

The Hydrodynamic Limit for the Reaction Diffusion Equation—An Approach in Terms of the GPV Method

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We study the hydrodynamic limit of the reaction diffusion process by means of the GPV technique (Guo *et al.*⁽⁴⁾). To this end, we first derive *a priori* bounds on the moments of the occupation numbers using the local central limit theorem and results of stochastic analysis. The result of De Masi and Presutti⁽²⁾ for the hydrodynamic limit of the reaction diffusion process is generalized here.

KEY WORDS: Reaction diffusion process; hydrodynamic limit; reaction diffusion equation; local central limit theorem.

1. INTRODUCTION

The one-dimensional zero-range process is one of the simplest particle systems (Liggett⁽⁵⁾) which describes the motion of indistinguishable particles on \mathbb{Z} . The particles move according to the following law: if a site x is occupied by k particles, one of the particles jumps to one of the two neighboring sites $x - 1$ and $x + 1$ with a rate $c(k)$. In the case $c(k) = k$ this corresponds to an independent motion of particles, and it is easy to describe the hydrodynamic behavior. Perturbations of the independent particle system were considered by De Masi and Presutti⁽²⁾ in the hydrodynamic limit. In particular, they studied a model for the reaction

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diffusion equation (RDE) and for the Carleman equation etc. The reader is referred to the book of De Masi and Presutti⁽²⁾ for details (see also Spohn⁽⁷⁾). The essential technique developed in De Masi and Presutti⁽²⁾ for investigating the hydrodynamic limit of interacting particle systems is to study the BBGKY hierarchical equations for the correlation functions. De Masi and Presutti⁽²⁾ asked whether it is possible to derive the RDE by the GPV technique (Guo *et al.*⁽⁴⁾) rather than the hierarchical equations for the correlation functions. In this paper, we give a positive answer to this question.

We apply some results of stochastic analysis and the local central limit theorem to obtain *a priori* bounds for the moments of the occupation numbers of the reaction diffusion process without using the BBGKY hierarchical equations. Then the hydrodynamic limit of the RDE is deduced by applying the GPV technique. We generalize the result in De Masi and Presutti⁽²⁾ for the RDE in one aspect: we do not need their restriction on the form of the reaction part except some growth conditions (see (Q) and (Q)') in Section 2). In the correlation functions approach, the reaction part was expressed in terms of Poisson polynomials and so had to be a polynomial. The idea behind our approach is that we try to separate the perturbation part (reaction part) from the independent particle system (diffusion part). For the reaction part we get a bound by standard results of stochastic analysis; an estimate for the diffusion part is obtained by means of the local central limit theorem. As we pointed out at the beginning, there are a number of different hydrodynamic models which can be considered as perturbations of the independent particle system. We believe that the techniques of this paper can also be applied to these other models. For example, we expect that our technique can be adopted to study the Carleman equation etc. (Yau⁽⁸⁾).

2. MAIN RESULTS

We start with introducing the model and some assumptions. Most of the notation is taken from De Masi and Presutti.⁽²⁾

Let $\varepsilon > 0$ be a parameter. The hydrodynamic limit corresponds to letting $\varepsilon \rightarrow 0$. Without loss of generality we assume that ε^{-1} is an integer, and we set $\mathbb{Z}_\varepsilon = \mathbb{Z}$ modulo ε^{-1} , i.e., we consider the model with periodic boundary conditions. The configuration space is $\mathbb{N}^{\mathbb{Z}_\varepsilon}$. We write $\eta(x)$ for the number of particles at a site x . The intensity for a jump of a particle at site x to the right or left is $\frac{1}{2}\eta(x)$ for $\eta = (\eta(x), x \in \mathbb{Z}_\varepsilon) \in \mathbb{N}^{\mathbb{Z}_\varepsilon}$. In addition to the jumps, we allow the birth and death of particles at x with rates $q_+(\cdot)$ respectively $q_-(\cdot)$.

We assume

$$q_-(0) = 0, \quad q_+(x) > 0, \quad q_-(x) > 0 \quad \text{for all } x > 0$$

and (Q)

$$q_+(x) + q_-(x) < B_1[-q_+(x) + q_-(x)] + B_2$$

for some positive numbers B_1 and B_2 independent of $x \in \mathbb{R}^+$.

