Available online at www.sciencedirect.com
ScienceDirect
Mathematical and Computer Modelling 46 (2007) 680–698
MATHEMATICAL AND COMPUTER MODELLING
www.elsevier.com/locate/mcm

Optimal movement control models of Langevin and Hamiltonian types

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Received 6 November 2006; accepted 29 November 2006

Abstract

We study a class of optimal stochastic control problems arising from the control of movements. Exact solutions are first presented for linear cases for both the during- and post-movement control problem, depending on a parameter $\alpha > 0$. It is found that for the Langevin type equation and for the post-movement control case, a non-degenerate solution exists only when $\alpha > 1/2$. For the Langevin type equation and for the during-movement control, a non-degenerate solution is found when $\alpha > 1$. For the post-movement control and the Hamiltonian type equation, an optimal control signal is obtained and is non-degenerate when $\alpha > 1/2$. Again for the during-movement control, we find an optimal non-degenerate control signal when $\alpha > 1$. All results are then generalized to nonlinear control cases (the first order perturbation of linear cases). Numerical examples are included to illustrate the applications of our results.

Keywords: Movement control; Langevin equation; Saccadic movement; Poisson process

1. Introduction

How a biological system controls its development in its early stages and motor movements later on remains illusive and is an active research area in recent years due to the availability of modern techniques such as microarray gene data in development [3] and multi-unit recordings in nervous systems [9]. In the current paper, we will exclusively concentrate on movement control problems, although the control problem arising from both areas bears many common features.

One of the prominent features of the recorded data from vertebrate nervous systems is the stochasticity due to the fluctuation of single channels (molecular fluctuation) on the membrane of a neuron [4,12]. In the literature, there is a huge body of papers devoted to the issue exploring the functional meaning of the randomness, for example the stochastic resonance approach. In the scenario of stochastic resonance, it is required that the noise should be very

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0895-7177/$ - see front matter
doi:10.1016/j.mcm.2006.11.033

small [5]. However, it is well accepted that the noise in nervous systems is proportional to the signal strength [4, 12]. Hence stochastic resonance results are interesting theoretically, but are limited when applying them to explaining phenomena observed in the nervous system. A few years ago, aiming to reveal the possible functional roles of noise in the nervous system, in [7] the authors proposed a model of optimal stochastic control of movement (saccadic and arm movements) and they found that some observed experimental principles coincide with their model outcomes. The model is now well accepted in the literature and might be of paramount importance in understanding the functional role of noise and in designing humanoid robots. Nevertheless, a systematic and rigorous mathematical analysis of the model and related problems has been missing. In the current paper, we present such a study.

Two classes of model are considered. The first class is the control problem of Langevin type equations and the second is of Hamiltonian type. Assume that the movement is performed during time period 0 to T. We consider two types of control problem: during-movement control, i.e. the control is carried out during a period $[T-R, T] \subset [0, T]$ with $T > R > 0$, and post-movement control, i.e. the control is performed during a period $[T, T+R]$ with $R > 0$. In our approach, a parameter $\alpha > 0$ is introduced to reflect the randomness of the input signal. For post-movement control problems, the optimal control signal is analytically obtained for all cases, but is degenerate when $\alpha \le 1/2$. For during-movement control problems, we again find the optimal control signal for the Langevin type equation and the Hamiltonian type equation, but the optimal control signal is degenerate when $\alpha \le 1/2$ and is asymptotically degenerate when $1/2 < \alpha \le 1$. Furthermore we also extend our results to nonlinear cases (perturbation of linear cases).

The study of optimal stochastic control problems has a long history with wide and successful applications, for example, in financial mathematics [10,8]. However, the problems we address here are novel and different from the issues investigated in the optimal stochastic control literature. In financial mathematics, usually only control problems with Langevin type equation are taken into account. The control problem with Hamiltonian type equations is theoretically more involved and is actually more interesting in some application areas such as robotic control.

2. Models and main results

We consider the stochastic differential equation

$$\begin{cases} \mathrm{d}x(t) = [-\lambda x + u(t) + \varepsilon f(x)]\mathrm{d}t + |u|^{\alpha}(t)\mathrm{d}B(t) \\ x(0) = 0, \end{cases} \tag{2.1}$$

where $B(t)$ is a standard Gaussian white noise, the function $f(x)$ satisfies the Lipschitz conditions:

$$|f(x_1) - f(x_2)| \le C|x_1 - x_2| \tag{2.2}$$

$\alpha > 0$ is a parameter which characterizes the randomness of the input control signal, and the term $u(t)\mathrm{d}t + |u|^{\alpha}\mathrm{d}B(t)$ is the control signal. It is well known in the literature that when $\alpha = 1/2$, the input control signal corresponds to a Poisson process. When $\alpha > 1/2$, the input signal is generated by a supra-Poisson process and $\alpha < 1/2$ is a sub-Poisson process [2,6,12].

The optimal control problem we consider here is to minimize the functional

$$\Phi_{T_1,T_2}[u] = \int_{T_1}^{T_2} D_{1,1}[u](t)\mathrm{d}t, \tag{2.3}$$

where $0 < T_1 < T_2$ are two constants (time),

$$D_{1,1}[u](t) \equiv Ex^2(t) - (Ex(t))^2 \tag{2.4}$$

and $x(t)$ is the solution of (2.1), under the condition

$$M_1[u](T) \equiv Ex(T) = D \tag{2.5}$$

for a constant D (target), $0 < T < T_2$.

For any control signal u we have

$$\Phi_{T_1,T_2}[u] \ge 0. \tag{2.6}$$

Definition 1. If a sequence of $u^{(n)}$ satisfies

$$\lim_{n\to\infty} \Phi_{T_1,T_2}[u^{(n)}] = 0 \tag{2.7}$$

then the sequence $u^{(n)}$ is called asymptotically optimal.

For a constant $R > 0$, we consider the following two cases:

(I) $T_1 = T$, $T_2 = T + R$ i.e. post movement control;
(II) $T_1 = T - R \geq 0$, $T_2 = T$ i.e. during movement control.

For case (I), we want to minimize the variance after we reach the target; for case (II), the control (to minimize the variance) is performed during its movement.

The problem posed above is for Langevin type equations, which is a simplification of the following Hamiltonian type equation Eq. (2.8). Surely the Hamiltonian type equation is more realistic for the actual control of the movement.

$$\begin{cases} \mathrm{d}x(t) = v(t)\mathrm{d}t \\ \mathrm{d}v(t) = (ax(t) + bv(t) + u(t) + \varepsilon f(x(t)))\mathrm{d}t + |u|^{\alpha}(t)\mathrm{d}B(t) \\ x(0) = 0, \quad v(0) = 0. \end{cases} \tag{2.8}$$

In [7], only the model defined by Eq. (2.8) with $\alpha = 1$ and $\epsilon = 0$ is numerically investigated.

Theorem 1. *Consider the optimal control problem defined by Eqs. (2.1)–(2.5) for $\varepsilon = 0$. In both cases* (I) *and* (II) *for $\alpha \leq \frac{1}{2}$ the solution of the problem (2.1)–(2.5) is a δ-function, which is obviously an optimal control signal. For any $\alpha > \frac{1}{2}$ in the case* (I) *and for any $\alpha > 1$ in the case* (II) *there is a unique solution given by*

$$u_0(t) = \begin{cases} 0, & t \notin [0, T] \cap [0, T_2], \\ C\left[\exp(\lambda(t-T))/(2\alpha\varphi_{T_1,T_2}(t))\right]^{1/(2\alpha-1)}, & t \in [0, T] \cap [0, T_2], \end{cases} \tag{2.9}$$

where

$$C = D(2\alpha)^{1/(2\alpha-1)} \left[\int_0^T \exp[2\alpha\lambda(s-T)/(2\alpha-1)]\varphi_{T_1,T_2}^{-1/(2\alpha-1)}(s)\mathrm{d}s\right]^{-1}$$

and

$$\varphi_{T_1,T_2}(s) = (2\lambda)^{-1}\left[\theta(T_1 - s)\mathrm{e}^{2\lambda s}(\mathrm{e}^{-2\lambda T_1} - \mathrm{e}^{-2\lambda T_2}) + \theta(s - T_1)(1 - \mathrm{e}^{2\lambda(s-T_2)})\right].$$

with $\theta(x) = 1$ when $x \geq 0$ and 0 when $x < 0$. For the case (II) *and $1/2 < \alpha \leq 1$, for $t \in [0, T]$, the following sequence*

$$u_0^{(n)}(t) = \begin{cases} C_n(\log n)^{-1}(n^{-1} + T - t)^{-1} & \text{when } \alpha = 1 \\ C_n n \exp[-n(T-t)] & \text{when } 1/2 < \alpha < 1 \end{cases} \tag{2.10}$$

with C_n being a bounded constant satisfying Eq. (2.5) is asymptotically optimal.

Remark 1. For post-movement control and $\alpha > 1/2$, we have

$$u_0(t) = \begin{cases} 0, & t \in [T, T+R], \\ K(\alpha)\mathrm{e}^{-\lambda(t-T)/(2\alpha-1)}, & t \in [0, T], \end{cases} \tag{2.11}$$

where

$$K(\alpha) = \frac{D(2\alpha-2)\lambda}{(2\alpha-1)(1-\mathrm{e}^{-\lambda T(2\alpha-2)/(2\alpha-1)})}.$$

When $\alpha = 1$, the constant $K(1) = \lim_{\alpha\to 1} K(\alpha) = D/T$.

Theorem 2. *Consider the problem Eqs. (2.3)–(2.8) for $\varepsilon = 0$. In both cases* (I) *and* (II) *for $\alpha \le \frac{1}{2}$ the solution of the problem (2.3)–(2.8) is a δ-function. In the case* (I) *with $\alpha > 1/2$ and case* (II) *with $\alpha > 1$, there is a unique solution given by*

$$u_0(t) = \begin{cases} 0, & t \notin [0,T] \cap [0,T_2], \\ C\left[\varphi_1(T-t)/\varphi_{T_1,T_2}(t)\right]^{1/(2\alpha-1)} & t \in [0,T] \cap [0,T_2], \end{cases} \tag{2.12}$$

where

$$\tilde{\varphi}_1(t) = \begin{cases} \dfrac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, & \lambda_1 \ne \lambda_2 \\ t e^{\lambda_1 t}, & \lambda_1 = \lambda_2, \end{cases} \tag{2.13}$$

with λ_1, λ_2 being the eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}, \qquad \lambda_{1,2} = \frac{1}{2}\left(b \pm \sqrt{b^2 + 4a}\right) \tag{2.14}$$

the function $\tilde{\varphi}_{T_1,T_2}(t)$ has the form

$$\tilde{\varphi}_{T_1,T_2}(t) \equiv \theta(T_1 - t)\int_{T_1}^{T_2} \tilde{\varphi}_1^2(s-t)ds + \theta(t - T_1)\int_t^{T_2} \tilde{\varphi}_1^2(s-t)ds, \tag{2.15}$$

and the constant C is defined as

$$C = D\left(\int_0^T ds\,\tilde{\varphi}_1^{2\alpha/(2\alpha-1)}(T-s)\tilde{\varphi}_{T_1,T_2}^{-1/(2\alpha-1)}(s)\right)^{-1}. \tag{2.16}$$

For the case (II) *and $1/2 < \alpha \le 1$, the sequence defined by Eq. (2.10) is asymptotically optimal.*

In the following we will only consider $\alpha > 1/2$.

