COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **7**, Number **2**, March **2008**

Website: http://AIMsciences.org

pp. –

DYNAMICAL BEHAVIOUR OF A LARGE COMPLEX SYSTEM

Jianfeng Feng

Department of Computer Science and Mathematics
Warwick University, Coventry CV4 7AL, United Kingdom

Mariya Shcherbina

Institute for Low Temperature Physics
Lenin ave 47, 61103, Ukraine

Brunello Tirozzi

Department of Physics, Rome Univ. "La Sapienza"
P. Aldo Moro 5, 00185 Roma, Italy

(Communicated by Vieri Mastropietro)

Abstract. Limit theorems for a linear dynamical system with random interactions are established. The theorems enable us to characterize the dynamics of a large complex system in details and assess whether a large complex system is weakly stable or unstable (see Definition 1 below).

1. Introduction.

Consider a dynamical system with random interactions (so-called a complex system in [9]) defined by

$$\overline{x}' = -\kappa\overline{x} + \mathbf{A}\overline{x} \tag{1}$$

where $\overline{x} \in \mathbb{R}^n$, κ is a real number and $\mathbf{A}$ is an $n \times n$ real random matrix with entries

$$A_{ij} = n^{-1/2} W_{ij}. \tag{2}$$

The question we ask here is *under what conditions on* $\mathbf{A}$, $\overline{x}$ *is stable* as $n \to \infty$.

Not surprisingly, this simple question has been extensively discussed in the literature and has wide applications in various areas. Early in 1972 [9], Robert May 'answered' the question in one of his Nature papers without proof. May's arguments are based upon the asymptotical behaviour of the maximal eigenvalue of the matrix $\mathbf{A}$. Using results related to Wigner's semi-circle, he concluded that $\overline{x}$ is stable if

$$\kappa > w$$

and unstable if

$$\kappa < w$$

provided that $W_{ij}, i, j = 1, \cdots, n$ are i.i.d. random variables, where w is the finite standard deviation of W_{ij}. In a nice paper published 12 years later Cohen and Newman [2], after a careful investigation of various more complex situations of the matrix $\mathbf{A}$, pointed out that May's criteria above could be false when A does not

2000 *Mathematics Subject Classification.* Primary: 60-xx, 93-xx; Secondary: 93Dxx.

Key words and phrases. Limit theorems, random matrix, Wigner's law.

This work has been supported by a collaboration of the University of Rome and the Ukrainian Academy of Science of Kharkov.

vary with n only by scaling (not Eq. (2)). At the end of their paper (page 309), they emphasized that the question asked at the beginning of the current paper remains open. Furthermore from May's criteria the stability at the critical point $\kappa_c = w$ is not clear.

In the current paper, we aim to establish limit theorems for $\overline{x}$ and shed new lights into the issue discussed above. To facilitate our discussion, we first introduce some notation. We are interested in the statistical distribution of $\{x_i(t)\}_{i=1}^n$ on the real line. To study this distribution for any fixed time t we define a normalized counting function of x_i.

$$N_n(\lambda, t) = n^{-1}\sharp\{x_i(t) \le \lambda\} = n^{-1}\sum_{i=1}^{n} \theta(\lambda - x_i(t)), \tag{3}$$

where $\theta(x)$ is a standard Heaviside function. This function is a distribution of the random discrete measure on the real line. Our goal is to study the behavior of this measure in the limit $n \to \infty$. More precisely, we prove that this measure becomes non-random, as $n \to \infty$ (i.e. the variance of $N_n(\lambda, t)$ tends to zero) and the mean value coincides with the function

$$\lim_{n\to\infty} E\{N_n(\lambda, t)\} = \int_{-\infty}^{\lambda} dx \frac{e^{-(x-a(t))^2/2\sigma(t)}}{\sqrt{2\pi\sigma(t)}}. \tag{4}$$

This means that $N_n(\lambda, t)$ becomes a normal distribution with mean value $a(t)$ and variance $\sigma(t)$. Hence we naturally introduce the following definition.

Definition 1.1. The dynamics $\overline{x}$ is (weakly) stable if and only if $\lim_{t\to\infty} \sigma(t) < \infty$.

We present a necessary and sufficient condition for $\overline{x}$ to be stable. When the matrix $\mathbf{A}$ has elements of i.i.d. random variables in additional to some minor restrictions, $N_n(\lambda, t)$ converges to the normal distribution with mean $a(t) = \exp(-\kappa t)$ and variance

$$\sigma(t) = e^{-2\kappa t} \sum_{m=1}^{\infty} \frac{(wt)^{2m}}{m!m!} \tag{5}$$

where $\overline{x}(0) = (1, 1, \cdots, 1)$. Hence $\overline{x}(t)$ is stable if and only if

$$\lim_{t\to\infty} \sigma(t) < \infty$$

The above results are then proved for the case of symmetric matrix $\mathbf{A}$ ($\sigma(t)$ and $a(t)$ take slightly more complex forms) and generalized to arbitrary initial conditions. Our proofs heavily rely on techniques recently developed in mathematical physics, in particular in the treatment of the Spin Glass model and the Hopfield model[7]. We first establish that the system we consider here has a self-average property and each single variable of the system is normally distributed. Based upon these properties and the homogeneity of the system, our proof is finally reduced to a simple calculation of the mean and variance of a single variable.

The applications of our theorems could be considerably wide, in the current research interests of network properties arising from social science (actor networks, authors networks etc.), biology (gene networks, protein networks, metabolic networks, and neuronal networks etc.) and computer science (internet connections) [4]. For example, we could directly apply our results to networks of neurons, extend our results to networks where the interactions have a long-tail distribution or are dependent such as in small-world networks and scale-free networks. Locally, we can extend our results to nonlinear dynamics which exhibit more rich dynamical activities such as limit cycles and chaos [3, 6].

2. **Results.** Let us consider the system of ordinary differential equations defined by Eq. (1) and (2) with W_{ij} satisfying conditions

$$E\{W_{ij}\} = 0, \quad E\{W_{ij}^2\} = w^2 \tag{6}$$

and there exists $\alpha > 0$ such that

$$\text{Prob}\{|W_{ij}| > \lambda\} \leq Ce^{-C\lambda^\alpha}, \quad (\forall \lambda > 0) \tag{7}$$

Supply the system by the following initial conditions:

$$\overline{x}(0) = \overline{c}, \quad \overline{c} = (1, \ldots, 1) \in \mathbb{R}^n \tag{8}$$

Note that the solution $\overline{x}$ of the dynamics (1) obviously depends on n. We drop the index n whenever there is no confusion. We then have the following theorem.