Note that the assumption (Q) is weaker than that in De Masi and Presutti⁽²⁾ (Chapter IV, see also assumption (Q') later). We do not need any restriction on the concrete form of the functions q_+ and q_- . Assumption (Q) is certainly satisfied in the case when q_+ and q_- are both polynomials with $\text{deg}(q_-) > \text{deg}(q_+)$. It is also satisfied when $\text{deg}(q_-) = \text{deg}(q_+)$ and the coefficient of the highest order of q_- is larger than that of q_+ . The essence of this assumption is that $q_-(x)$ increases to infinity faster than $q_+(x)$ as x tends to infinity. It is easy to give examples for nonpolynomial functions satisfying assumption (Q).

The process $\eta(x, t)$ we consider in this paper is a Markov process with the generator

$$L^\varepsilon f(\eta) = \varepsilon^{-2} L_0 f(\eta) + L_c f(\eta) \tag{2.1}$$

for bounded functions f on $\mathbb{N}^{\mathbb{Z}^d}$. Here L_0 is the generator of the symmetric independent process, namely

$$L_0 f(\eta) = \frac{1}{2} \sum_x \eta(x) [f(\eta + \delta_{x+1} - \delta_x) + f(\eta + \delta_{x-1} - \delta_x) - 2f(\eta)]$$

and L_c is the generator of the birth-and-death process

$$L_c f(\eta) = \sum_x \{ q_+(\eta(x)) [f(\eta + \delta_x) - f(\eta)] + q_-(\eta(x)) [f(\eta - \delta_x) - f(\eta)] \}$$

$\delta_x \in \mathbb{N}^{\mathbb{Z}^d}$ is the configuration with just one particle at the position x , and sums and differences of configurations are defined componentwise.

By a standard result of stochastic analysis (Protter⁽⁶⁾), we know that the process $\eta(x, t)$ can be expressed as

$$\eta(x, t) - \eta(x, 0) = \int_0^t L\eta(x, s) ds + M(x, t) \tag{2.2}$$

where $M(x, t)$ is a L_2 martingale with respect to the filtration generated by the process $\eta(x, t)$. In fact, it is known that $E\eta^k(x, t) < \infty$ for some $k \in \mathbb{N}$ depending on ε (see for example, Chen⁽¹⁾). Equation (2.2) is the starting point for our analysis.

In order to apply the GPV technique to our case, we need *a priori* bounds on $E\eta^k(x, t)$ which are independent of ε .

Let

$$\begin{aligned} \eta(t) &= (\eta(1, t), \eta(2, t), \dots, \eta(\varepsilon^{-1}, t))' \\ L_c \eta(t) &= (L_c \eta(1, t), L_c \eta(2, t), \dots, L_c \eta(\varepsilon^{-1}, t))' \end{aligned}$$

and

$$M(t) = (M(1, t), M(2, t), \dots, M(\varepsilon^{-1}, t))'$$

where $(\cdot)'$ is the transpose of a vector. So we can rewrite the Eq. (2.2) in the form

$$\eta(t) - \eta(0) = \int_0^t A\eta(s) ds + \int_0^t L_c \eta(s) ds + M(t) \tag{2.3}$$

where A is the $\varepsilon^{-1} \times \varepsilon^{-1}$ Q-matrix of the symmetric independent process, i.e.,

$$A = \varepsilon^{-2} \begin{pmatrix} -1 & 1/2 & 0 & \dots & 0 & 1/2 \\ 1/2 & -1 & 1/2 & \dots & 0 & 0 \\ & & & \dots & & \\ 1/2 & 0 & 0 & \dots & 1/2 & -1 \end{pmatrix}$$

By the variation of constant (Protter⁽⁶⁾), we obtain

$$\eta(t) = e^{At} \eta(0) + \int_0^t e^{A(t-s)} L_c \eta(s) ds + \int_0^t e^{A(t-s)} dM(s) \tag{2.4}$$

Since A is a finite matrix, we have

$$e^{At} = I + At + \dots + \frac{(At)^n}{n!} + \dots$$

Note that the matrix e^{At} is the transition matrix for the simple symmetric random walk on \mathbb{Z} at time $\varepsilon^{-2}t$. For fixed x , let u be the x -th unit vector of $\mathbb{R}^{\varepsilon^{-1}}$, and $\langle \cdot, \cdot \rangle$ the inner product of two vectors in $\mathbb{R}^{\varepsilon^{-1}}$. Then Eq. (2.4) implies that

$$\eta(t, x) = \langle e^{At} \eta(0), u \rangle + \int_0^t \langle e^{A(t-s)} L_c \eta(s), u \rangle ds + \int_0^t \langle e^{A(t-s)} dM(s), u \rangle \tag{2.5}$$