Theorem 3. *The solution of the problem (2.1)–(2.5) in the case* (I) *up to the first order perturbation terms with respect to ε in the interval $[0, T]$ has the form $u(t) = u_0(t) + \varepsilon u_1(t)$, where $u_0(t)$ is defined by (2.11) and*

$$u_1(t) = \varphi_{T_1,T_2}^{-1}(t)\frac{u_0^{2-2\alpha}(t)}{4\alpha(2\alpha-1)}\left(C^{2\alpha-1}\phi^{(1)}(t;u_0) + 2\alpha C^{2\alpha-1}u_0^{2\alpha-1}(t)\varphi^{(1)}(t;u_0) + C_1 e^{\lambda(t-T)} - 4\alpha u_0^{2\alpha-1}(t)\varphi_{T_1,T_2}^{(1)}(t;u_0) - \phi_{T_1,T_2}^{(1)}(t;u_0)\right), \tag{2.17}$$

where

$$\begin{cases} \varphi_{T_1,T_2}^{(1)}(\tau;u) = \displaystyle\int_\tau^{T_2} e^{2\lambda(\tau-s)}\varphi_{T_1,T_2}(s)\frac{\partial}{\partial\sigma}G_2(\sigma(s),a(s))ds \\ \phi_{T_1,T_2}^{(1)}(\tau;u) = \displaystyle\int_\tau^{T_2} e^{2\lambda(\tau-s)}\varphi_{T_1,T_2}(s)\frac{\partial}{\partial a}G_2(\sigma(s),a(s))ds \\ \varphi^{(1)}(\tau;u) = \displaystyle\int_\tau^{T} e^{\lambda(2\tau-s-T)}\frac{\partial}{\partial\sigma}G_1(\sigma(s),a(s))ds \\ \phi^{(1)}(\tau;u) = \displaystyle\int_\tau^{T} e^{\lambda(\tau-T)}\frac{\partial}{\partial a}G_1(\sigma(s),a(s))ds \end{cases} \tag{2.18}$$

with

$$\varphi_{T_1,T_2}(s) \equiv (2\lambda)^{-1}\left(\theta(T_1-s)e^{2\lambda s}(e^{-2\lambda T_1} - e^{-2\lambda T_2}) + \theta(s-T_1)(1 - e^{2\lambda(s-T_2)})\right), \tag{2.19}$$

$$G_1(\sigma,a) = \frac{1}{\sqrt{2\pi\sigma}}\int e^{-(x-a)^2/2\sigma}f(x)dx, \qquad G_2(\sigma,a) = \frac{1}{\sqrt{2\pi\sigma}}\int (x-a)e^{-(x-a)^2/2\sigma}f(x)dx, \tag{2.20}$$

$$\sigma(t) = \int_0^t \mathrm{e}^{-2\lambda(t-s)}|u_0|^{2\alpha}(s)\mathrm{d}s, \qquad a(t) = \int_0^t \mathrm{e}^{-\lambda(t-s)}u_0(s)\mathrm{d}s \tag{2.21}$$

and the constant C_1 is defined from the condition

$$\int_0^T \left(\mathrm{e}^{\lambda(\tau-T)}u_1(\tau) + 2\alpha\varphi^{(1)}(\tau;u_0)|u_0|^{2\alpha}(\tau) + \phi^{(1)}(\tau;u_0)u_0(\tau)\right)\mathrm{d}\tau = 0. \tag{2.22}$$

For $t \in [T, T+R]$, $u(t) = \varepsilon^{1/(2\alpha-1)}\overline{u}_1(t)$, where

$$\overline{u}_1(t) = -\operatorname{sign}\phi^{(1)}_{T_1,T_2}(t;u_0)\left(\frac{|\phi^{(1)}_{T_1,T_2}(t;u_0)|}{\alpha|\varphi_{T_1,T_2}(t)|}\right)^{1/(2\alpha-1)}. \tag{2.23}$$

For $\alpha > 1$ and case (II)*, the solution of the problem Eqs.* (2.1)–(2.5) *has the form $u(t) = u_0(t) + \varepsilon u_1(t)$, where $u_0(t)$ is defined by* (2.9) *and $u_1(t)$ is defined by* (2.17) *with $T_1 = T - R$, $T_2 = T$.*

Theorem 4. *The solution of the problem Eqs.* (2.3)–(2.8) *in the case* (I) *up to the first order perturbation terms with respect to ε in the interval $[0, T]$ has the form $u(t) = u_0(t) + \varepsilon u_1(t)$, where $u_0(t)$ is defined by* (2.12) *and*

$$\begin{aligned} u_1(t) = {} & \tilde{\varphi}^{-1}_{T_1,T_2}(t)\frac{u_0^{2-2\alpha}(t)}{4\alpha(2\alpha-1)}\Big(C^{2\alpha-1}\tilde{\phi}^{(1)}(t;u_0) + 2\alpha C^{2\alpha-1}u_0^{2\alpha-1}(t)\tilde{\varphi}^{(1)}(t;u_0) \\ & + C_1(\mathrm{e}^{A(t-s)})_{1,1}(\mathrm{e}^{A(t-s)})_{1,2} - 4\alpha u_0^{2\alpha-1}(t)\tilde{\varphi}^{(1)}_{T_1,T_2}(t;u_0) - \tilde{\phi}^{(1)}_{T_1,T_2}(t;u_0)\Big), \end{aligned} \tag{2.24}$$

where

$$\begin{cases} \tilde{\varphi}^{(1)}_{T_1,T_2}(\tau;u) = \displaystyle\int_\tau^{T_2}(\mathrm{e}^{A(s-\tau)})^2_{1,2}\left[\left(\psi_{T_1,T_2}(s) + \tilde{\psi}_{T_1,T_2}(s)\frac{\sigma_{12}(s)}{\sigma_{11}(s)}\right)\frac{\partial}{\partial\sigma}G_2(\sigma_{11}(s),a_1(s))\mathrm{d}s \right. \\ \quad \left. - \tilde{\psi}_{T_1,T_2}(s)\dfrac{\sigma_{12}(s)}{\sigma_{11}(s)}G_2(\sigma_{11}(s),a_1(s))\right]\mathrm{d}s + \displaystyle\int_\tau^{T_2}(\mathrm{e}^{A(s-\tau)})_{1,2}(\mathrm{e}^{A(s-\tau)})_{1,1}\frac{\tilde{\psi}_{T_1,T_2}(s)}{\sigma^2_{11}(s)}G_2(\sigma_{11}(s),a_1(s)) \\ \tilde{\phi}^{(1)}_{T_1,T_2}(\tau;u) = \displaystyle\int_\tau^{T_2}(\mathrm{e}^{A(s-\tau)})_{1,2}\left(\psi_{T_1,T_2}(s) + \tilde{\psi}_{T_1,T_2}(s)\frac{\sigma_{12}(s)}{\sigma_{11}(s)}\right)\frac{\partial}{\partial a}G_2(\sigma_{11}(s),a_1(s))\mathrm{d}s \\ \tilde{\varphi}^{(1)}(\tau;u) = \displaystyle\int_\tau^{T}(\mathrm{e}^{A(T-s)})_{1,2}(\mathrm{e}^{A(s-\tau)})^2_{1,2}\frac{\partial}{\partial\sigma}G_1(\sigma_{11}(s),a_1(s))\mathrm{d}s \\ \tilde{\phi}^{(1)}(\tau;u) = \displaystyle\int_\tau^{T}(\mathrm{e}^{A(T-s)})_{1,2}(\mathrm{e}^{A(s-\tau)})_{1,2}\frac{\partial}{\partial a}G_1(\sigma_{11}(s),a_1(s))\mathrm{d}s \end{cases} \tag{2.25}$$

with $G_{1,2}(\sigma, a)$ defined by (2.20) *and*

$$\psi_{T_1,T_2}(s) = \int_{T_1}^{T_2}\theta(t-s)(\mathrm{e}^{A(t-s)})_{1,1}(\mathrm{e}^{A(t-s)})_{1,2}\mathrm{d}t, \qquad \tilde{\psi}_{T_1,T_2}(s) = \int_{T_1}^{T_2}\theta(t-s)(\mathrm{e}^{A(t-s)})^2_{1,2}\mathrm{d}t, \tag{2.26}$$

$$\sigma_{11}(t) = \int_0^t (\mathrm{e}^{A(t-s)})_{1,2}(\mathrm{e}^{A(t-s)})_{1,2}|u_0|^{2\alpha}(s)\mathrm{d}s, \qquad a_1(t) = \int_0^t(\mathrm{e}^{A(t-s)})_{1,2}u_0(s)\mathrm{d}s. \tag{2.27}$$

The constant C_1 should be defined from the condition

$$\int_0^T\left((\mathrm{e}^{A(T-s)})_{1,2}u_1(\tau) + 2\alpha\tilde{\varphi}^{(1)}(\tau;u_0)|u_0|^{2\alpha}(\tau) + \tilde{\phi}^{(1)}(\tau;u_0)u_0(\tau)\right)\mathrm{d}\tau = 0. \tag{2.28}$$

For $t \in [T, T+R]$ $u(t) = \varepsilon^{1/(2\alpha-1)}\overline{u}_1(t)$, where

$$\overline{u}_1(t) = -\operatorname{sign}\tilde{\phi}^{(1)}_{T_1,T_2}(t;u_0)\left(\frac{|\tilde{\phi}^{(1)}_{T_1,T_2}(t;u_0)|}{\alpha|\tilde{\varphi}_{T_1,T_2}(t)|}\right)^{1/(2\alpha-1)}. \tag{2.29}$$

For $\alpha > 1$ *and case* (II)*, the solution of the problem Eqs.* (2.3)–(2.8) *has the form* $u(t) = u_0(t) + \varepsilon u_1(t)$*, where* $u_0(t)$ *is defined by* (2.12) *and* $u_1(t)$ *is defined by* (2.24) *with* $T_1 = T - R$, $T_2 = T$.