Theorem 2.1. *Consider the system (1) with a matrix* $\mathbf{A}$ *of the form (2) under conditions (6)and (7), and supply this system by the initial conditions (8). Then for any* $t > 0$, $N_n(\lambda, t)$ *defined by (3) converges in probability to the normal distribution* $N(a(t), \sigma(t))$ *(4) with the mean value*

$$a(t) = e^{-\kappa t} \tag{9}$$

and variance

$$\sigma(t) = e^{-2\kappa t} \sum_{m=1}^{\infty} \frac{(wt)^{2m}}{m!m!}. \tag{10}$$

Remark 1. We know that the series in the expression of $\sigma(t)$ is the Bessel function and it behaves as $\exp(2wt)$. Therefore $\kappa_c = w$, where κ_c is the critical point of the stability of the dynamics (1). On the other hand, it is readily seen from the expression of $\sigma(t)$ that when $\kappa = w$ we have $\lim_{t\to\infty} \sigma(t) < \infty$. Hence $\overline{x}$ is stable iff $\kappa_c \geq w$.

We study also the same problem in the real symmetric case, i.e. the case when $\mathbf{A}$ is a real symmetric matrix ($A_{ij} = A_{ji}$) of the form (2) and W_{ij} ($i \leq j$) are i.i.d. random variables, satisfying conditions (6) and (7).

Theorem 2.2. *Consider the system (1) with a real symmetric matrix* $\mathbf{A}$ ($A_{ij} = A_{ji}$) *of the form (2) under conditions (6)and (7), and supply this system by the initial conditions (8). Then for any* $t > 0$, $N_n(\lambda, t)$ *defined by (3) converges in probability to the normal distribution* $N(a_s(t), \sigma_s(t))$ *(4) with the mean value*

$$a_s(t) = \frac{1}{2\pi w} \int_{-2w}^{2w} \exp\{-\kappa t + \lambda t\} \sqrt{4w^2 - \lambda^2} d\lambda \tag{11}$$

and variance

$$\begin{aligned} \sigma_s(t) &= \frac{1}{2\pi w} \int_{-2w}^{2w} \exp\{-2\kappa t + 2\lambda t\} \sqrt{4w^2 - \lambda^2} d\lambda \\ &\quad - \left(\frac{1}{2\pi w} \int_{-2w}^{2w} \exp\{-\kappa t + \lambda t\} \sqrt{4w^2 - \lambda^2} d\lambda \right)^2 \end{aligned} \tag{12}$$

Our last result is a generalizations of Theorems 2.1 and 2.2 to the case of arbitrary initial distribution. More precisely we study the system (1) in both symmetric and nonsymmetric cases with initial condition

$$\overline{x}(0) = \overline{\xi} = (\xi_1, \dots, \xi_n), \tag{13}$$

where $\{\xi_i\}_{i=1}^n$ are i.i.d. random variables independent of $\{W_{ij}\}_{i,j=1}^n$ with

$$E\{\xi_i\} = a_0, \quad E\{\xi_i^2\} = w_0^2 \neq a_0^2, \quad E\{\xi_i^4\} \leq C. \tag{14}$$

Theorem 2.3. *Consider the system (1) with nonsymmetric matrix* $\mathbf{A}$ $(A_{ij} \neq A_{ji})$ *of the form (2) under conditions (6) and (7), and supply this system by the initial conditions (13) with (14). Then for any* $t > 0$, $N_n(\lambda, t)$ *defined by (3) converges in probability to the distribution of the random variable of the form*

$$y(t) = e^{-\kappa t}\xi_1 + w_0\sigma^{1/2}(t)z \tag{15}$$

where z *is a standard normal random variable independent of* ξ_1 *and* $\sigma(t)$ *is defined by (10).*

If $\mathbf{A}$ *in (1) is a real symmetric matrix* $(A_{ij} = A_{ji})$ *of the form (2) under conditions (6) and (7), then under the initial conditions (13) and (14) and for any* $t > 0$, $N_n(\lambda, t)$ *(defined by (3)) converges in probability to the distribution of the random variable of the form*

$$y_s(t) = a_s(t)\xi_1 + w_0\sigma_s^{1/2}(t)z, \tag{16}$$

where z *is a standard normal random variable independent of* ξ_1 *and* $a_s(t)$ *and* $\sigma_s(t)$ *are defined by (11) and (10) respectively.*

Remark 2. From results above, we see that in a sense our results are more general than May's criteria. We actually completely characterize the dynamical behaviour of $\overline{x}$, independent of whether it is stable or not.

3. **Proofs.**

Remark 3. Let us observe that the change of variables $\tilde{x}_i(t) = e^{-\kappa t}x_i(t)$ allows us everywhere below consider without loss of generality the system (1) with $\kappa = 0$.

Proof of Theorem 2.1. The first step is the proof of the self averaging property of $N_n(\lambda, t)$, as $n \to \infty$, i.e. we prove that for any real λ and $t > 0$

$$\lim_{n\to\infty} E\Big\{\Big(N_n(\lambda, t) - E\{N_n(\lambda, t)\}\Big)^2\Big\} = 0. \tag{17}$$

According to the standard theory of measure, for this aim it is enough to prove that $g_n(z, t)$ – the Stieltjes transform of the distribution $N_n(\lambda, t)$

$$g_n(z, t) = \int \frac{dN_n(\lambda, t)}{\lambda - z} = n^{-1}\sum_{i=1}^n \frac{1}{x_i(t) - z}, \quad (\Im z \neq 0), \tag{18}$$

for any $z : \Im z \neq 0$ possesses a self averaging property, i.e.

$$\lim_{n\to\infty} E\Big\{\Big|g_n(z, t) - E\{g_n(z, t)\}\Big|^2\Big\} = 0 \tag{19}$$

where $\Im z$ is the imaginary part of z. We prove (19) by using a standard method, based on the martingale differences. This method was proposed initially in [5, 11] to prove the self averaging property of the free energy of the Sherrington-Kirkpatrick model of spin glasses. We use it in the form:

Theorem 3.1. *Consider the function* $f(\overline{\xi}_1,\dots,\overline{\xi}_p)$ $f:\mathbb{R}^{\nu_1+\dots+\nu_p}\to\mathbb{C}$, *where* $\overline{\xi}_1\in\mathbb{R}^{\nu_1},\dots,\overline{\xi}_p\in\mathbb{R}^{\nu_p}$ *are independent random vectors. If for any* $k=1,\dots,p$ *there exists a function* $\psi_k(\overline{\xi}_1,\dots,\overline{\xi}_{k-1},\overline{\xi}_{k+1},\dots\overline{\xi}_p)$ *(independent of* $\overline{\xi}_k$*) and such that*

$$E\bigg\{\bigg|f(\overline{\xi}_1,\dots,\overline{\xi}_p)-\psi_k(\overline{\xi}_1,\dots,\overline{\xi}_p)\}\bigg|^2\bigg\}\le C_k, \tag{20}$$

then

$$E\bigg\{\bigg|p^{-1}f(\overline{\xi}_1,\dots,\overline{\xi}_p)-E\{p^{-1}f(\overline{\xi}_1,\dots,\overline{\xi}_p)\}\bigg|^2\bigg\}\le 4p^{-2}\sum_{k=1}^p C_k. \tag{21}$$

Proof of Theorem 3.1

This theorem was proven in [10], but since the proof is very simple we repeat it here for the sake of completeness. Denote E_k the averaging with respect to the random vectors $\overline{\xi}_1,\dots,\overline{\xi}_k$, $E_p=E$ and E_0 means the absence of averaging. Then it is evident that

$$p^{-1}f(\overline{\xi}_1,\dots,\overline{\xi}_p)-E\{p^{-1}f(\overline{\xi}_1,\dots,\overline{\xi}_p)\}=p^{-1}\sum_{k=1}^p\Delta_k,$$

where

$$\Delta_k=E_k\{f(\overline{\xi}_1,\dots,\overline{\xi}_p)\}-E_{k-1}\{f(\overline{\xi}_1,\dots,\overline{\xi}_p)\}.$$

Since evidently for $k<j$ $E\{\Delta_k\overline{\Delta}_j\}=0$, we obtain immediately that

$$\begin{aligned}E\bigg\{\bigg|p^{-1}f(\overline{\xi}_1,\dots,\overline{\xi}_p)-E\{p^{-1}f(\overline{\xi}_1,\dots,\overline{\xi}_p)\}\bigg|^2\bigg\}&=p^{-2}\sum_{k=1}^pE\{|\Delta_k|^2\}\\ \le 2p^{-2}\sum_{k=1}^pE\bigg\{\bigg|E_k\{f(\overline{\xi}_1,\dots,\overline{\xi}_p)-\psi_k(\overline{\xi}_1,\dots,\overline{\xi}_p)\}\bigg|^2\bigg\}&\\ +2p^{-2}\sum_{k=1}^pE\bigg\{\bigg|E_{k-1}\{f(\overline{\xi}_1,\dots,\overline{\xi}_p)-\psi_k(\overline{\xi}_1,\dots,\overline{\xi}_p)\}\bigg|^2\bigg\}&\le 4p^{-2}\sum_{k=1}^pC_k.\end{aligned}$$

Theorem 3.1 is proven. □

Now we use Theorem 3.1 for the proof of (19). Then $p=n$, $\overline{\xi}_k=(W_{k1},\dots,W_{kn})$ and $f=ng_n(z,t)$.

Let us take

$$\psi_k=\sum_{j=1}^n\frac{1}{x_j^{(k)}(t)-z},$$

where $x_j^{(k)}(t)$ are the solutions of the system

$$\overline{x}'=\mathbf{A}^{(k)}\overline{x},\quad \overline{x}(0)=\overline{c}, \tag{22}$$

with the matrix $\mathbf{A}^{(k)}$, whose entries coincide with A_{ij}, if $i\ne k$, $j\ne k$ and are equal to zeros otherwise. It is evident, that ψ_k does not depend on $(W_{k1},\dots,W_{kn})$. So we are left to prove the bound (20). Due to the symmetry of the problem it is enough to prove (20) for $k=1$.

According to the standard theory of differential equations, considering the terms $A_{j1}x_1(t)$ as known functions, we can write for $j=2,\dots,n$

$$x_j(t)=(e^{t\mathbf{A}^{(1)}}\overline{c})_j+\int_0^t\sum_{i=2}^n(e^{(t-s)\mathbf{A}^{(1)}})_{ji}A_{i1}x_1(s)ds=x_j^{(1)}(t)+\tilde{\Delta}_j(t). \tag{23}$$

Let us represent

$$\frac{1}{x_j^{(1)}(t)-z}-\frac{1}{x_j(t)-z}=\frac{\tilde{\Delta}_j(t)}{(x_j^{(1)}(t)-z)^2}-\frac{\tilde{\Delta}_j^2(t)}{(x_j^{(1)}(t)-z)^2(x_j(t)-z)}$$

Then, using this representation for all terms of $(\psi^{(1)}-ng_n(z,t))$, except the first one, we write

$$\psi_1-ng_n(z,t)=\frac{1}{x_1^{(1)}(t)-z}-\frac{1}{x_1(t)-z}+I+II \tag{24}$$

and

$$E\{|I|^2\}=E\bigg\{\int_0^t\int_0^t ds_1ds_2$$

$$\sum_{j_1,j_2,i_1,i_2=2}^n\frac{(e^{(t-s_1)\mathbf{A}^{(1)}})_{j_1i_1}A_{i_11}}{(x_{j_1}^{(1)}(t)-z)^2}\frac{(e^{(t-s_2)\mathbf{A}^{(1)}})_{j_2i_2}A_{i_21}}{(x_{j_2}^{(1)}(t)-z)^2}x_1(s_1)x_1(s_2)\bigg\}$$

$$\le C(t)E^{1/2}\bigg\{\int_0^t x_1^4(s)ds\bigg\}E^{1/2}\bigg\{\sum_{j_1,j_2,i_1,i_2=2}^n\ \sum_{j_1',j_2',i_1',i_2'=2}^n\int_0^t\int_0^t ds_1ds_2$$

$$\frac{(e^{(t-s_1)\mathbf{A}^{(1)}})_{j_1i_1}A_{i_11}}{(x_{j_1}^{(1)}(t)-z)^2}\frac{(e^{(t-s_2)\mathbf{A}^{(1)}})_{j_2i_2}A_{i_21}}{(x_{j_2}^{(1)}(t)-z)^2}\frac{(e^{(t-s_1)\mathbf{A}^{(1)}})_{j_1'i_1'}A_{i_1'1}}{(x_{j_1'}^{(1)}(t)-z)^2}$$

$$\frac{(e^{(t-s_2)\mathbf{A}^{(1)}})_{j_2'i_2'}A_{i_2'1}}{(x_{j_2'}^{(1)}(t)-z)^2}\bigg\} \tag{25}$$

Here and below we use notations $C(t)$ for some independent of n positive functions, which satisfy the bound $C(t)\le e^{ct}$ with some positive t-independent constant c. These functions can be different in different formulas.