Lemma 2.1. Let $B = B_2/(1 + B_1)$ and $t \geq 0$. Then

$$\begin{aligned} \int_0^t \langle e^{A(t-s)} dM(s), u \rangle &= \int_0^t \langle dM(s), e^{A(t-s)} u \rangle \\ \langle e^{At} \eta(0), u \rangle &\leq \max_y \eta(0, y) \\ \int_0^t \langle e^{A(t-s)} L_c \eta(s), u \rangle ds &\leq Bt \end{aligned}$$

Proof. The first equation follows from the symmetry of the matrix A . Similarly, we have

$$\begin{aligned} \langle e^{At} \eta(0), u \rangle &= \langle \eta(0), e^{At} u \rangle \\ &= \sum_y \eta(0, y) P_x(\xi_{t-s} = y) \\ &\leq \max_y \eta(0, y) \end{aligned}$$

where ξ_t is the symmetric random walk on \mathbb{Z}_ε and P_x the probability distribution of the symmetric random walk starting from x . We need to emphasize here that $P_x(\xi_{t-s} = y)$ depends on ε , we omit it only for the simplicity of notation.

For the last inequality in the lemma, we conclude from the assumption (Q)

$$(1 + B_1)(q_+ - q_-) \leq B_2 - 2q_-$$

and

$$e^{A(t-s)} L_c \eta(x, s) = E_x L_c \eta(\xi_{t-s}, s)$$

that

$$\int_0^t \langle e^{A(t-s)} L_c \eta(s), u \rangle ds \leq Bt$$

According to Lemma 2.1 and after taking squares on both sides of the Eq. (2.5) we see that

$$\eta(t, x)^2 \leq 3 \max_y \eta^2(0, y) + 3B^2 t^2 + 3 \left(\int_0^t \langle dM(s), e^{A(t-s)} u \rangle \right)^2 \quad (2.6)$$

Hence to obtain a bound for the occupation numbers of the process we only need to estimate the last term in the inequality in Eq. (2.6).

For any function $g(t, x)$ we introduce the process

$$Y(g)(t) = \langle \eta(t), g(t) \rangle = \sum \eta(x, t) g(x, t)$$

for $g(t) = (g(1, t), \dots, g(\varepsilon^{-1}, t))'$. Note that if $g(t, x) = \phi(\varepsilon x)$, then $\varepsilon Y(g)$ is the density field defined in De Masi and Presutti.⁽²⁾ For the Markov process $Y(g)(t)$, we have

Lemma 2.2. For $t \geq 0$

$$Y(g)(t) - Y(g)(0) - \int_0^t \gamma_1(s) ds = \int_0^t \langle dM(s), g(s) \rangle$$

$$\left(\int_0^t \langle dM(s), g(s) \rangle \right)^2 - \int_0^t \gamma_2(s) ds = N(t)$$

where

$$\gamma_1(s) = LY(g)(s) + \frac{dY(g)}{dt}(s)$$

$$\gamma_2(s) = LY^2(g)(s) - 2Y(g)(s)LY(g)(s)$$

and $N(t)$ is a martingale.

Proof. The conclusion of this lemma is just (2.18), (2.19), (2.21), and (2.22) in De Masi and Presutti.⁽²⁾ □

Letting $g(s, x) = e^{A(t-s)u}$, we obtain from Lemma 2.2 by some elementary computations that

$$\begin{aligned} \gamma_2(s) &= \frac{1}{4} \sum_y \eta(y, s) \left[\left(\frac{P_x(\xi_{t-s} = y+1) - P_x(\xi_{t-s} = y)}{\varepsilon} \right)^2 \right. \\ &\quad \left. + \left(\frac{P_x(\xi_{t-s} = y-1) - P_x(\xi_{t-s} = y)}{\varepsilon} \right)^2 \right] \\ &\quad + \sum_y [q_-(\eta(y, s)) + q_+(\eta(y, s))] P_x(\xi_{t-s} = y) \\ &= \gamma_2^0(s) + \gamma_2^c(s) \end{aligned} \tag{2.7}$$

(For the first term here, see (3.12) in De Masi and Presutti⁽²⁾).

From Lemma 2.2, inequality in Eq. (2.6) and equality in Eq. (2.7) we see that

$$E_\eta \eta(x, t)^2 \leq 3 \max_y \eta^2(y, 0) + 3B^2 t^2 + 3E_\eta \int_0^t (\gamma_2^0(s) + \gamma_2^c(s)) ds \quad (2.8)$$

Here E_η is the expectation for the process $\eta(x, t)$ starting from η .