3. Proofs

Proof of Theorem 1. Let $p(x, t \mid y, s)$ be the probability density of transition from y at time s to x at time t of the diffusion process generated by solutions of Eq. (2.1):

$$\mathrm{Prob}\{x(t) \in \Delta \subset \mathbf{R}, \mid x(s) = y\} = \int_{\Delta} p(x, t \mid y, s)\mathrm{d}x. \tag{3.1}$$

Then one can write the direct Kolmogorov equation (or the Fokker–Planck equation, see, e.g. [1]) for the function $p(x, t) = p(x, t \mid 0, 0)$:

$$\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x}((-\lambda x + \varepsilon f(x) + u(t))\, p(x, t)) + \frac{|u|^{2\alpha}(t)}{2}\frac{\partial^2}{\partial x^2} p(x, t) \tag{3.2}$$

with the initial condition

$$p(x, 0) = \delta(x). \tag{3.3}$$

Taking the Fourier transform

$$\hat{p}(k, t) = \int \mathrm{e}^{\mathrm{i}kx} p(x, t)\mathrm{d}x,$$

we obtain from (3.2) the equation for $\hat{p}(k, t)$

$$\frac{\partial}{\partial t}\hat{p}(k, t) = -\lambda k \frac{\partial}{\partial k}\hat{p}(k, t) + \left(\mathrm{i}ku(t) - \frac{k^2}{2}|u|^{2\alpha}(t)\right)\hat{p}(k, t) + \varepsilon \hat{F}(k, t), \tag{3.4}$$

where we denote

$$F(x, t) = -\frac{\partial}{\partial x}(f(x)p(x, t)) \Rightarrow \hat{F}(k, t) = \mathrm{i}k \int \hat{f}(k - k')\hat{p}(k', t)\mathrm{d}k' = \mathrm{i}k\varepsilon \int \hat{p}(k - k', t)\hat{f}(k')\mathrm{d}k'.$$

Let us observe that

$$M_1(t) = \int x p(x, t)\mathrm{d}x = -\mathrm{i}\frac{\partial}{\partial k}\hat{p}(k, t)\bigg|_{k=0}, \tag{3.5}$$

$$M_{1,1}(t) = \int x^2 p(x, t)\mathrm{d}x = -\frac{\partial^2}{\partial k^2}\hat{p}(k, t)\bigg|_{k=0}. \tag{3.6}$$

Thus, taking $-\mathrm{i}\frac{\partial}{\partial k}$ from both parts of (3.4) and then putting $k = 0$, we obtain the equation:

$$M_1'(t) = -\lambda M_1(t) + u(t) + \varepsilon \langle f(x)\rangle (t) \Rightarrow M_1(t) = \int_0^t \mathrm{e}^{-\lambda(t-s)}u(s)\mathrm{d}s + \varepsilon \int_0^t \mathrm{e}^{-\lambda(t-s)} \langle f(x)\rangle (s)\mathrm{d}s. \tag{3.7}$$

Here and below

$$\langle(\ldots)\rangle (t) = \int (\ldots)p(x, t)\mathrm{d}x.$$

Then, taking $\frac{\mathrm{d}^2}{\mathrm{d}k^2}$ from both parts of (3.4) and then putting $k = 0$, we get

$$M_{1,1}'(t) = -2\lambda M_{1,1}(t) + 2u(t)M_1 + 2\varepsilon \langle xf(x)\rangle (t) + |u|^{2\alpha}(t). \tag{3.8}$$

Combining (3.7) with (3.8), we obtain the differential equation for $D_{1,1}$:

$$D_{1,1}'(t) = -2\lambda D_{1,1}(t) + |u|^{2\alpha}(t) + \varepsilon \langle f(x)(x - \langle x\rangle)\rangle (t) \tag{3.9}$$

so that

$$D_{1,1}(t) = \int_0^t \mathrm{d}s \mathrm{e}^{-2\lambda(t-s)} |u|^{2\alpha}(s) + \varepsilon \int_0^t \mathrm{d}s \mathrm{e}^{-2\lambda(t-s)} \langle f(x)(x - \langle x \rangle) \rangle (s).$$

Thus for $\varepsilon = 0$ condition (2.3) takes the form

$$\int_0^T \mathrm{e}^{\lambda(s-T)} u(s)\mathrm{d}s = D, \tag{3.10}$$

and the functional $\Phi_{T_1,T_2}(u)$ takes the form

$$\begin{aligned}\int_{T_1}^{T_2} D_{1,1}[u](t)\mathrm{d}t &= \int_{T_1}^{T_2} \mathrm{d}t \int_0^t \mathrm{d}s \mathrm{e}^{2\lambda(s-t)} |u|^{2\alpha}(s) = \int_0^{T_2}\int_0^{T_2} \mathrm{d}s\mathrm{d}t \theta(t - T_1)\theta(t - s)\mathrm{e}^{2\lambda(s-t)} |u|^{2\alpha}(s) \\ &= \int_0^{T_2} \mathrm{d}s |u|^{2\alpha}(s) \varphi_{T_1,T_2}(s),\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}\varphi_{T_1,T_2}(s) &\equiv \int_{T_1}^{T_2} \mathrm{d}t \mathrm{e}^{2\lambda(s-t)} \theta(t - s)\mathrm{d}t \\ &= (2\lambda)^{-1}(\theta(T_1 - s)\mathrm{e}^{2\lambda s}(\mathrm{e}^{-2\lambda T_1} - \mathrm{e}^{-2\lambda T_2}) + \theta(s - T_1)(1 - \mathrm{e}^{2\lambda(s-T_2)})).\end{aligned}$$

Now to solve the problem (2.1)–(2.5) we use the Lagrange method, according to which we should find such a function $u(\tau)$ and such a constant $\tilde{C}$, that give the minimum to a functional of the form

$$\mathcal{F}[u, \tilde{C}] = \Phi_{T_1,T_2}[u] - \tilde{C}(M_1[u](T) - D). \tag{3.12}$$

Now, taking the derivative of $\mathcal{F}[u, \tilde{C}]$ in the direction $\tilde{u}$ and the derivative with respect to $\tilde{C}$ we get two conditions

$$\begin{cases} \dfrac{\partial}{\partial \mu} \Phi_{T_1,T_2}[u + \mu\tilde{u}]\Big|_{\mu=0} - \tilde{C} \dfrac{\partial}{\partial \mu} M[u + \mu\tilde{u}](T)\Big|_{\mu=0} = 0; \\ M_1[u](T) - D = 0. \end{cases} \tag{3.13}$$

Substituting in the above formulas $\Phi_{T_1,T_2}[u]$ by (3.11) and $M_1[u](T)$ by (3.7) with $\varepsilon = 0$, we obtain that for any $\tilde{u}(s)$

$$2\alpha \int_0^{T_2} \mathrm{d}s\ \mathrm{sign}(u_0(s))|u_0|^{2\alpha-1}(s)\varphi_{T_1,T_2}(s)\tilde{u}(s) - \tilde{C}\int_0^T \mathrm{e}^{\lambda(s-T)}\tilde{u}(s)\mathrm{d}s = 0 \tag{3.14}$$

and taking the derivative with respect to $\tilde{C}$, we get

$$\int_0^T \mathrm{e}^{\lambda(s-T)}\tilde{u}(s)\mathrm{d}s = D. \tag{3.15}$$

Now, making the change of variables $\tilde{C} = C^{2\alpha-1}$, one can see easily that (3.14) implies that

$$\mathrm{sign}(u_0(s))|u_0|^{2\alpha-1}(s) = \begin{cases} 0, & s \notin [0, T] \cap [0, T_2], \\ C^{2\alpha-1}\mathrm{e}^{\lambda(s-T)}/[2\alpha\varphi_{T_1,T_2}(s)] & s \in [0, T] \cap [0, T_2], \end{cases} \tag{3.16}$$

where the constant C is defined by the equation

$$C(2\alpha)^{-1/(2\alpha-1)} \int_0^T \mathrm{e}^{2\alpha\lambda(s-T)/(2\alpha-1)}\varphi_{T_1,T_2}^{-1/(2\alpha-1)}(s)\mathrm{d}s = D. \tag{3.17}$$

Thus in the case (I) we get immediately (2.11) for any $\alpha > \frac{1}{2}$. And in the case (II), since $T_2 = T$, we have that $u(s) \sim |T - s|^{-1/(2\alpha-1)}$. But to have $|u|^{2\alpha}\varphi_{T_1,T_2} \in L[0, T]$ we need

$$-\frac{2\alpha}{2\alpha - 1} + 1 > -1,$$

which implies $\alpha > 1$. The remaining conclusions can be easily verified. □

Proof of Theorem 2. Let us again write the direct Kolmogorov equation for the probability density of transition from (y_1, y_2) at time s to (x_1, x_2) at time t of the diffusion process generated by solutions of Eq. (2.1):

$$\mathrm{Prob}\{\overline{x}(t) \in \Delta_1 \times \Delta_2 \subset \mathbb{R}^2, \mid x(s) = y_1,\ v(s) = y_2\} = \int_{\Delta_1\times\Delta_2} p(\overline{x}, t \mid \overline{y}, s)\mathrm{d}\overline{x}. \tag{3.18}$$

Then one can write the direct Kolmogorov equation (or the Fokker–Planck equation, see, e.g. [1]) for the function $p(\overline{x}, t) = p(\overline{x}, t \mid 0, 0, 0)$:

$$\frac{\partial}{\partial t} p(\overline{x}, t) = -(\nabla_x, p(\overline{x}, t)A\overline{x}) - (u(t) + \varepsilon f(x_1))\frac{\partial}{\partial x_2} p(\overline{x}, t) + \frac{|u|^{2\alpha}(t)}{2}\frac{\partial^2}{\partial x_2^2} p(\overline{x}, t). \tag{3.19}$$

Here the matrix A is defined by (2.14) and $\nabla_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$.

We supply (3.19) with the initial condition

$$p(\overline{x}, 0) = \delta(x_1)\delta(x_2). \tag{3.20}$$

Taking the Fourier transform

$$\hat{p}(\overline{k}, t) = \int \mathrm{e}^{\mathrm{i}(\overline{k},\overline{x})} p(\overline{x}, t)\mathrm{d}\overline{x},$$

we obtain from (3.19) the equation for $\hat{p}(\overline{k}, t)$

$$\frac{\partial}{\partial t}\hat{p}(\overline{k}, t) = (\overline{k}, A\nabla_k \hat{p}(k, t)) + \left(\mathrm{i}k_2 u(t) - \frac{k_2^2}{2}|u|^{2\alpha}(t)\right)\hat{p}(\overline{k}, t) + \varepsilon\hat{F}(\overline{k}, t), \tag{3.21}$$

where

$$F(x, t) = f(x_1)\frac{\partial}{\partial x_2} p(x, t) \Rightarrow \hat{F}(\overline{k}, t) = -\mathrm{i}k_2 \int \hat{p}(\overline{k} - \overline{k}', t) f(k_1')\mathrm{d}\overline{k}'.$$

Below we use also the notations

$$M_i(t) = \int x_i p(\overline{x}, t)\mathrm{d}\overline{x}, \quad M_{i,j}(t) = \int x_i x_j p(\overline{x}, t)\mathrm{d}\overline{x}, \quad D_{i,j}(t) = M_{i,j}(t) - M_i(t)M_j(t), \quad (i, j = 1, 2).$$

One can see easily that

$$M_i(t) = -\mathrm{i}\frac{\partial}{\partial k_i}\hat{p}(\overline{k}, t)\bigg|_{\overline{k}=0}, \qquad M_{i,j}(t) = -\frac{\partial^2}{\partial k_i \partial k_j}\hat{p}(\overline{k}, t)\bigg|_{\overline{k}=0}. \tag{3.22}$$

Hence, taking $-\mathrm{i}\frac{\partial}{\partial k_i}$ from both parts of (3.21), and putting then $\overline{k} = 0$ we get the system of differential equations

$$\begin{cases} M_1'(t) = A_{1,1}M_1(t) + A_{1,2}M_2(t) \\ M_2'(t) = A_{2,1}M_1(t) + A_{2,2}M_2(t) + u(t) + \varepsilon\langle f(x_1)\rangle(t), \\ M_i(t) = \int_0^t (\mathrm{e}^{A(t-s)})_{i,2}u(s)\mathrm{d}s + \varepsilon\int_0^t (\mathrm{e}^{A(t-s)})_{i,2}\langle f(x_1)\rangle(s)\mathrm{d}s \end{cases} \tag{3.23}$$

where the matrix A is defined by (2.14).