Now, since $\mathbf{A}^{(1)}$ and $x_j^{(1)}(t)$ do not depend on A_{i1}, the averaging with respect to all A_{i1} gives us that we have nonzero terms in the last sum only if i_1,i_2,i_1',i_2' are pairwise equal, e.g., $i_1=i_1'$, $i_2=i_2'$. Then, denoting

$$D_j=\frac{1}{(x_j^{(1)}(t)-z)^2},$$

after the summation with respect to i_1,i_2,i_1',i_2' we get

$$E\{|I|^2\}\le C(t)E^{1/2}\bigg\{\int_0^t x_1^4(s)ds\bigg\}E^{1/2}\bigg\{n^{-2}\sum_{j_1,j_2,j_1',j_2'=2}^n\int_0^t\int_0^t ds_1ds_2$$

$$(e^{(t-s_1)\mathbf{A}^{(1)T}}(t)e^{(t-s_1)\mathbf{A}^{(1)}})_{j_1'j_1}D_{j_1}D_{j_1'}(e^{(t-s_2)\mathbf{A}^{(1)T}}e^{(t-s_2)\mathbf{A}^{(1)}})_{j_2'j_2}D_{j_2}D_{j_2'}\bigg\}$$

$$\le C(t)E^{1/2}\bigg\{\int_0^t x_1^4(s)ds\bigg\}E^{1/2}\bigg\{n^{-2}|D|^4e^{4t||\mathbf{A}^{(1)}||}\bigg\}, \tag{26}$$

where $\mathbf{A}^T$ means the transposed matrix of $\mathbf{A}$. Now we use the result of [1], according to which under condition (7) for Hermitian matrix $\mathbf{A}$ with i.i.d. complex elements, such that $A_{ij}=\overline{A}_{ji}$ and $E\{A_{ij}\}=0$, $E\{|A_{ij}|^2\}=w^2$

$$\text{Prob}\{||\mathbf{A}||>2w+\varepsilon\}\le Ce^{-Cn^{\gamma}\varepsilon^{\gamma_1}},\quad \gamma=\frac{\alpha}{2(1+\alpha)},\quad \gamma_1=\frac{\alpha+6}{\alpha+4} \tag{27}$$

So, for non symmetric matrix $\mathbf{A}$ we can write $\mathbf{A} = \mathbf{A}_1 + i\mathbf{A}_2$ with $\mathbf{A}_1 = \frac{1}{2}(\mathbf{A} + \mathbf{A}^*)$ and $\mathbf{A}_2 = \frac{1}{2i}(\mathbf{A} - \mathbf{A}^*)$ being Hermitian matrices with i.i.d. elements, satisfying (7). Then, since $||\mathbf{A}|| \le ||\mathbf{A}_1|| + ||\mathbf{A}_2||$, we can derive from (27) that in non symmetric case

$$\text{Prob}\{||\mathbf{A}|| > 4w + \lambda\} \le Ce^{-Cn^\gamma \lambda^{\gamma_1}}, \quad \gamma = \frac{\alpha}{2(1+\alpha)}, \quad \gamma_1 = \frac{\alpha+6}{\alpha+4} \tag{28}$$

This estimate is rather crude, because it is known that $||\mathbf{A}|| \to 2w$, as $n \to \infty$ (see [8], [12], where the large deviation type bounds was found for $\text{Prob}\{||\mathbf{A}\mathbf{A}^*|| > 4w^2 + \varepsilon\}$ in the case $\alpha \ge 2$ or [13] for the case $a_{ij} = w \pm 1$). But it is enough for our purposes.

Remark 4. Inequality (28) allows us to use $||\mathbf{A}||$ in our considerations like a bounded random variables. Indeed, since, e.g., $|x_1(t)| \le ne^{t||\mathbf{A}||}$, denoting $P_n(\lambda) = Prob\{||\mathbf{A}|| > 4w + \lambda\}$ and using (28), we can write for any fixed t and $m, s << n^\gamma / \log n$

$$\begin{aligned} E\{|x_1(t)|^m e^{s||\mathbf{A}||}\} &\le e^{s(4w+\epsilon)} E\{|x_1(t)|^m \theta(4w + \epsilon - ||\mathbf{A}||)\} \\ &\quad + n^m E\{e^{(s+mt)||\mathbf{A}||} \theta(||\mathbf{A}|| - 4w - 2\epsilon)\} \\ &\le e^{s(4w+\epsilon)} E\{|x_1(t)|^m\} + n^m \int_{\lambda > \epsilon} e^{(s+mt)\lambda} dP_n(\lambda) \\ &\le e^{s(4w+\epsilon)} E\{|x_1(t)|^m\} + O(e^{-Cn^\gamma \varepsilon/2}) \end{aligned}$$

Hence, below we use $||\mathbf{A}||$ as a bounded variable without additional explanations.

Using (28) and the evident bound $|D_j| \le |\Im z|^{-2}$, we get

$$E\{|I|^2\} \le C(t) E^{1/2}\Big\{ \int_0^t x_1^4(s) ds \Big\}. \tag{29}$$

Besides, evidently

$$\begin{aligned} |II| &\le |\Im z|^{-3} \sum_{j=2}^n \tilde{\Delta}_j(t)^2 \\ &= |\Im z|^{-3} \int_0^t \int_0^t x_1(s_1) x_1(s_2) ds_1 ds_2 \sum_{i_1, i_2 = 2}^n (e^{(t-s_1)\mathbf{A}^{(1)T}} e^{(t-s_2)\mathbf{A}^{(1)}})_{i_1 i_2} A_{i_1 1} A_{i_2 1} \\ &\le |\Im z|^{-3} t \int_0^t x_1^2(s) ds\, e^{2t||\mathbf{A}^{(1)}||} n^{-1} \sum_i^n W_{i1}^2. \end{aligned} \tag{30}$$

Thus we get

$$E\{|II|^2\} \le C(t) E^{1/2}\Big\{ \int_0^t x_1^4(s) ds \Big\}. \tag{31}$$

Now we need the following lemma

Lemma 3.2. *Under conditions of Theorem 2.1*

$$E\{x_1^4(t)\} \le C(t) \tag{32}$$

and $x_1(t)$ converges in distribution to a Gaussian random variable.