Lemma 2.3. For $\gamma_2^0(s)$ and $\gamma_2^c(s)$ defined by the equality in Eq. (2.7) we have

$$\begin{aligned} E_\eta \int_0^t \gamma_2^0(s) ds &\leq B_3 \varepsilon^2 \left(\varepsilon \sum_x \eta(x, 0) + Bt^2/2 \right) \\ \left(E_\eta \int_0^t \gamma_2^c(s) ds \right)^2 &\leq 8B_1^2 E_\eta \eta^2(t, x) + 8B_1^2 \max_y \eta^2(0, y) \\ &\quad + 8E_\eta \int_0^t \gamma_2^0(s) ds + 4B_1^2 + 8B_2^2 t^2 \end{aligned}$$

for the constants B_1, B_2 from the assumption (Q), $B = B_2/(1 + B_1)$ and a constant B_3 , which is independent of ε .

Proof. From equality in Eq. (2.7) we have

$$\begin{aligned} E_\eta \int_0^t \gamma_2^0(s) ds &= E_\eta \int_0^t \frac{1}{4} \sum_y \eta(s, y) \left[\left(\frac{P_x(\xi_{t-s} = y + 1) - P_x(\xi_{t-s} = y)}{\varepsilon} \right)^2 \right. \\ &\quad \left. + \left(\frac{P_x(\xi_{t-s} = y - 1) - P_x(\xi_{t-s} = y)}{\varepsilon} \right)^2 \right] ds \end{aligned}$$

In terms of the local central limit theorem (Feller⁽³⁾; and De Masi and Presutti⁽²⁾),

$$P_x(\xi_t = y) \sim \frac{\varepsilon}{\sqrt{2\pi t}} \exp(-\varepsilon^4(y - x)^2/t^2)$$

we obtain that

$$\left(\frac{P_x(\xi_{t-s} = y + 1) - P_x(\xi_{t-s} = y)}{\varepsilon} \right)^2 \leq B_3 \varepsilon$$

Furthermore, Eq. (2.1) yields that

$$\begin{aligned} \varepsilon \sum_y E_\eta \eta(y, t) - \varepsilon \sum_y E_\eta \eta(y, 0) &= \int_0^t E_\eta L_c \varepsilon \sum_y \eta(y, s) ds \\ &\leq Bt \end{aligned}$$

Therefore

$$\begin{aligned}
 E_\eta \int_0^t \gamma_2^0(s) ds &\leq E_\eta \int_0^t \sum_y \eta(s, y) B_3 \varepsilon \\
 &\leq B_3 \varepsilon \left(\varepsilon \sum_x \eta(x, 0) + Bt^2/2 \right)
 \end{aligned}$$

For the estimate of the term $E_\eta \int_0^t \gamma_2^\varepsilon(s) ds$, we first note that

$$\begin{aligned}
 E_\eta \int_0^t \gamma_2^\varepsilon(s) ds &= E_\eta \int_0^t \sum_y [q_-(\eta(s, y)) + q_+(\eta(s, y))] P_x(\xi_{t-s} = y) ds \\
 &\leq E_\eta \int_0^t -B_1 \left[\sum_y [q_+(\eta(s, y)) - q_-(\eta(s, y))] \right. \\
 &\quad \left. \times P_x(\xi_{t-s} = y) ds + B_2 t \right] \\
 &= -B_1 E_\eta \int_0^t \langle L_c \eta(y, s) ds, e^{A(t-s)} u \rangle ds + B_2 t \\
 &\leq B_1 E_\eta \left| \eta(t, x) - \langle e^{At} \eta(0), u \rangle + \int_0^t \langle dM(s), e^{A(t-s)} u \rangle \right| + B_2 t
 \end{aligned} \tag{2.9}$$

where the first inequality follows from assumption (Q) and the second equation from Eq. (2.4). Taking squares on both sides of Eq. (2.9), using the definition of $\gamma_2(s)$ and the elementary inequality $a^2 \leq b + ac \Rightarrow a^2 \leq 2b + 4c$ we arrive at the conclusion. \square

Let ν_ρ denote the Poisson distribution on \mathbb{N} with parameter ρ , i.e.,

$$\nu_\rho(k) = \frac{\rho^k e^{-\rho}}{k!}$$

Let \mathcal{S} be the unit circle, $\rho(r)$ a smooth positive function $\rho^* = \max_{r \in \mathcal{S}} \rho(r)$, and μ^ε the product measure on $\mathbb{N}^{\mathbb{Z}^\varepsilon}$ with the property

$$\mu^\varepsilon(\eta(x) = k) = \nu_{\rho(\varepsilon x)}(k)$$

for all $x \in \mathbb{Z}^\varepsilon$.