Similarly, applying the operation $-\frac{\partial^2}{\partial k_i \partial k_j}$ to both parts of (3.21), and then putting $\overline{k} = 0$, we get the system of differential equations

$$\begin{cases} M_{1,1}'(t) = 2A_{1,1}M_{1,1}(t) + 2A_{1,2}M_{1,2}(t) \\ M_{1,2}'(t) = A_{2,1}M_{1,1}(t) + (A_{2,2} + A_{1,2})M_{1,2}(t) + A_{1,2}M_{2,2}(t) + u(t)M_1(t) + \varepsilon\langle f(x_1)\rangle(t) \\ M_{2,2}'(t) = 2A_{2,1}M_{1,2}(t) + 2A_{2,2}M_{2,2}(t) + 2u(t)M_2(t) + |u|^{2\alpha}(t) + 2\varepsilon\langle f(x_1)\rangle(t). \end{cases} \tag{3.24}$$

Using (3.23) and (3.24), we get

$$\begin{cases} D_{1,1}'(t) = 2A_{1,1}D_{1,1}(t) + 2A_{1,2}D_{1,2}(t) \\ D_{1,2}'(t) = A_{2,1}D_{1,1}(t) + (A_{2,2} + A_{1,2})D_{1,2}(t) + A_{1,2}D_{2,2}(t) + \varepsilon\langle f(x_1)(x_1 - \langle x_1\rangle)\rangle(t) \\ D_{2,2}'(t) = 2A_{2,1}D_{1,2}(t) + 2A_{2,2}D_{2,2}(t) + |u|^{2\alpha}(t) + 2\varepsilon\langle f(x_1)(x_2 - \langle x_2\rangle)\rangle(t). \end{cases} \tag{3.25}$$

Proposition 1. *Define the matrix*

$$\tilde{A} = \begin{pmatrix} 2A_{1,1} & 2A_{1,2} & 0 \\ A_{2,1} & A_{2,2} + A_{1,2} & A_{1,2} \\ 0 & 2A_{2,1} & 2A_{2,2} \end{pmatrix}. \tag{3.26}$$

Define also the operation $\mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^3$

$$\overline{v}^{(1)} \cdot \overline{v}^{(2)} = \left(2v_1^{(1)}v_1^{(2)}, v_1^{(1)}v_2^{(2)} + v_2^{(1)}v_1^{(2)}, 2v_2^{(1)}v_2^{(2)}\right).$$

Then

$$\tilde{A}\overline{v}^{(1)} \cdot \overline{v}^{(2)} = A\overline{v}^{(1)} \cdot \overline{v}^{(2)} + \overline{v}^{(1)} \cdot A\overline{v}^{(2)} \Rightarrow \mathrm{e}^{t\tilde{A}}\overline{v}^{(1)} \cdot \overline{v}^{(2)} = \mathrm{e}^{tA}\overline{v}^{(1)} \cdot \mathrm{e}^{tA}\overline{v}^{(2)}.$$

The proof is straightforward.

Now, introducing $\overline{D}(t) = (D_{1,1}(t), D_{1,2}(t), D_{2,2}(t))$ and $\overline{e}_1 = (1, 0)$, $\overline{e}_2 = (0, 1)$ we rewrite (3.25) as

$$\begin{aligned} \overline{D}'(t) &= \tilde{A}D(t) + \frac{1}{2}|u|^{2\alpha}(t)\overline{e}_2 \cdot \overline{e}_2 + \varepsilon \langle f(x_1)\overline{e}_2 \cdot (\overline{x} - \langle \overline{x} \rangle)\rangle (t) \\ \Rightarrow \overline{D}(t) &= \frac{1}{2}\int_0^t \mathrm{d}s|u|^{2\alpha}(s)\mathrm{e}^{(t-s)\tilde{A}}\overline{e}_2 \cdot \overline{e}_2 = \frac{1}{2}\int_0^t \mathrm{d}s|u|^{2\alpha}(s)\mathrm{e}^{(t-s)A}\overline{e}_2 \cdot \mathrm{e}^{(t-s)A}\overline{e}_2 \\ &\quad + \varepsilon \int_0^t \mathrm{d}s\mathrm{e}^{(t-s)A}\overline{e}_2 \cdot \mathrm{e}^{(t-s)A} \langle f(x_1)(\overline{x} - \langle \overline{x} \rangle)\rangle \\ \Rightarrow D_{1,1}(t) &= \int_0^t \mathrm{d}s(\mathrm{e}^{(t-s)A})_{1,2}^2|u|^{2\alpha}(s) + \varepsilon \int_0^t \mathrm{d}s(\mathrm{e}^{(t-s)A})_{1,2}\left(\mathrm{e}^{(t-s)A}\langle f(x_1)(\overline{x} - \langle \overline{x} \rangle)\rangle (s)\right)_1. \end{aligned} \tag{3.27}$$

Hence, for $\varepsilon = 0$ (3.23) implies that (2.3) takes the form

$$\int_0^T (\mathrm{e}^{A(T-s)})_{1,2}u(s)\mathrm{d}s = D \tag{3.28}$$

and (3.27) implies that the functional $\Phi_{T_1,T_2}[u]$ takes the form

$$\Phi_{T_1,T_2}[u] = \int_{T_1}^{T_2} D_{1,1}(t)\mathrm{d}t = \int_{T_1}^{T_2}\mathrm{d}t\int_0^t (\mathrm{e}^{A(t-s)})_{1,2}^2|u|^{2\alpha}(s)\mathrm{d}s = \int_0^{T_2}\tilde{\varphi}_{T_1,T_2}(s)|u|^{2\alpha}(s)\mathrm{d}s \tag{3.29}$$

where

$$\tilde{\varphi}_{T_1,T_2}(s) \equiv \theta(T_1 - s)\int_{T_1}^{T_2}\tilde{\varphi}_1^2(t-s)\mathrm{d}t + \theta(s - T_1)\int_s^{T_2}\tilde{\varphi}_1^2(t-s)\mathrm{d}t,$$

with

$$\tilde{\varphi}_1(t) = \begin{cases} \dfrac{\mathrm{e}^{\lambda_1 t} - \mathrm{e}^{\lambda_2 t}}{\lambda_1 - \lambda_2}, & \lambda_1 \neq \lambda_2 \\ t\mathrm{e}^{\lambda_1 t}, & \lambda_1 = \lambda_2. \end{cases} \tag{3.30}$$

Using the Lagrange method in the form (3.12) and (3.13), similarly to (3.14) and (3.15), we get that, if u_0 is a minimum point, then

$$\mathrm{sign}(u_0(s))|u_0|^{2\alpha-1}(s) = \begin{cases} 0, & s \notin [0, T] \cap [0, T_2], \\ C^{2\alpha-1}\tilde{\varphi}_1(T-s)/(2\alpha\tilde{\varphi}_{T_1,T_2}(s)) & s \in [0, T] \cap [0, T_2], \end{cases} \tag{3.31}$$

where the constant C is defined by the equation

$$C(2\alpha)^{-1/(2\alpha-1)}\int_0^T \tilde{\varphi}_1^{2\alpha/(2\alpha-1)}(T-s)\tilde{\varphi}_{T_1,T_2}^{-1/(2\alpha-1)}(s)\mathrm{d}s = D. \tag{3.32}$$

Thus we get the representation (2.12)–(2.16) for $u_0(t)$. One can see easily, that in the case (I) ($T_2 > T$) the function $\tilde{\varphi}_{T_1,T_2}(s)$ defined by (2.19) has no zeros in $[0, T]$ and so $u_0(t)$ is a bounded function. But in the case (II) ($T_2 = T$) $\tilde{\varphi}_{T_1,T_2}(s) \sim |s-T|^3$ and $\tilde{\varphi}_1(s-T) \sim |s-T|$ as $s \to T$. Therefore $u_0(s) \sim |s-T|^{-2/(2\alpha-1)}$, and we need:

$$\int_{T_1}^{T_2} D_{1,1}[u](t)\mathrm{d}t = \int_0^T \mathrm{d}s\tilde{\varphi}^2_{T_1,T_2}(s)|u|_0^{2\alpha}(s) < \infty,$$
$$\Rightarrow \int_0^T \mathrm{d}s|s-T|^{2-4\alpha/(2\alpha-1)+1} < \infty$$
$$\Rightarrow 2 - 4\alpha/(2\alpha-1) + 1 > -1,$$

which gives us $\alpha > 1$. The remaining conclusions can be easily verified. □

Proof of Theorem 3. Let us study the equation

$$\frac{\partial}{\partial t}\hat{p}(k,t) = -\lambda k\frac{\partial}{\partial k}\hat{p}(k,t) + \left(\mathrm{i}ku(t) - \frac{k^2}{2}|u|^{2\alpha}(t)\right)\hat{p}(k,t) + \varepsilon\hat{F}(k,t), \tag{3.33}$$

where $\hat{F}(k,t)$ is the Fourier transform of some function belonging to $L_1(\mathbf{R})$ for any $t > 0$.

Lemma 1. *For $\varepsilon = 0$ the solution of (3.33) has the form:*

$$\hat{p}_0(k,t) = \exp\left\{-\frac{1}{2}k^2\sigma(t) + \mathrm{i}ka(t)\right\}, \tag{3.34}$$

where

$$\sigma(t) = \int_0^t \mathrm{e}^{-2\lambda(t-s)}|u|^{2\alpha}(s)\mathrm{d}s, \qquad a(t) = \int_0^t \mathrm{e}^{-\lambda(t-s)}u(s)\mathrm{d}s. \tag{3.35}$$

For $\varepsilon \neq 0$ the solution of (3.33) has the form

$$\begin{aligned}\hat{p}(k,t) &= \hat{p}_0(k,t) + \varepsilon\int_0^t \frac{p_0(k,t)}{p_0(k\mathrm{e}^{\lambda(s-t)},s)}\hat{F}(k\mathrm{e}^{\lambda(s-t)},s)\mathrm{d}s \\ &= \hat{p}_0(k,t) + \varepsilon\int_0^t \exp\left\{-\frac{k^2}{2}\sigma(t,s) + \mathrm{i}ka(t,s)\right\}\hat{F}(k\mathrm{e}^{\lambda(s-t)},s)\mathrm{d}s,\end{aligned} \tag{3.36}$$

where

$$\sigma(t,s) = \sigma(t) - \sigma(s)\mathrm{e}^{2\lambda(s-t)}, \qquad a(t,s) = a(t) - a(s)\mathrm{e}^{\lambda(s-t)}.$$

Proof of Lemma 1. Differentiating with respect to t, we get easily

$$\frac{\partial}{\partial t}\hat{p}_0(k,t) = \left(-\frac{k^2}{2}\sigma'(t) + \mathrm{i}ka'(t)\right)\hat{p}_0(k,t) = \left(-\frac{k^2}{2}|u|^{2\alpha}(t) + \mathrm{i}ku(t) + \lambda k^2\sigma(t) - \mathrm{i}k\lambda a(t)\right)\hat{p}_0(k,t).$$

On the other hand, differentiating with respect to k, we find

$$-\lambda k\frac{\partial}{\partial k}\hat{p}_0(k,t) = (\lambda k^2\sigma(t) - \mathrm{i}k\lambda a(t))\hat{p}_0(k,t).$$

Thus we have proved (3.34).