Proof of Lemma 3.2

Using the first equation in (22) for $k = 1$ and the representation (23), we get

$$x_1'(t) = \sum_{j=2}^{n} \frac{W_{1j}}{n^{1/2}} x_j^{(1)}(t) + \int_0^t ds\, K_n(t-s)x_1(s), \tag{33}$$

where

$$K_n(t-s) = n^{-1} \sum_{i,j=2}^{n} (e^{(t-s)\mathbf{A}^{(1)}})_{ij} W_{1i} W_{j1}. \tag{34}$$

Hence

$$x_1(t) = \phi(t) + \int_0^t ds\, S_n(t-s)x_1(s), \tag{35}$$

with

$$\phi(t) = 1 + \sum_{j=2}^{n} \frac{W_{1j}}{n^{1/2}} d_j(t), \quad d_j(t) = \int_0^t ds\, x_j^{(1)}(s), \quad S_n(t) = \int_0^t d\tau\, K_n(\tau)$$

Making iteration in (35) we get

$$x_1(t) = \phi(t) + \sum_{m=1}^{n_1} \int_0^t ds\, S_n^{(m)}(t-s)\phi(s) + \int_0^t ds\, S_n^{(n_1+1)}(t-s)x_1(s), \tag{36}$$

where $n_1 = [\log^2 n]$ ($[x]$ is the integer part of x) and $S_n^{(m)}(t)$ is defined as

$$S_n^{(1)}(t) = S_n(t), \quad S_n^{(m)}(t) = \int_0^t S_n(t-s) S_n^{(m-1)}(s) ds.$$

Since evidently

$$|S_n(t)| \le \frac{e^{t||\mathbf{A}^{(1)}||} - 1}{||\mathbf{A}^{(1)}||} \Big(n^{-1} \sum_{i=2}^{n} W_{1i}^2 \Big)^{1/2} \Big(n^{-1} \sum_{i=2}^{n} W_{i1}^2 \Big)^{1/2} = \mathbf{K}, \tag{37}$$

we have

$$|S_n^{(m)}(t)| \le \frac{\mathbf{K}^m}{(m-1)!},$$

and so for any $m \ge 2$

$$\Big| \int_0^t ds\, S_n^{(m)}(t-s)\phi(s) \Big| \le \frac{\mathbf{K}^{m-1}}{(m-2)!} \Big| \int_0^t ds\, (S_n(t-s))^2 \Big|^{1/2} \Big| \int_0^t ds\, \phi^2(s) \Big|^{1/2}.$$

Therefore

$$\begin{aligned} & E\Big\{ \Big(\int_0^t ds\, S_n^{(m)}(t-s)\phi(s) \Big)^4 \Big\} \\ \le\ & \frac{1}{((m-2)!)^4} E^{1/2} \Big\{ \Big| \int_0^t ds\, (S_n(t-s))^2 \Big|^4 \Big| \int_0^t ds\, \phi^2(s) \Big|^2 \Big\} \\ & E^{1/2} \Big\{ \Big| \int_0^t ds\, \phi^2(s) \Big|^2 \mathbf{K}^{4(m-1)} \Big\}. \end{aligned} \tag{38}$$

But using definitions (34),(35) and taking into account that $\mathbf{A}^{(1)}$ and $\phi(t)$ do not depend on A_{i1}, we get for any t

$$E\Big\{(S_n(t))^8\int_0^t \phi^4(s)ds\Big\} = E\Big\{\int_0^t d\tau\, n^{-4}\Big(\sum_{i,j=2}^n (e^{\tau\mathbf{A}^{(1)}})_{ij}W_{1i}W_{j1}\Big)^8\int_0^t \phi^4(s)ds\Big\}$$

$$\le n^{-4}C(t)E\Big\{\Big(n^{-1}\sum_{i=2} W_{1i}^2\Big)^4\int_0^t \phi^4(s)ds\Big\} \le n^{-4}C(t)E\Big\{\int_0^t \phi^4(s)ds\Big\}. \quad (39)$$

Hence, it follows from (38), (39) and Remark 4 that for $m \le \log^2 n$

$$E\Big\{\Big(\int_0^t ds\, S_n^{(m)}(t-s)\phi(s)\Big)^4\Big\} \le n^{-2}\frac{C^{4m}(t)}{((m-1)!)^4}E\Big\{\int_0^t \phi^4(s)ds\Big\}. \quad (40)$$

Similarly, using the trivial bound $|x_1(t)| \le ne^{t||\mathbf{A}||}$, we get

$$E\Big\{\Big(\int_0^t ds\, S_n^{(n_1+1)}(t-s)x_1(s)\Big)^4\Big\} \le n^2\frac{C^{4(n_1+1)}(t)}{(n_1!)^4} \le O(n^{-2}). \quad (41)$$

Now, using (40) and the Hölder inequality, we obtain

$$E\Big\{\Big(\sum_{m=1}^{n_1-1}\int_0^t ds\, S_n^{(m)}(t-s)\phi(s)\Big)^4\Big\} \le O(n^{-2})E\Big\{\int_0^t \phi^4(s)ds\Big\}, \quad (42)$$

and so it follows from (36) and (41)

$$E\{x_1^4(t)\} \le CE\{\phi^4(t)\} + O(n^{-2}) \le C + Cn^{-1}\sum_{j=2}^n E\Big\{\Big(\int_0^t ds\, x_j^{(1)}(s)\Big)^2\Big\}+$$

$$C\Big(n^{-1}\sum_{j=2}^n E\Big\{\Big(\int_0^t ds\, x_j^{(1)}(s)\Big)^2\Big\}\Big)^2 + Cn^{-2}\sum_{j=2}^n E\Big\{\Big(\int_0^t ds\, x_j^{(1)}(s)\Big)^4\Big\}+O(n^{-2}). \quad (43)$$

But

$$n^{-1}\sum_{j=2}^n E\Big\{\Big(\int_0^t ds\, x_j^{(1)}(s)\Big)^2\Big\} \le tn^{-1}\int_0^t ds \sum_{j=2}^n E\{(x_j^{(1)}(s))^2\}$$

$$\le tE\Big\{\int_0^t ds\, n^{-1}\sum_{i,j=2}^n (e^{s\mathbf{A}^{(1)T}}e^{s\mathbf{A}^{(1)}})_{ij}\Big\} \le tE\Big\{\int_0^t ds\, e^{2ts||\mathbf{A}^{(1)}||}\Big\} \le C(t)$$

and (23) implies that

$$E\{(x_j^{(1)}(t))^4\} \le C(t)E\{x_j^4(t)\} + C(t)\int_0^t E\{x_1^4(s)\}ds$$

Substituting these bounds in (43) and taking into account that (due to the symmetry) $E\{x_j^4(t)\} = E\{x_1^4(t)\}$, we get

$$E\{x_1^4(t)\} \le C(t) + C(t)n^{-1}\int_0^t E\{x_1^4(s)\}ds$$

So

$$\max_{0\le s\le t} E\{x_1^4(s)\} \le C(t) + tC(t)n^{-1}\max_{0\le s\le t} E\{x_1^4(s)\}$$

Hence, we have proved (32).