Lemma 2.4 (*A priori bounds*). For any positive integer k and $t \geq 0$, we have

$$\sup_{0 \leq s \leq t} E_{\mu^\varepsilon} \eta^{2k}(s, y) \leq B_4 E_{\nu_\rho} \eta^{2k}(x) + B_5$$

where E_{μ^ε} is the expectation with the initial distribution μ^ε and B_4, B_5 are two constants independent of ε .

Proof. From Eq. (2.5) we have that

$$\begin{aligned} \eta(t, x)^{2k} &= \left(\langle e^{A_t} \eta(0), u \rangle + \int_0^t \langle e^{A(t-s)} L_c \eta(s), u \rangle ds \right. \\ &\quad \left. + \int_0^t \langle e^{A(t-s)} dM(s), u \rangle \right)^{2k} \\ &\leq B_6 (\langle e^{A_t} \eta(0), u \rangle)^{2k} + B_6 \left(\int_0^t \langle e^{A(t-s)} L_c \eta(s), u \rangle ds \right)^{2k} \\ &\quad + B_6 \left(\int_0^t \langle e^{A(t-s)} dM(s), u \rangle \right)^{2k} \end{aligned}$$

where B_6 is a constant which depends only on k . From the Itô formula we conclude that

$$E_\eta \left(\int_0^t \langle e^{A(t-s)} dM(s), u \rangle \right)^{2k} \leq C_1 E_\eta \left(\int_0^t \gamma_2(s) ds \right)^k + C_2$$

for some constants C_1 and C_2 independent of ε . Hence we obtain by the same estimate as in the case $k = 1$ that

$$\sup_{0 \leq s \leq t} E_{\mu^\varepsilon} \eta^{2k}(x, s) \leq B'_4 E_{\nu_p} \max_x \eta^{2k}(x) + B_5$$

By the independence of $\eta(x)$ and Doob's inequality we obtain the lemma. □

In order to establish the boundedness of $E_{\nu_p} q_+^2(\eta(1))$ and $E_{\nu_p} q_-^2(\eta(1))$ we need the following assumption, which is of course true in the case considered by De Masi and Presutti.⁽²⁾

$$\exists k > 0, \text{ such that } q_-(\eta(1)) \leq B_6 \eta^k(1) \text{ for some positive constant } B_6 \tag{Q'}$$

$$E_{\nu_p} \eta^{2k}(1) < \infty$$

and

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log E_{\mu^\varepsilon} \exp \left(\sum_x \eta(x) \right) < \infty$$

The density field $X_t^\varepsilon(\phi)$ is defined as

$$X_t^\varepsilon(\phi) = \varepsilon \sum_{x \in \mathbb{Z}_\varepsilon} \phi(\varepsilon x) \eta(x, t)$$

for $\phi \in \mathcal{S}$, the Schwartz space of all smooth functions on the unit circle \mathcal{S} . Let P^ε be its law on $\mathcal{D}(\mathbb{R}_+, \mathcal{S}')$ (see De Masi and Presutti⁽²⁾).

We have the following theorem which is proved, by the BBGKY hierarchical equations for the correlation functions, in Chapter IV in De Masi and Presutti⁽²⁾ for the case that q_+ and q_- are both polynomials and $\deg q_+ < \deg q_-$.

Theorem 2.1. Under the assumptions (Q) and (Q') and the uniqueness of the solution of the following PDE (2.10), the law P^ε on $\mathcal{D}(\mathbb{R}_+, \mathcal{S}')$ converges weakly to the measure P which is supported on the distribution valued trajectory $\rho(r, t)$ satisfying

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial^2 \rho}{\partial r^2} + F_+(\rho) - F_-(\rho) \tag{2.10}$$

$$\rho(r, 0) = \rho(r)$$

with

$$F_+(\rho) = E_{\nu_\rho} q_+(\eta(1)) \text{ and } F_-(\rho) = E_{\nu_\rho} q_-(\eta(1))$$

The proof of the Theorem will be given in the next section.