To prove (3.36) we remark first that, since $p_0(x,t)$ satisfies the initial condition (3.3), we need to find the solution of (3.4), which satisfies the zero initial condition. Let us seek this solution in the form

$$\hat{p}(k,t) = \hat{p}_0(k,t)\tilde{p}(k,t).$$

Then we get

$$\tilde{p}(k,t)\frac{\partial}{\partial t}\hat{p}_0(k,t)+\hat{p}_0(k,t)\frac{\partial}{\partial t}\tilde{p}(k,t)=\left[-\lambda k\frac{\partial}{\partial k}\hat{p}_0(k,t)+\left(-\frac{k^2}{2}|u|^{2\alpha}(t)+\mathrm{i}ku(t)\right)\hat{p}_0(k,t)\right]\tilde{p}(k,t)$$
$$-\lambda k\frac{\partial\tilde{p}}{\partial k}\hat{p}_0(k,t)+\varepsilon\hat{F}(k,t).$$

Thus, using (3.4), we obtain

$$\frac{\partial}{\partial t}\tilde{p}(k,t)=-\lambda k\frac{\partial\tilde{p}}{\partial k}+\varepsilon\hat{F}(k,t)\hat{p}_0^{-1}(k,t). \tag{3.37}$$

Now, by direct calculation one can check easily, that

$$\tilde{p}(k,t)=\varepsilon\int_0^t\hat{F}(k\mathrm{e}^{\lambda(s-t)},s)\hat{p}_0^{-1}(k\mathrm{e}^{\lambda(s-t)},s)\mathrm{d}s.$$

Lemma 1 follows. □

This lemma and formula (3.7) allow us to conclude that to find $M_1[u]$ up to the first order with respect to ε we should replace $\langle f(x)\rangle$ in (3.7) by $\int f(x)p_0(x,t)\mathrm{d}x$. Now, using Lemma 1, we obtain

$$M_1[u](t)=a(t)+\varepsilon\int_0^t\mathrm{e}^{\lambda(s-t)}\mathrm{d}s\int\mathrm{d}x p_0(x,s)f(x)=a(t)+\varepsilon\int_0^t\mathrm{e}^{\lambda(s-t)}G_1(\sigma(s),a(s))+O(\varepsilon^2),$$

where $G_1(\sigma,a)$ is defined by (2.20).

Similarly

$$\Phi_{T_1,T_2}[u]=\int_{T_1}^{T_2}D_{1,1}[u](t)\mathrm{d}t=\int_{T_1}^{T_2}\sigma(t)\mathrm{d}t+2\varepsilon\int_{T_1}^{T_2}\mathrm{d}t\int_0^t\mathrm{e}^{2\lambda(s-t)}G_2(\sigma(s),a(s))\mathrm{d}s+O(\varepsilon^2)$$
$$=\int_{T_1}^{T_2}\sigma(t)\mathrm{d}t+2\varepsilon\int_0^{T_2}\varphi_{T_1,T_2}(s)G_2(\sigma(s),a(s))\mathrm{d}s+O(\varepsilon^2), \tag{3.38}$$

where $G_2(\sigma,a)$ is defined by (2.20) and $\varphi_{T_1,T_2}(s)$ is defined by (2.19). Now to solve the problem (2.1)–(2.5) we use again the Lagrange method, according to which we should find such a function $u(\tau)$ and such a constant $\tilde{C}$, that give the minimum to a functional of the form

$$\mathcal{F}[u,\tilde{C}]=\Phi_{T_1,T_2}[u]-\tilde{C}(M_1[u](T)-D).$$

Thus, we get two conditions:

$$\frac{\partial}{\partial\mu}\Phi_{T_1,T_2}[u+\mu\tilde{u}]\bigg|_{\mu=0}-\tilde{C}\frac{\partial}{\partial\mu}M_1[u+\mu\tilde{u}](T)\bigg|_{\mu=0}=0,\qquad M_1[u](T)-D=0.$$

Taking the above derivative and keeping the terms of order 0 and 1 with respect to ε, we get

$$\frac{\partial}{\partial\mu}\Phi_{T_1,T_2}[u+\mu\tilde{u}]\bigg|_{\mu=0}=2\alpha\int_0^{T_2}\varphi_{T_1,T_2}(\tau)\mathrm{sign}\,u(\tau)|u|^{2\alpha-1}(\tau)\tilde{u}(\tau)\mathrm{d}\tau$$
$$+4\alpha\varepsilon\int_0^{T_2}\mathrm{sign}\,u(\tau)|u|^{2\alpha-1}(\tau)\tilde{u}(\tau)\mathrm{d}\tau\int_0^{T_2}\theta(s-\tau)\mathrm{d}s\varphi_{T_1,T_2}(s)\frac{\partial}{\partial\sigma}G_2(\sigma(s),a(s))\mathrm{e}^{2\lambda(\tau-s)}$$
$$+2\varepsilon\int_0^{T_2}\tilde{u}(\tau)\mathrm{d}\tau\int_0^{T_2}\theta(s-\tau)\varphi_{T_1,T_2}(s)\cdot\frac{\partial}{\partial a}G_2(\sigma(s),a(s))\mathrm{e}^{\lambda(\tau-s)}\mathrm{d}s+O(\varepsilon^2)$$

with $\varphi_{T_1,T_2}(\tau)$ as defined in (3.11). Similarly

$$\frac{\partial}{\partial\mu}M_{u+\mu\tilde{u}}(T)\bigg|_{\mu=0}=\int_0^T\left[\mathrm{e}^{\lambda(\tau-T)}+2\alpha\varepsilon\varphi^{(1)}(\tau;u)\mathrm{sign}\,u(\tau)|u|^{2\alpha-1}(\tau)+\varepsilon\phi^{(1)}(\tau;u)\right]\tilde{u}(\tau)\mathrm{d}\tau. \tag{3.39}$$

Hence, solving the problem (2.1)–(2.5) to first order with respect to ε, we need to find the function u and the constant $\tilde{C}$ such that for any $\tilde{u}$

$$\int_0^T \left\{\left[\left(\varphi_{T_1,T_2}(\tau) + 2\varepsilon\varphi^{(1)}_{T_1,T_2}(\tau;u)\right) 2\alpha\,\mathrm{sign}\,u(\tau)|u|^{2\alpha-1}(\tau) + 2\varepsilon\phi^{(1)}_{T_1,T_2}(\tau;u)\right]\right.$$
$$\left. - \tilde{C}\left[\mathrm{e}^{\lambda(\tau-T)} + 2\alpha\varepsilon\varphi^{(1)}(\tau;u)\mathrm{sign}\,u(\tau)|u|^{2\alpha-1}(\tau) + \varepsilon\phi^{(1)}(\tau;u)\right]\right\}\tilde{u}(\tau)\mathrm{d}\tau$$
$$+ \int_T^{T_2}\left[\left(\varphi_{T_1,T_2}(\tau) + 2\varepsilon\varphi^{(1)}_{T_1,T_2}(\tau;u)\right) 2\alpha\,\mathrm{sign}\,u(\tau)|u|^{2\alpha-1}(\tau) + 2\varepsilon\phi^{(1)}_{T_1,T_2}(\tau;u)\right]\tilde{u}(\tau)\mathrm{d}\tau = O(\varepsilon^2),$$
$$\int_0^T \mathrm{e}^{\lambda(s-T)}(1 + G_1(\sigma(s), a(s)))\mathrm{d}s = D + O(\varepsilon^2) \tag{3.40}$$

where the functions $\varphi^{(1)}_{T_1,T_2}(\tau;u)$, $\phi^{(1)}_{T_1,T_2}(\tau;u)$, $\phi^{(1)}(s;u)$ and $\varphi^{(1)}(\tau;u)$ are given by (2.18). Since we know the zero order approximation for the solution of this problem, it is natural to seek $\tilde{C}$ in the form $\tilde{C} = C^{2\alpha-1} + \varepsilon C_1$, where the constant C is defined from (3.17). Thus, similarly to the proof of Theorem 1, we conclude that for $\tau \in [0, T]$

$$\left(\varphi_{T_1,T_2}(\tau) + 2\varepsilon\varphi^{(1)}_{T_1,T_2}(\tau;u)\right) 2\alpha\,\mathrm{sign}\,u(\tau)|u|^{2\alpha-1}(\tau) + 2\varepsilon\phi^{(1)}_{T_1,T_2}(\tau;u)$$
$$= (C^{2\alpha-1} + \varepsilon C_1)\left(\mathrm{e}^{\lambda(\tau-T)} + 2\alpha\varepsilon\varphi^{(1)}(\tau;u)\mathrm{sign}\,u(\tau)|u|^{2\alpha-1}(\tau) + \varepsilon\phi^{(1)}(\tau;u)\right) + O(\varepsilon^2), \tag{3.41}$$

and for $\tau \in [T, T+R]$

$$\left(\varphi_{T_1,T_2}(\tau) + 2\varepsilon\varphi^{(1)}_{T_1,T_2}(\tau;u)\right) 2\alpha\,\mathrm{sign}\,u(\tau)|u|^{2\alpha-1}(\tau) + 2\varepsilon\phi^{(1)}_{T_1,T_2}(\tau;u) = O(\varepsilon^2)$$
$$\Rightarrow \mathrm{sign}\,u(\tau)|u|^{2\alpha-1}(\tau) = -\varepsilon\alpha^{-1}\varphi^{(1)}_{T_1,T_2}(\tau;u)\varphi^{-1}_{T_1,T_2}(\tau) + O(\varepsilon^2). \tag{3.42}$$

Thus we conclude that the first order approximation terms have the form (2.17) and (2.23) in the intervals $[0, T]$ and $(T, T_2]$ respectively. Now substituting this solution in the last line of (3.40), and keeping terms of order 0 and 1 with respect to ε, we obtain the condition

$$\int_0^T \left(\mathrm{e}^{\lambda(\tau-T)} + 2\alpha\varepsilon\varphi^{(1)}(\tau;u_0)\mathrm{sign}\,u_0(\tau)|u_0|^{2\alpha-1}(\tau) + \varepsilon\phi^{(1)}(\tau;u_0)\right)(u_0(\tau) + \varepsilon u_1(\tau))\mathrm{d}\tau = D + O(\varepsilon^2). \tag{3.43}$$

Hence we get (2.22).

Let us remark that since by the definition (2.19), $\varphi_{T_1,T_2}(\tau) \sim (T_2 - \tau)$, as $\tau \to T_2$, one can see easily from the definition (3.30) that

$$\phi^{(1)}_{T_1,T_2}(\tau;u) \sim (T_2-\tau)^2, \quad (\tau \to T_2).$$

Therefore we can state that $\overline{u}_1(t)$ is a bounded function.

