A5}$$

If $j > p$,

$$\det M_{jp} = (-1)^{s-j} \det D_{pj}(K \setminus \{j\}), \tag{A3$'$}$$

and since p is the p-th member of $K \setminus \{j\}$,

$$\det D_{pj}(K \setminus \{j\}) = (-1)^{p+s} \sum_{g \in G_{pj}(K \setminus \{j\})} \pi(g). \tag{A4$'$}$$

And again (A5) results.

If $j = p$,

$$\det M_{pp} = \det(\boldsymbol{I} - \boldsymbol{P}_{K \setminus \{p\}}),$$

and in the place of (A4) we get

$$\det(\boldsymbol{I} - \boldsymbol{P}_{K \setminus \{p\}}) = \sum_{g \in G(K \setminus \{p\})} \pi(g),$$

$$(\boldsymbol{I} - \boldsymbol{P}_K)^{-1}_{pp} = \frac{\sum_{g \in G(K \setminus \{p\})} \pi(g)}{\det(\boldsymbol{I} - \boldsymbol{P}_K)}. \tag{A5$'$}$$

Finally by (A1),

$$\begin{aligned} &P(X_{\tau(K)} = q | X_0 = p) \\ &= \frac{\sum_{j=1, j \neq p}^{s} \sum_{g \in G_{pj}(K \setminus \{j\})} \pi(g) p(j,q) + \sum_{g \in G(K \setminus \{p\})} \pi(g) p(p,q)}{\det(\boldsymbol{I} - \boldsymbol{P}_K)} \\ &= \frac{\sum_{g \in G_{pq}(K)} \pi(g)}{\det(\boldsymbol{I} - \boldsymbol{P}_K)} = \frac{(-1)^{p+q} \det D_{pq}(K)}{\det(\boldsymbol{I} - \boldsymbol{P}_K)}. \end{aligned}$$

The last equality holds because of (1.3). Q.E.D.

Proof of Lemma 1.3.

$$E_p \tau(K) = \sum_{j=1}^{s} (\boldsymbol{I} - \boldsymbol{P}_K)^{-1}_{p,j} = \frac{\sum_{j=1}^{s} (-1)^{p+j} \det M_{j,p}}{\det(\boldsymbol{I} - \boldsymbol{P}_K)},$$

where $M_{j,p}$ is obtained from $\boldsymbol{I} - \boldsymbol{P}_K$ by deleting the j-th row and p-th column. But

$$\det M_{j,p} = \begin{cases} \sum\limits_{g\in G(K\setminus\{p\})} \pi(g) & \text{if} \quad j=p; \\ (-1)^{p-j} \sum\limits_{g\in G_{pj}(K)\setminus\{j\})} \pi(g) & \text{if} \quad j\neq p \end{cases}$$

and $\det(\boldsymbol{I}-\boldsymbol{P}_K) = \sum\limits_{g\in G(K)} \pi(g)$. Q.E.D.

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