3. PROOF OF THE THEOREM

We follow the proof of the hydrodynamic limit for nonlinear diffusion equations in De Masi and Presutti⁽²⁾ where the GPV technique is applied.

Lemma 2.5. For any $\delta > 0$ and $T > 0$ the following equality holds

$$\lim_{\varepsilon \rightarrow 0} P_{\mu^\varepsilon}^\varepsilon \left(\sup_{t \leq T} \left| X_t^\varepsilon(\phi) - X_0^\varepsilon(\phi) - \int_0^t ds \left[\frac{1}{2} \varepsilon \sum_x \phi''(\varepsilon x) \eta(x, s) - \varepsilon \sum_x \phi(\varepsilon x) (q_+(\eta(x, s)) - q_-(\eta(x, s))) \right] \right| > \delta \right) = 0$$

Proof. To prove the lemma it is enough for us to check the boundedness of

$$\begin{aligned}
 & E_{\mu^\varepsilon} [\varepsilon^{-2} L_0 X_i^\varepsilon(\phi) + L_c X_i^\varepsilon(\phi)]^2 \\
 & \leq 2E_{\mu^\varepsilon} \left[\varepsilon^{-2} \varepsilon \sum_{x \in \mathbb{Z}_\varepsilon} \eta(x, t) \cdot \left[\frac{1}{2} \phi(\varepsilon x + \varepsilon) + \frac{1}{2} \phi(\varepsilon x - \varepsilon) - \phi(\varepsilon x) \right] \right]^2 \\
 & \quad + 2E_{\mu^\varepsilon} \left[\varepsilon \sum_{x \in \mathbb{Z}_\varepsilon} (q_+(\eta(x, t)) - q_-(\eta(x, t))) \cdot \phi(\varepsilon x) \right]^2
 \end{aligned}$$

and

$$\begin{aligned}
 & E_{\mu^\varepsilon} [\varepsilon^{-2} L_0 X_i^\varepsilon(\phi)^2 - 2X_i^\varepsilon(\phi) \varepsilon^{-2} L_0 X_i^\varepsilon(\phi) + L_c X_i^\varepsilon(\phi)^2 - 2X_i^\varepsilon(\phi) L_c X_i^\varepsilon(\phi)]^2 \\
 & \leq 2E_{\mu^\varepsilon} \left[\frac{1}{4} \varepsilon^2 \sum_{x \in \mathbb{Z}_\varepsilon} \eta(x, t) \right] \left[\left(\frac{\phi(\varepsilon x + \varepsilon) - \phi(\varepsilon x)}{\varepsilon} \right)^2 \right. \\
 & \quad \left. + \left(\frac{\phi(\varepsilon x - \varepsilon) - \phi(\varepsilon x)}{\varepsilon} \right)^2 \right]^2 \\
 & \quad + 2E_{\mu^\varepsilon} \left[\varepsilon^2 \sum_{x \in \mathbb{Z}_\varepsilon} [q_+(\eta(x, t)) + q_-(\eta(x, t))] \phi(\varepsilon x) \right]^2
 \end{aligned}$$

which is a consequence of the following inequalities

$$\begin{aligned}
 & \sup_{0 \leq s \leq t} E_{\mu^\varepsilon} \eta(x, t)^2 \leq B_4 E_{\mu_\rho} \eta(x)^2 + B_5 \\
 & \sup_{0 \leq s \leq t} E_{\mu^\varepsilon} q_+(\eta(x, t))^2 \leq B_4 B_6 [B_1 E_{\mu_\rho} \eta(x)^{2k} + B_2] + B_5 \\
 & \sup_{0 \leq s \leq t} E_{\mu^\varepsilon} q_-(\eta(x, t))^2 \leq B_4 B_6 E_{\mu_\rho} \eta(x)^{2k} + B_5 \quad \square
 \end{aligned}$$

The essential step in the proof of the Theorem is to replace the terms in Lemma 2.5 by functions of the density field. First of all, we prove that q_+ and q_- can be replaced by some bounded functions.