In the case (II) for $\alpha \le 1$, the function $u = u_0 + \varepsilon u_1$, with u_0 given by (2.9) and u_1 given by (2.18), still gives us $\Phi_{T_1,T_2}[u] = \infty$ and so does not correspond to the minimum point of $\Phi_{T_1,T_2}[u]$. But for $\alpha > 1$ this function gives us the solution of our variational problem. □

Proof of Theorem 4. Let us study the equation

$$\frac{\partial}{\partial t}\hat{p}_0(\bar{k},t) = (\bar{k}, A\nabla_k\hat{p}_0(k,t)) + \left(\mathrm{i}k_2 u(t) - \frac{k_2^2}{2}|u|^{2\alpha}(t)\right)\hat{p}_0(\bar{k},t) + \varepsilon\hat{F}(\bar{k},t), \tag{3.44}$$

where $\hat{F}(\bar{k},t)$ is the Fourier transform of some function, belonging in $L_1(\mathbf{R}^2)$ uniformly in t.

Lemma 2. *For $\varepsilon = 0$ the solution of (3.44) has the form*

$$\hat{p}_0(\bar{k},t) = \exp\left\{-\frac{1}{2}(\hat{\sigma}(t)\bar{k},\bar{k}) + \mathrm{i}(\bar{k},\bar{a}(t))\right\}, \tag{3.45}$$

where the matrix elements of $\hat{\sigma}(t)$ have the form

$$\sigma_{ij}(t) = \int_0^t (e^{A(t-s)})_{i,2}(e^{A(t-s)})_{j,2}|u|^{2\alpha}(s)ds, \tag{3.46}$$

and the vector $\overline{a}(t)$ has the components

$$a_i(t) = \int_0^t (e^{A(t-s)})_{i,2}u(s)ds. \tag{3.47}$$

For $\varepsilon = 0$ the solution of (3.44) has the form

$$\begin{aligned} \hat{p}(\overline{k},t) &= \hat{p}_0(\overline{k},t) + \varepsilon \int_0^t \frac{\hat{p}_0(k,t)}{\hat{p}_0(e^{A^\dagger(t-s)}\overline{k},s)} \hat{F}(e^{A^\dagger(t-s)}\overline{k},s)ds \\ &= \hat{p}_0(\overline{k},t) + \varepsilon \int_0^t \exp\left\{-\frac{1}{2}(\hat{\sigma}(t,s)\overline{k},\overline{k}) + i(\overline{k},\overline{a}(t,s))\right\} \hat{F}(e^{A^\dagger(t-s)}\overline{k},s)ds, \end{aligned} \tag{3.48}$$

where $\hat{p}_0(\overline{k},t)$ is defined by (3.45), $A^\dagger$ is the adjoint matrix of A (see (2.14)) and $\hat{\sigma}(t,s)$ and $a(t,s)$ are defined by the relations

$$\hat{\sigma}(t,s) = \hat{\sigma}(t) - e^{A(t-s)}\hat{\sigma}(s)e^{A^\dagger(t-s)}, \qquad \overline{a}(t,s) = \overline{a}(t) - e^{A(t-s)}\overline{a}(s),$$

with $\hat{\sigma}(t)$ and $\overline{a}(t)$ defined above.

Proof. Differentiating with respect to t, we get

$$\frac{\partial}{\partial t}\hat{p}_0(k,t) = \left(-\frac{k_2^2}{2}|u|^{2\alpha}(t) + ik_2u(t) - (A\hat{\sigma}(t)\overline{k},\overline{k}) + i(A\overline{a}(t),k)\right) p_0(\overline{k},t).$$

And differentiating with respect to $\overline{k}$ gives us

$$(\overline{k}, A\nabla_k \hat{p}_0(k,t)) = \left(-(A\hat{\sigma}(t)\overline{k},\overline{k}) + i(A\overline{a}(t),k)\right) p_0(\overline{k},t).$$

Thus we have proved (3.45).

To prove (3.48) let us, as in the case of Eq. (3.33), seek the solution of (3.44) in the form $\hat{p}(\overline{k},t) = \hat{p}_0(\overline{k},t)\tilde{p}(\overline{k},t)$. Then for $\tilde{p}(\overline{k},t)$ we get the equation:

$$\frac{\partial}{\partial t}\tilde{p}_0(\overline{k},t) = (\overline{k}, A\nabla_k \tilde{p}_0(k,t)) + \varepsilon \hat{F}(\overline{k},t)\hat{p}_0^{-1}(\overline{k},t). \tag{3.49}$$

Now direct calculations show us that

$$\tilde{p}(\overline{k},t) = \varepsilon \int_0^t \frac{\hat{F}(e^{A^\dagger(t-s)}\overline{k},s)}{\hat{p}_0(e^{A^\dagger(t-s)}\overline{k},s)}ds.$$

Lemma 2 follows. □

Now, using (3.23) and Lemma 2, we obtain that to find $M_1[u](t)$ up to the first order with respect to ε, we have just to replace $\langle f(x_1)\rangle$ in (3.23) by $\int f(x_1)p_0(\overline{x},t)dx$. Then we get

$$M_1[u](t) = a_1(t) + \varepsilon \int_0^t (e^{A(t-s)})_{1,2}G_1(\hat{\sigma}_{11}(s), a_1(s)) + O(\varepsilon^2),$$

where $G_1(\sigma,a)$ is defined by (2.20).

Similarly, from (3.27) and Lemma 2 we obtain

$$\begin{aligned} M_u^{(2)}(t) &\equiv \int x_1^2 p(x,t) = -\frac{\partial^2}{\partial k_1^2}\hat{p}(\overline{k},t)\Big|_{k=0} = \sigma_{11}(t) + a_1^2(t) - 2i\varepsilon \int_0^t (e^{A(t-s)})_{1,2}ds \\ &\times \int \left(ia_1(t) + (\hat{\sigma}(s)e^{A^\dagger(t-s)})_{1,1}k_1'\right) \exp\left\{-\frac{k_1'^2}{2}\sigma_{11}(s) - ia_1(s)k_1'\right\} \hat{f}(k_1')dk_1' + O(\varepsilon^2) \end{aligned}$$

$$= \sigma_{11}(t) + a_1^2(t) + 2\varepsilon a_1(t)\int_0^t (\mathrm{e}^{A(t-s)})_{1,2} G_1(\sigma_{11}(s), a_1(s))$$
$$+ 2\varepsilon \int_0^t (\mathrm{e}^{A(t-s)})_{1,2} \frac{(\hat\sigma(s)\mathrm{e}^{A^\dagger(t-s)})_{1,1}}{\sigma_{11}} G_2(\sigma_{11}(s), a_1(s))\mathrm{d}s + O(\varepsilon^2),$$

where $G_2(\sigma, a)$ is defined by (2.20). So we obtain

$$\Phi_{T_1,T_2}[u] = \int_{T_1}^{T_2} \sigma_{11}(t)\mathrm{d}t + 2\varepsilon \int_{T_1}^{T_2} \mathrm{d}t \int_0^t (\mathrm{e}^{A(t-s)})_{1,2} \frac{(\hat\sigma(s)\mathrm{e}^{A^\dagger(t-s)})_{1,1}}{\sigma_{11}} G_2(\sigma_{11}(s), a_1(s))\mathrm{d}s + O(\varepsilon^2)$$
$$= \int_{T_1}^{T_2} \sigma_{11}(t)\mathrm{d}t + 2\varepsilon \int_0^{T_2} \left(\psi_{T_1,T_2}(s) + \tilde\psi_{T_1,T_2}(s)\frac{\sigma_{12}(s)}{\sigma_{11}(s)}\right) G_2(\sigma_{11}(s), a_1(s))\mathrm{d}s + O(\varepsilon^2), \tag{3.50}$$

where

$$\psi_{T_1,T_2}(s) = \int_{T_1}^{T_2} \theta(t-s)(\mathrm{e}^{A(t-s)})_{1,1}(\mathrm{e}^{A(t-s)})_{1,2}\mathrm{d}t, \qquad \tilde\psi_{T_1,T_2}(s) = \int_{T_1}^{T_2} \theta(t-s)(\mathrm{e}^{A(t-s)})_{1,2}^2\mathrm{d}t.$$

We use again the Lagrange method in the form (3.12) and (3.13). Taking the derivative of $\Phi_{T_1,T_2}[u]$ in the direction $\tilde u$ and keeping the terms of order 0 and 1 with respect to ε, we get

$$\frac{\partial}{\partial\mu}\Phi_{T_1,T_2}[u+\mu\tilde u]\bigg|_{\mu=0} = 2\alpha \int_0^{T_2} \tilde\varphi_{T_1,T_2}(\tau)\operatorname{sign} u(\tau)|u|^{2\alpha-1}(\tau)\tilde u(\tau)\mathrm{d}\tau$$
$$+ 4\alpha\varepsilon \int_0^{T_2} \operatorname{sign} u(\tau)|u|^{2\alpha-1}(\tau)\tilde u(\tau)\mathrm{d}\tau \int_\tau^{T_2} (\mathrm{e}^{A(s-\tau)})_{1,2}^2$$
$$\times \left[\left(\psi_{T_1,T_2}(s) + \tilde\psi_{T_1,T_2}(s)\frac{\sigma_{12}(s)}{\sigma_{11}(s)}\right)\frac{\partial}{\partial\sigma}G_2(\sigma_{11}(s), a_1(s)) - \tilde\psi_{T_1,T_2}(s)\frac{\sigma_{12}(s)}{\sigma_{11}^2(s)}G_2(\sigma_{11}(s), a_1(s))\right]\mathrm{d}s$$
$$+ 4\alpha\varepsilon \int_0^{T_2} \operatorname{sign} u(\tau)|u|^{2\alpha-1}(\tau)\tilde u(\tau)\mathrm{d}\tau \int_\tau^{T_2} (\mathrm{e}^{A(s-\tau)})_{1,2}(\mathrm{e}^{A(s-\tau)})_{1,1}\tilde\psi_{T_1,T_2}(s)\frac{\tilde\psi_{T_1,T_2}(s)}{\sigma_{11}(s)}\mathrm{d}s$$
$$+ 2\varepsilon \int_0^{T_2} \tilde u(\tau)\mathrm{d}\tau \int_\tau^{T_2} (\mathrm{e}^{A(s-\tau)})_{1,2}\left(\psi_{T_1,T_2}(s) + \tilde\psi_{T_1,T_2}(s)\frac{\sigma_{12}(s)}{\sigma_{11}(s)}\right)\frac{\partial}{\partial a}G_2(\sigma_{11}(s), a_1(s))\mathrm{d}s + O(\varepsilon^2)$$

with $\varphi_{T_1,T_2}(\tau)$ defined in (3.11).