The second statement of Lemma 3.2 follows from representation (36), which now, using the bounds (41) and (42), we rewrite as

$$x_1(t) = 1 + \sum_{j=2}^{n} \frac{W_{1j}}{n^{1/2}} d_j(t) + r_n(t), \quad d_j(t) = \int_0^t ds\, x_j^{(1)}(s), \tag{44}$$

where

$$d_j(t) = \int_0^t ds\, x_j^{(1)}(s), \quad E\{r_n^2(t)\} \le C(t) n^{-1}.$$

Now we can apply the central limit theorem, because $d_j(t)$ are independent of $\{W_{1i}\}_{i=2}^n$ and, according to the above considerations,

$$\begin{aligned} n^{-1} \sum_{j=2}^{n} E\{(d_j(t))^4\} &\le C(t) n^{-1} \sum_{j=2}^{n} \int_0^t ds\, E\{(x_j^{(1)}(s))^4)\} \\ &\le C(t) n^{-1} \sum_{j=2}^{n} \int_0^t ds\, E\{x_j^4(s)\} = C(t) \int_0^t ds\, E\{x_1^4(s)\} \le C(t), \end{aligned} \tag{45}$$

so $d_j(t)$ satisfy some kind of the Lindeberg condition. Lemma 3.2 is proven. □

Using Lemma 3.2, one can easily derive (20) from (24), (29) and (31). Thus, we have proved the self averaging of $g_n(z,t)$ (19) and so also the self averaging of $N_n(\lambda, t)$ (17).

Hence, we need to study only $E\{N_n(\lambda, t)\}$. But due to the symmetry of the problem it is easy to see that

$$E\{N_n(\lambda, t)\} = E\{\theta(\lambda - x_1(t))\}.$$

So, $E\{N_n(\lambda, t)\}$ coincides with the distribution $x_1(t)$. But, according to Lemma 3.2, $x_1(t)$ converges in distribution, as $n \to \infty$, to a Gaussian random variable. So, we are left only to find the mean value and the variance of $x_1(t)$.

Using the bound (see the proof of Lemma 3.2)

$$\begin{aligned} & E\Big\{ \Big(\int_0^t ds\, S_n(t-s) x_1(s) \Big)^2 \Big\} \\ \le \quad & E^{1/2}\Big\{ \int_0^t ds\, (S_n(t-s))^4 \Big\} E^{1/2}\Big\{ \int_0^t x_1^4(s) ds \Big\} \le C(t) n^{-1}, \end{aligned} \tag{46}$$

we derive from (35) that

$$E\{x_1(t)\} = 1 + O(n^{-1/2}), \tag{47}$$

So we have proved (9) for $\kappa = 0$. Now using the remark in the beginning of the section, one can easily get (9) for any $\kappa \ne 0$.

To prove (10) define

$$R_n(t,s) = E\{x_1(t) x_1(s)\} = E\{x_j(t) x_j(s)\}. \tag{48}$$

Using representation (35) for $x_1(t)$ and $x_1(s)$ and the bound (46), we obtain

$$R_n(t,s) = 1 + \int_0^t \int_0^s dt' ds' R_n^{(1)}(t', s') + O(n^{-1}), \tag{49}$$

where we denote

$$R_n^{(1)}(t,s) = n^{-1} \sum_{j=2}^{n} E\{x_j^{(1)}(t) x_j^{(1)}(s)\}. \tag{50}$$

But from representation (23) and the inequality (30) we get easily

$$\begin{aligned}|R_n(t,s)-R_n^{(1)}(t,s)| &\le C(s)E^{1/2}\Big\{n^{-1}\sum_{j=2}^n\tilde\Delta_j^2(t)\Big\}+C(t)E^{1/2}\Big\{n^{-1}\sum_{j=2}^n\tilde\Delta_j^2(s)\Big\}\\ &\le (C(t)+C(s))n^{-1/2}.\end{aligned}$$

Thus, we obtain from (49) the equation

$$R_n(t,s)=1+\int_0^t\int_0^s dt'ds'R_n(t',s')+O(n^{-1/2}), \tag{51}$$

Iterating this equation, we find easily

$$\lim_{n\to\infty}R_n(t,s)=1+\sum_{m=1}^\infty\frac{(wt)^m(ws)^m}{m!m!}. \tag{52}$$

In particular,

$$\lim_{n\to\infty}E\{x_1^2(s)\}=1+\sum_{m=1}^\infty\frac{(wt)^{2m}}{m!m!}. \tag{53}$$

Now, using (47), we get (10) for $\kappa=0$. Then, using again the remark in the beginning of the section, it is easy to obtain (10) for any κ. Theorem 2.1 is proven. □

Proof of Theorem 2.2. The first step here is again to prove the self averaging of $N_n(\lambda)$, i.e. the proof of (17) or equivalently (19). This proof almost coincides with that in Theorem 2.1 and therefore we omit it. The only difference is in the proof of the analog of Lemma 3.2.

Lemma 3.3. *Under conditions of Theorem 2.1*

$$E\{x_1^4(t)\}\le C(t), \tag{54}$$

and $x_1(t)$ converges in distribution to a Gaussian random variable.