Lemma 2.6. For any $l > 0$, $\delta > 0$ and $\phi \in \mathcal{S}$, we have that

$$\begin{aligned}
 & \limsup_{R \rightarrow \infty} \limsup_{l \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} P_{\mu^\varepsilon} \left(\int_0^T dt \left| \varepsilon \sum_x \phi(\varepsilon x) q_+(\eta(x, t)) \right. \right. \\
 & \quad \left. \left. - \varepsilon \sum_x \phi(\varepsilon x) \frac{1}{\varepsilon^{-1} l} \sum_{|y-x| \leq (1/2) \varepsilon^{-1} l} q_{+R}(\eta(y, t)) \right| > \delta \right) = 0
 \end{aligned}$$

$$\limsup_{R \rightarrow \infty} \limsup_{I \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} P_{\mu^\varepsilon} \left(\int_0^T dt \left| \varepsilon \sum_x \phi(\varepsilon x) q_-(\eta(x, t)) - \varepsilon \sum_x \phi(\varepsilon x) \frac{1}{\varepsilon^{-1}I} \sum_{|y-x| \leq (1/2)\varepsilon^{-1}I} q_{-R}(\eta(y, t)) \right| > \delta \right) = 0$$

where $q_{+R}(k) = \min(q_+(k), R)$ and $q_{-R}(k) = \min(q_-(k), R)$.

Proof. We only prove the conclusion for function q_- , the proof of the other result is analogous. Note that

$$\begin{aligned} E_{\mu^\varepsilon}(q_-(\eta(x, t)) - q_{-R}(\eta(x, t))) &\leq E_{\mu^\varepsilon} I_{\{\eta(x, t) \geq R\}} q_-(\eta(x, t)) \\ &\leq (E_{\mu^\varepsilon} I_{\{\eta(x, t) \geq R\}})^{1/2} (E_{\mu^\varepsilon} q_-^2(\eta(x, t)))^{1/2} \\ &\leq ([E_{\mu^\varepsilon} \eta(x, t)]/R)^{1/2} (E_{\mu^\varepsilon} q_-^2(\eta(x, t)))^{1/2} \end{aligned}$$

Because of Lemma 2.4, we deduce that $[E_{\mu^\varepsilon} \eta(x, t)]^{1/2}$ and $(E_{\mu^\varepsilon} q_-^2(\eta(x, t)))^{1/2}$ are both bounded uniformly in ε , and therefore

$$E_{\mu^\varepsilon}(q_-(\eta(x, t)) - q_{-R}(\eta(x, t))) \rightarrow 0$$

uniformly in ε and $t \in [0, T]$ as $R \rightarrow \infty$. □

For any space interval I in \mathbb{Z}_ε and bounded cylindrical function f define

$$\begin{aligned} \mathcal{E}(f, I, t) &= |I|^{-1} \sum_{x \in I} f(\eta(x, t)) \\ \tilde{\mathcal{E}}(f, I, t) &= E_{\nu_{z(t)}} f(\eta(1)) \end{aligned}$$

for

$$z(t) = \sum_{k=1}^{\infty} k \nu_{\mathcal{E}(I, t)}(k)$$

and

$$\mathcal{E}(I, t) = |I|^{-1} \sum_{x \in I} \eta(x, t)$$

Lemma 2.7. For any a and t ,

$$\limsup_{I \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log E_{\nu_{\rho^*}} \left(\exp \left(\frac{2}{t} \int_0^t ds [a\varepsilon^{-1} W_s^c - \lambda N_s] \right) \right) \leq C$$

where C is a constant independent of a ,

$$W_s^e = \varepsilon \sum_{x \in \mathbb{Z}_\varepsilon} |\mathcal{E}(f, \tau_x I, t) - \tilde{\mathcal{E}}(f, \tau_x I, t)|$$

for $I = [1, [\varepsilon^{-1}l]]$, τ_x is the translation by x in \mathbb{Z}_ε and $N_s = \sum_{x \in \mathbb{Z}_\varepsilon} \eta(x, s)$.

Proof. The proof is divided into several steps.

Step 1. From the Feynman–Kac formula it is easily seen that

$$\begin{aligned} E_{\nu_{\rho^*}} \exp \left(\frac{2}{t} \int_0^t ds [a\varepsilon^{-1}W_s^e - \lambda N_s] \right) \\ \leq \exp(\varepsilon^{-1}M(te^{-1}L_0 + 2aW_0^e - 2\lambda\varepsilon N_0 + \varepsilon tL_c)) \end{aligned}$$

where $M(\cdot)$ is the maximal eigenvalue of (\cdot) .