Hence, solving the problem (2.1)–(2.5) in the first order with respect to ε, we need to find a function u and a constant $\tilde C$ such that for any $\tilde u$

$$\int_0^{T_2}\left[\left(\tilde\varphi_{T_1,T_2}(\tau) + 2\varepsilon\tilde\varphi^{(1)}_{T_1,T_2}(\tau;u)\right)2\alpha\operatorname{sign} u(\tau)|u|^{2\alpha-1}(\tau) + 2\varepsilon\tilde\phi^{(1)}_{T_1,T_2}(\tau;u)\right]\tilde u(\tau)\mathrm{d}\tau$$
$$- \tilde C\int_0^T \left[(\mathrm{e}^{A(T-\tau)})_{1,2} + 2\alpha\varepsilon\tilde\varphi^{(1)}(\tau;u)\operatorname{sign} u(\tau)|u|^{2\alpha-1}(\tau) + \varepsilon\tilde\phi^{(1)}(\tau;u)\right]\tilde u(\tau)\mathrm{d}\tau = O(\varepsilon^2),$$
$$\int_0^T \left((\mathrm{e}^{A(T-\tau)})_{1,2}u(s) + \varepsilon G_1(\sigma_{11}(u), a_1(u))\right)\mathrm{d}s = D + O(\varepsilon^2) \tag{3.51}$$

where the functions $\tilde\varphi^{(1)}_{T_1,T_2}(\tau;u)$, $\tilde\phi^{(1)}_{T_1,T_2}(\tau;u)$, $\tilde\phi^{(1)}(s;u)$ and $\tilde\varphi^{(1)}(\tau;u)$ are given by (2.25). Similarly to the proof of Theorem 3 we seek $\tilde C = C^{2\alpha-1} + \varepsilon C_1$, where C is defined by (3.31). Then it is easy to conclude that for $\tau\in[0,T]$

$$\left(\tilde\varphi_{T_1,T_2}(\tau) + 2\varepsilon\tilde\varphi^{(1)}_{T_1,T_2}(\tau;u)\right)2\alpha\operatorname{sign} u(\tau)|u|^{2\alpha-1}(\tau) + 2\varepsilon\tilde\phi^{(1)}_{T_1,T_2}(\tau;u)$$
$$= (C^{2\alpha-1} + \varepsilon C_1)\left(\mathrm{e}^{\lambda(\tau-T)} + 2\alpha\varepsilon\tilde\varphi^{(1)}(\tau;u)\operatorname{sign} u(\tau)|u|^{2\alpha-1}(\tau) + \varepsilon\tilde\phi^{(1)}(\tau;u)\right) + O(\varepsilon^2), \tag{3.52}$$

and for $\tau\in[T, T+R]$

$$\alpha\tilde\varphi_{T_1,T_2}(\tau)\operatorname{sign} u(\tau)|u|^{2\alpha-1}(\tau) + \varepsilon\tilde\phi^{(1)}_{T_1,T_2}(\tau;u) = O(\varepsilon^2). \tag{3.53}$$

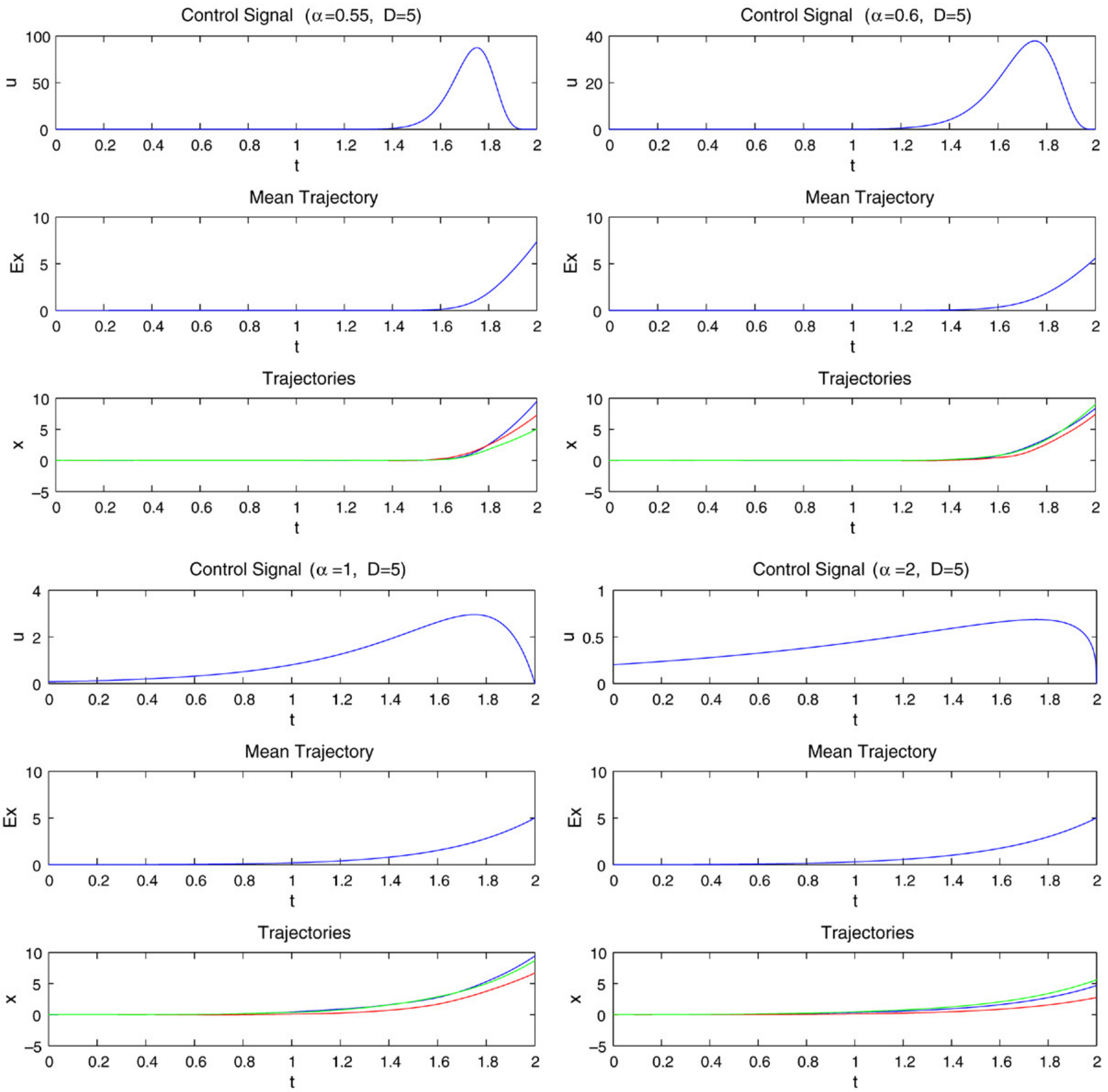

Fig. 1. Post-movement control signal $u(t)$, mean trajectory $Ex(t)$, trajectories $x(t)$ for $\alpha = 0.55, 0.6, 1, 2$ for Hamiltonian type equations. Other parameters are as specified in the context.

If we substitute $u = u_0(t) + \varepsilon u_1(1)$ in the last line of (3.51) then

$$\int_0^T \left[(\mathrm{e}^{A(T-\tau)})_{1,2} + 2\alpha\varepsilon\varphi^{(1)}(\tau; u_0)\operatorname{sign} u_0(\tau)|u_0|^{2\alpha-1}(\tau) + \varepsilon\phi^{(1)}(\tau; u_0) \right] (u_0(\tau) + \varepsilon u_1(\tau))\mathrm{d}\tau$$
$$= D + O(\varepsilon^2). \tag{3.54}$$

Finally, (3.51)–(3.54) in the case (I) give us (2.24) and (2.28).

In the case (II) one can see easily that

$$\tilde{\varphi}_{T_1,T_2}(\tau; u)\tilde{\varphi}^{(1)}_{T_1,T_2}(\tau; u),\ \tilde{\phi}^{(1)}_{T_1,T_2}(\tau; u),\ \tilde{\varphi}^{(1)}(\tau; u), \tilde{\phi}^{(1)}(\tau; u) \sim (T-\tau)^3, \quad \tau \to T.$$

Thus, similarly to Theorem 2, in the case (II) for $\alpha \le 1$ the function $u = u_0 + \varepsilon u_1$ with u_0 given by (2.12) and u_1 given by (2.25) gives us $\Phi_{T_1,T_2}[u] = \infty$ and so does not correspond to the minimum point of $\Phi_{T_1,T_2}[u]$. But for $\alpha > 1$ this function gives us the solution of our variational problem. □

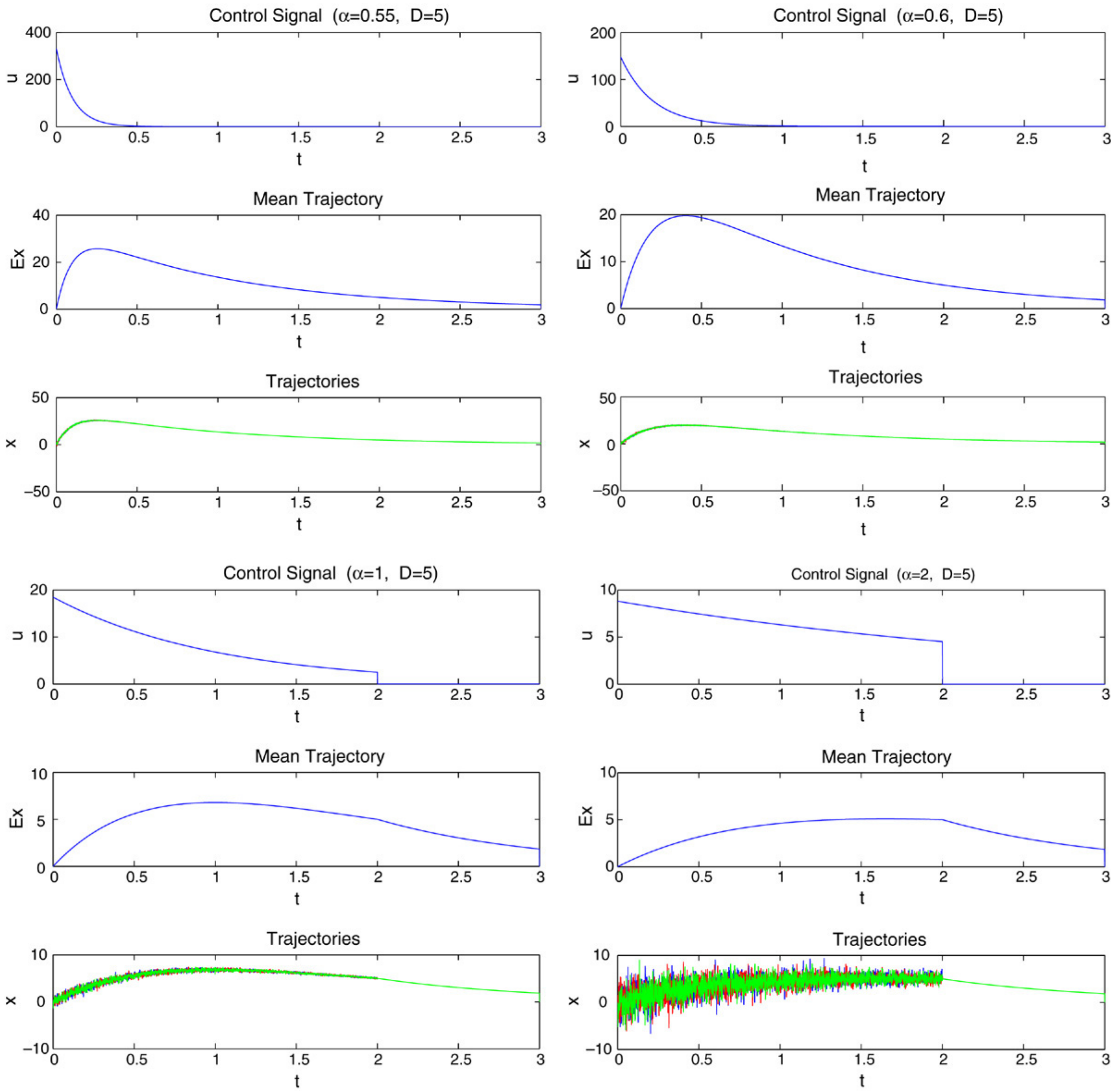

Fig. 2. Control signal $u(t)$, mean trajectory $Ex(t)$, trajectories $x(t)$ for $\alpha = 0.55, 0.6, 1, 2$ for Langevin type equations. Other parameters are as specified in the context.