Proof of Lemma 3.3

As in the case of Lemma 3.2, we use the equation, which can be obtained, if we use the last $n-1$ equations to express $x_j(t)$ $(j=2,\dots,n)$ via $x_1(t)$.

$$x_1'(t)=\sum_{j=2}^n\frac{W_{1j}}{n^{1/2}}x_j^{(1)}(t)+\int_0^t ds\,K_n(t-s)x_1(s)ds, \tag{55}$$

where $x_j^{(1)}(t)$ are the solutions of (22) in the symmetric case with $k=1$,

$$\begin{aligned}K_n(t)&=K_n^0(t)+\tilde K_n(t)\\ K_n^0(t)&=n^{-1}\sum_{i=2}^n(e^{t\mathbf{A}^{(1)}})_{ii}w^2,\\ \tilde K_n(t)&=n^{-1}\sum_{i,j=2,i\ne j}^n(e^{t\mathbf{A}^{(1)}})_{ij}W_{1i}W_{1j}+n^{-1}\sum_{i=2}^n(e^{t\mathbf{A}^{(1)}})_{ii}(W_{1i}^2-w^2),\end{aligned} \tag{56}$$

and here and below $A_{ij}^{(1)}$ coincides with A_{ij}, if $i\ne1, j\ne1$ and is equal to zero otherwise. Hence

$$x_1(t)=\phi(t)+\int_0^t ds\,S_n(t-s)x_1(s) \tag{57}$$

with

$$\begin{aligned} &\phi(t) = 1 + \sum_{j=2}^{n} \frac{W_{1j}}{n^{1/2}} d_j(t), \quad d_j(t) = \int_0^t ds\, x_j^{(1)}(s) ds, \\ &S_n(t) = S_n^0(t) + \tilde{S}_n(t), \quad S_n^0(t) = \int_0^t d\tau K_n^0(\tau), \quad \tilde{S}_n(t) = \int_0^t d\tau \tilde{K}_n(\tau). \end{aligned} \tag{58}$$

Iterating (57) n_1 times ($n_1 = [\log^2 n]$), we get

$$x_1(t) = \phi(t) + \sum_{m=1}^{n_1} \int_0^t S_n^{(m)}(t-s)\phi(s) ds + \int_0^t S_n^{(n_1+1)}(t-s) x_1(s) ds, \tag{59}$$

where $S_n^{(m)}(t)$ is defined in (64) and has the same bound (37). Repeating the conclusions of Lemma 3.2, we obtain finally

$$\begin{aligned} &x_1(t) = \phi(t) + \int_0^t ds\, \hat{S}_n^0(t-s)\phi(s) ds + \int_0^t R_n(t-s)\phi(s) ds + \tilde{r}_n(t), \\ &\tilde{r}_n(t) = \int_0^t S_n^{(n_1+1)}(t-s) x_1(s) ds \end{aligned} \tag{60}$$

where similarly to (41)

$$E\{\tilde{r}_n^4(t)\} \le O(n^{-2})$$

and we denote

$$\begin{aligned} &\hat{S}_n^0(t) = \sum_{m=1}^{n_1} S_n^{(0,m)}(t), \\ &S_n^{(0,1)}(t) = S_n^0(t), \quad S_n^{(0,m)}(t) = \int_0^t S_n^0(t-s) S_n^{(0,m-1)}(s) ds \end{aligned} \tag{61}$$

and $R_n(t)$ is the kernel of the remainder operator, which satisfies the bound

$$E\bigg\{ \bigg(\int_0^t ds\, R_n(t-s)\phi(s) \bigg)^4 \bigg\} \le C(t) E^{1/2} \bigg\{ \bigg| \int_0^t ds\, (\tilde{S}_n(t-s))^2 \bigg|^2 \bigg\}. \tag{62}$$

Here and below we use the result of [1], according to which in the symmetric case under conditions (6), (7) the bound (27) is valid.

But, using definitions (56),(59) and taking into account that $\mathbf{A}^{(1)}$ does not depend on A_{i1}, we get for any t

$$E\left\{ (\tilde{S}_n(t))^4 \right\} \le C(t) n^{-2}. \tag{63}$$

Hence, we derive from (60) and the fact that $\hat{K}_n^0(t)$ does not depend on A_{i1} that

$$\begin{aligned} E\{x_1^4(t)\} \quad \le \quad & C(t) \bigg(n^{-1} \sum_{j=2}^{n} E\bigg\{ \bigg(\int_0^t ds\, x_j^{(1)}(s) \bigg)^2 \bigg\} \bigg)^2 \\ & + C(t) n^{-2} \sum_{j=2}^{n} E\bigg\{ \bigg(\int_0^t ds\, x_j^{(1)}(s) \bigg)^4 \bigg\} + O(n^{-2}). \end{aligned} \tag{64}$$

Then the bound (54) follows by the same way as in Lemma 3.2.

The second statement of Lemma 3.3 follows from representation (60), by the same way as in Lemma 3.2, if we observe that

$$x_1(t) = \phi(t) + \int_0^t ds\, \hat{S}_n^0(t-s)\phi(s)ds + r_n(t)$$
$$= 1 + \int_0^t ds\, \hat{S}_n^0(t-s) + \sum_{j=2}^n \frac{W_{1j}}{n^{1/2}}\Big(d_j(t) + \int_0^t ds\, \hat{S}_n^0(t-s)d_j(s)\Big) + r_n(t), \quad (65)$$

where $d_j(t)$ and $\hat{S}_n^0(t)$ are independent of $\{W_{1i}\}i = 2^n$, $\hat{S}_n^0(t)$ is bounded and

$$E\{r_n^2(t)\} \le C(t)n^{-1}.$$

The analog of the Lindeberg condition follows from (45).

Lemma 3.3 is proven. □

Now, the proof of the self averaging property of $g_n(z,t)$ (19) and so also the self averaging property of $N_n(\lambda,t)$ (17) is similar to the proof of Theorem 2.1.

Hence, we need to study only $E\{N_n(\lambda,t)\}$. But due to the symmetry of the problem, $E\{N_n(\lambda,t)\}$ coincides with the distribution $x_1(t)$. And since, according to Lemma 3.3, $x_1(t)$ converges in distribution to a Gaussian random variable, to prove Theorem 2.2 we are left to find

$$\begin{aligned} a_{s,n}(t) &= E\{(e^{t\mathbf{A}}\overline{c})_1\} = E\{(e^{t\mathbf{A}})_{11}\} + \sum_{j=2}^n E\{(e^{t\mathbf{A}})_{1j}\} \\ \sigma_{s,n}(t) &= E\{x_1^2(t)\} - E^2\{x_1(t)\} \end{aligned} \quad (66)$$

Let us use the Cauchy formula, valid for any symmetric matrix $\mathbf{A}$,

$$(e^{t\mathbf{A}})_{1j} = \oint_L dz e^{zt} G_{1j}(z)dz, \quad (67)$$

where $\mathbf{G}(z) = (\mathbf{A} - z)^{-1}$ is the resolvent of the matrix A and the contour L is taken in such a way to contain inside the interval $[-2w, 2w]$, and the distance from L to $[-2w, 2w]$ is more than some constant d. According to the result [1] (see (27)), then with probability more than $1 - e^{-Cdn^\gamma}$ all eigenvalues of $\mathbf{A}$ are inside the contour and the distance from any of them to L is more than $d/2$. Hence with the same probability formula (67) is valid, and

$$\mathrm{Prob}\{||\mathbf{G}(z)|| \le \frac{d}{2}, \forall z \in L\} \ge 1 - e^{-Cdn^\gamma}. \quad (68)$$