Step 2. For any $\psi \in L_2(\mathbb{N}^{\mathbb{Z}_\varepsilon}, \nu_{\rho^*})$, we note that

$$\begin{aligned} M(te^{-1}L_0 + 2aW_0^e - 2\lambda\varepsilon N_0 + \varepsilon tL_c) \\ = \sup_{\langle \psi, \psi \rangle = 1} \langle \psi, (te^{-1}L_0 + 2aW_0^e - 2\lambda\varepsilon N_0 + \varepsilon tL_c) \psi \rangle \end{aligned}$$

Step 3. It is already proved in De Masi and Presutti⁽²⁾ that

$$\limsup_{l \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \sup_{\langle \psi, \psi \rangle = 1} \langle \psi, (te^{-1}L_0 + 2aW_0^e - 2\lambda\varepsilon N_0) \psi \rangle \leq 0$$

So what we need to establish is a bound for

$$\sup_{\langle \psi, \psi \rangle = 1} \langle \psi, \varepsilon tL_c \psi \rangle$$

In fact

$$\begin{aligned} \sup_{\langle \psi, \psi \rangle = 1} \langle \psi, \varepsilon tL_c \psi \rangle &= \sup_{\langle \psi, \psi \rangle = 1} \int dv_{\rho^*} \sum_x \varepsilon t [q_{+R}(\eta(x))(\psi(\eta + \delta_x) - \psi(\eta)) \\ &\quad - q_{-R}(\eta(x))(\psi(\eta - \delta_x) - \psi(\eta))] \psi(\eta) \\ &\leq C \end{aligned}$$

for some positive constant C independent of a . □

From Lemma 2.7 we get the following super-exponential estimate.

Lemma 2.8. For any positive t and δ

$$\limsup_{t \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \left(P_{\nu_\rho}^\varepsilon \left[\frac{1}{t} \int_0^t ds W_s^\varepsilon > \delta \right] \right) = -\infty$$

where W_s^ε is defined as in Lemma 2.7.

Proof. It is easily seen that

$$\begin{aligned} & P_{\nu_\rho}^\varepsilon \left[\frac{1}{t} \int_0^t ds W_s^\varepsilon > \delta \right] \\ & \leq \exp(-\varepsilon^{-1} a \delta) E_{\nu_\rho} \left(\exp \left(t^{-1} \int_0^t ds a \varepsilon^{-1} W_s^\varepsilon \right) \right) \\ & \leq \exp(-\varepsilon^{-1} a \delta) \left[E_{\nu_\rho} \left(\exp \left(2t^{-1} \int_0^t ds [a \varepsilon^{-1} W_s^\varepsilon - \lambda N_s] \right) \right) \right]^{1/2} \\ & \quad \cdot [E_{\nu_\rho}(\exp(2\lambda N_s))]^{1/2} \end{aligned}$$

where $N_s = \sum_x \eta(x, s)$. Note that

$$\varepsilon \log E_{\mu^\varepsilon} \exp(2\lambda N_s) \leq \varepsilon \log E_{\mu^\varepsilon} \exp(2\lambda N_0) + Bt < \infty$$

is bounded (assumption (Q')). Hence

$$\limsup_{t \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log \left(P_{\nu_\rho}^\varepsilon \left[\frac{1}{t} \int_0^t ds W_s^\varepsilon > \delta \right] \right) \leq -a\delta + C$$

for some constant C . Since a is arbitrary, the lemma follows. □

Proof of the Theorem 2.1. The super exponential estimate in Lemma 2.8 allows us to follow De Masi and Presutti⁽²⁾ without any essential difference. So we omit the details. □

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REFERENCES

1. Chen, M. F. (1986). *Jump Processes and Particle Systems*, Beijing Normal University Press (in Chinese).
2. De Masi, A., and Presutti, E. (1991). *Mathematical Methods for Hydrodynamic Limits*, LNM 1501, Springer-Verlag, Berlin.
3. Feller, W. (1957). *An Introduction to Probability and Its Applications*, Vol. I, J. Wiley and Sons, New York.
4. Guo, M. Z., Papanicolaou, G. C., and Varadhan, S. R. S. (1988). Non Linear Diffusion Limit for a System with nearest Neighbor Interactions. *Commun. Math. Phys.* **118**, 31–59.
5. Liggett, T. (1985). *Interacting Particle Systems*, Springer-Verlag, New York.
6. Protter, P. (1990). *Stochastic Integration and Differential Equations*, Springer-Verlag, New York.
7. Spohn, H. (1991). *Large Scale Dynamics of Interacting Particles*, Springer-Verlag, Berlin, Heidelberg.
8. Yau, H. T. (1991). *Relative Entropy and Hydrodynamics of Ginzburg–Landau Models*. *Letters Math. Phys.* **22**, 63–80.