4. Numerical examples

To elaborate the applications of our theory in the previous sections, we include numerical examples for different parameters, in particular α.

In Fig. 1, we plot the control signal $u(t)$, the mean trajectory $Ex(t)$ and actual trajectories $x(t)$ for $\alpha = 0.55, 0.6, 1, 2$ and $D = 5, T = 2, R = 1$. From Theorem 2, we know that when α approaches 0.5, the solution becomes degenerate, i.e. $u(t)$ is a delta function. This can be easily seen in Fig. 1. When $\alpha = 2$, the maximal value of $u(t)$ is less than 1 and is quite flat, but when $\alpha = 0.55$, $u(t)$ becomes larger and its maximal value is around 100, more like a delta function. In Fig. 1, only three trajectories are plotted for different α and the fluctuations are obvious for different realizations.

In Fig. 2, the optimal control signal, the mean trajectory and actual trajectories are depicted for Langevin type equations. Different from the Hamiltonian case, here the actual trajectories are very noisy (see for example, the case

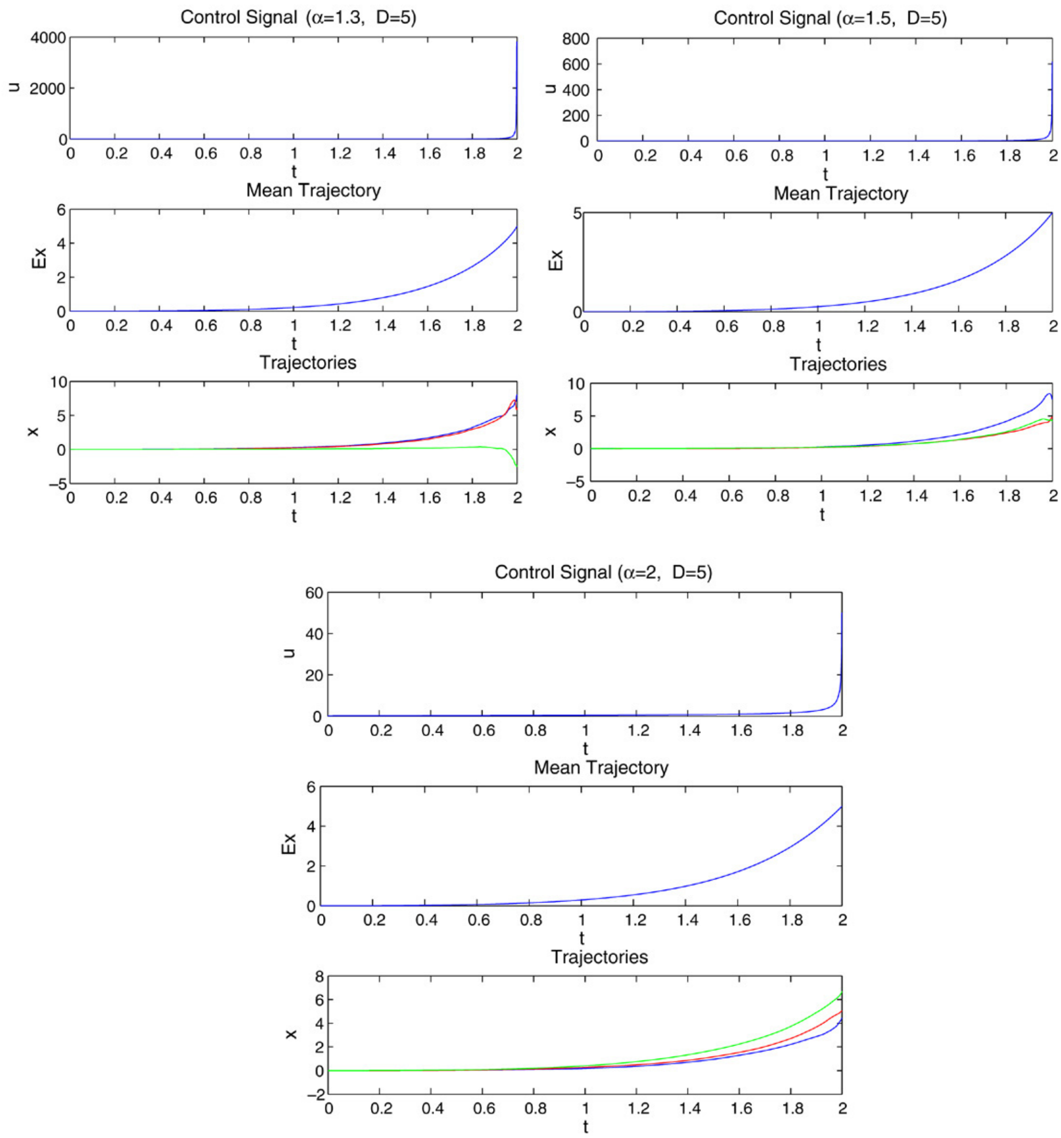

Fig. 3. During-movement control signal $u(t)$, mean trajectory $Ex(t)$, trajectories $x(t)$ for $\alpha = 1.3, 1.5, 2$ for Hamiltonian type equations. Other parameters are as specified in the context.

of $\alpha = 2$). This is due to the fact that the Langevin type equation is an approximation of the real physical system: the Hamiltonian equation. In the Hamiltonian case, the trajectory is an average (integral) of the noisy velocity and is much smoother already.

In Fig. 3, we plot the control signal $u(t)$, the mean trajectory $Ex(t)$ and actual trajectories $x(t)$ for $\alpha = 1.3, 1.5, 2$ and $D = 5, T = 2, R = 1$. From Theorem 2, we know when α approaches 1, the solution becomes degenerate, i.e. $u(t)$ is a delta function. This can be easily seen in Fig. 3 as well.

In Fig. 4, the optimal control signal, the mean trajectory and actual trajectories are depicted for Langevin type equations. From Figs. 1–4 we conclude that post-movement control is much more reliable than during-movement control, as partly revealed in Theorems 1 and 2. This could be an interesting issue and worth further exploring in real-world applications, since most, if not all, optimal control tasks are performed using during-movement control principles, but not post-movement control.

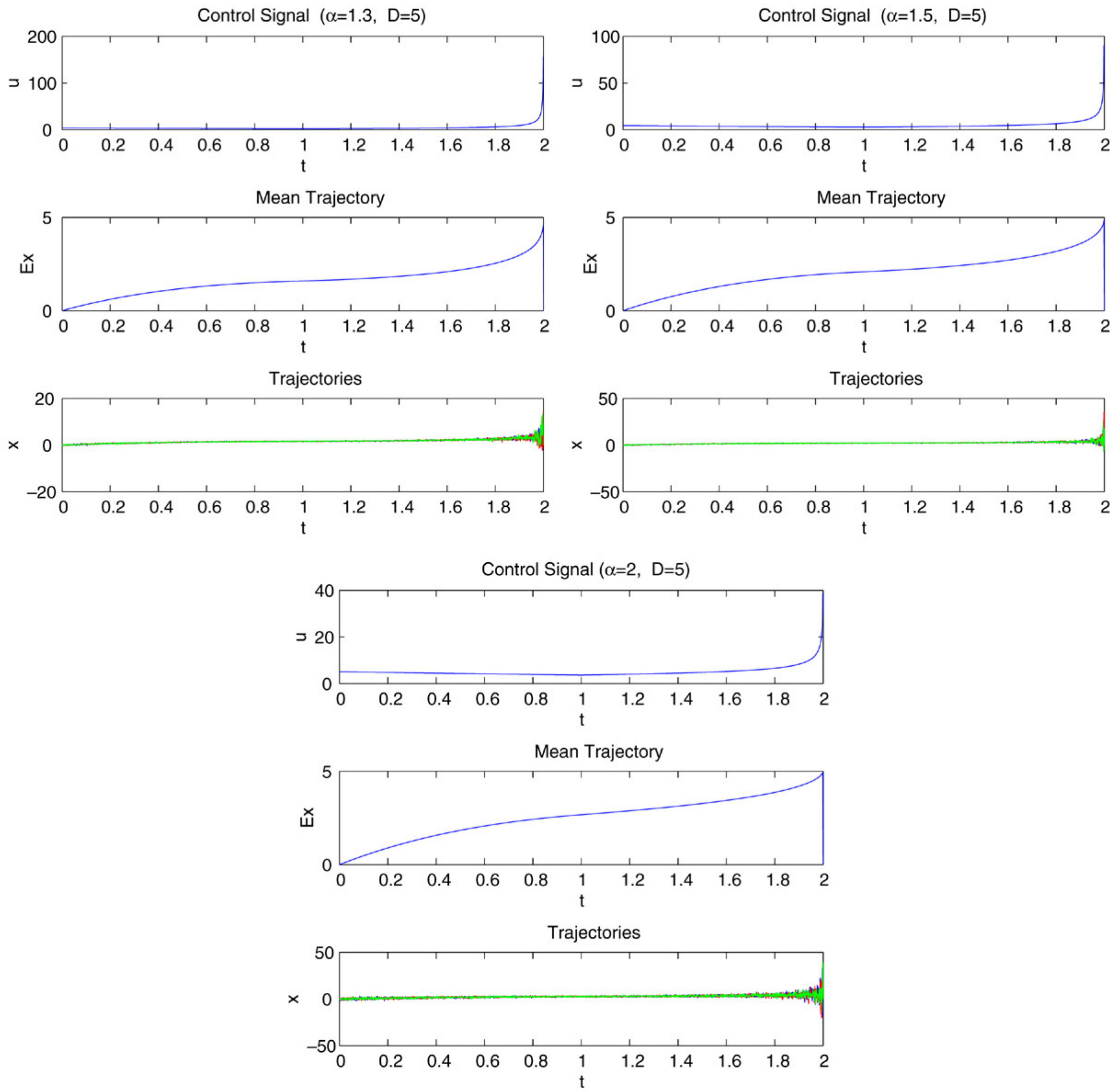

Fig. 4. During-movement control signal $u(t)$, mean trajectory $Ex(t)$, trajectories $x(t)$ for $\alpha = 1.3, 1.5, 2$ for Langevin type equations. Other parameters are as specified in the context.

5. Discussion

We presented a study of the optimal control for a class of models arising from movement control. Analytical solutions for the optimal control signal are obtained for post-movement controls for both the Langevin type and the Hamiltonian type equations. For the during-movement control, we obtained analytical solutions but when $0 < \alpha \le 1$ the solution is degenerate. For the during-movement control, the optimal control signal is degenerate when $\alpha \le 1/2$. All results were then generalized to nonlinear cases.

There are many problems we have not touched on in the current paper. For example, we have only considered open loop control here. A study of a similar model with a feedback control signal is obviously interesting [11]. From the application point of view, how to apply our results here to robotic control is a challenging issue.

Acknowledgement

JF was partially supported by grants from UK EPSRC (R54569), (S20574), (S30443), and (S63830).

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