We use also the following representation of the resolvent $\mathbf{G}(z)$:

$$G_{1j} = \Big(\sum_{i,i'=2}^n G_{ii'}^{(1)} A_{1i}A_{1i'} + z\Big)^{-1} \sum_{i=2}^n G_{ji}^{(1)} A_{1i}, \quad (j \ne 1),$$

where $\mathbf{G}^{(1)}(z) = (\mathbf{A}^{(1)} - z)^{-1}$ is the resolvent of $\mathbf{A}^{(1)}$. Hence, we can write

$$\sum_{j=2}^n G_{1j} = \Big(w^2\tilde{g}_n(z) + r_n(z) + z\Big)^{-1} \sum_{i,j=2}^n G_{ji}^{(1)} A_{1i}, \quad (69)$$

where

$$\tilde{g}_n(z) = n^{-1}\sum_{i=2}^{n} G_{ii}^{(1)},$$
$$r_n(z) = n^{-1}\sum_{i=2}^{n} G_{ii}^{(1)}(W_{1i}^2 - w^2) + n^{-1}\sum_{i,i'=2,i\neq i'}^{n} G_{ii'}^{(1)} W_{1i}W_{1i'}.$$

Using that $\mathbf{G}^{(1)}(z)$ does not depend on A_{1i}, and (68) is valid also for $||\mathbf{G}^{(1)}(z)||$, it is easy to get

$$\begin{aligned} E\{|r_n(z)|^2\} &\leq Cn^{-2}E\Big\{\sum_{i=2}^{n}|G_{ii}^{(1)}(z)|^2\Big\} + Cn^{-2}E\Big\{\sum_{i=2}^{n}(G^{(1)}(z)*G^{(1)}(\overline{z}))_{ii}\Big\} \\ &\leq Cn^{-1}. \end{aligned} \tag{70}$$

Hence, it follows from (69), that

$$\begin{aligned} E\Big\{\sum_{j=2}^{n} G_{1j}\Big\} &= E\Big\{(w^2\tilde{g}_n(z)+z)^{-1}\sum_{i,j=2}^{n} G_{ji}^{(1)}A_{1i}\Big\} \\ &\quad - E\Big\{r_n(z)(w^2\tilde{g}_n(z)+z)^{-1}(w^2\tilde{g}_n(z)+r_n(z)+z))^{-1}\sum_{i,j=2}^{n} G_{ji}^{(1)}A_{1i}\Big\} \\ &= I - II \end{aligned} \tag{71}$$

Since $\mathbf{G}^{(1)}(z)$ and $g_n(z)$ do not depend on A_{1i}, $I = 0$. Besides, since

$$|(w^2\tilde{g}_n(z)+z)^{-1}|, |(w^2\tilde{g}_n(z)+r_n(z)+z))^{-1}| \leq ||\mathbf{G}||,$$

combining the Schwartz inequality with (70), we obtain

$$|II| \leq CE^{1/2}\Big\{n^{-1}\sum_{i,j=2}^{n}\Big(G^{(1)}(z)*G^{(1)}(\overline{z})\Big)_{ij}\Big\}E^{1/2}\{|r_n(z)|^2\} \leq Cn^{-1/2}$$

So, it follows from (66)-(71) that

$$\sum_{j=2}^{n} E\{(e^{t\mathbf{A}})_{1j}\} = O(n^{-1/2}) \tag{72}$$

and so

$$a_{s,n}(t) = E\{(e^{t\mathbf{A}})_{11}\} + O(n^{-1/2}) = n^{-1}E\{\mathrm{Tr}\ e^{t\mathbf{A}}\} + O(n^{-1/2}).$$

Hence, according to the results of [14], we get

$$\lim_{n\to\infty} a_{sn}(t) = a_s(t) = \frac{1}{2\pi w}\int_{-2w}^{2w} e^{\lambda t}\sqrt{4w^2-\lambda^2}d\lambda$$

and so we have proved (11) for $\kappa = 0$. Using remark in the beginning of the section, now it is easy to obtain (11) for $\kappa \neq 0$.

To find $\sigma_{sn}(t)$ let us observe that, due to the symmetry,

$$\begin{aligned} E\{x_1^2(t)\} &= n^{-1}\sum_{i=1}^{n} E\{x_i^2(t)\} = n^{-1}\sum_{i=1}^{n} E\{(e^{tA}\overline{c})_i^2\} \\ &= n^{-1}\sum_{i,j=1}^{n} E\{(e^{2tA})_{ij}\} = E\{(e^{2tA}\overline{c})_1\} = E\{x_1(2t)\} \end{aligned}$$

Now it is easy to obtain (12) for any κ. □

Proof of Theorem 2.3. The proof of the fact that $N_n(\lambda, t)$ is a self averaging quantity and coincides in the limit in the distribution of $x_1(t)$ is the same as in Theorem 2.1, 2.2. Thus we are left to prove only that $x_1(t)$ can be represented in the form (15) in the non symmetric case or (16) in the symmetric case.

In the non symmetric case we get similarly to (44), that

$$x_1(t) = \xi_1 + \sum_{j=2}^{n} \frac{W_{1j}}{n^{1/2}} d_j(t) + r_n(t), \quad d_j(t) = \int_0^t ds\, x_j^{(1)}(s), \tag{73}$$

where

$$d_j(t) = \int_0^t ds\, x_j^{(1)}(s), \quad E\{r_n^2(t)\} \leq C(t)n^{-1}$$

and since $d_j(t)$ are independent on W_{1j} and ξ_1 and satisfy the inequality(45, we obtain that the second sum converges in probability to a normal random variable with zero mean and the variance

$$\sigma^{\xi}(t) = \lim_{n\to\infty} n^{-1} \sum E\{d_j^2(t)\} \tag{74}$$

Now, let us denote

$$R_n(t, s) = E\{x_1(t)x_1(s)\}.$$

Then repeating the conclusions (49)-(53) of Theorem 2.1, we get from (20)

$$R(t, s) = \lim_{n\to\infty} R_n(t, s) = w_0^2 \left(1 + \sum_{m=1}^{\infty} \frac{t^m s^m}{m!m!}\right) \tag{75}$$

Hence, by (74) and the symmetry of the problem, we get

$$\sigma^{\xi}(t) = \int_0^t \int_0^t dt_1 dt_2 R(t_1, t_2) = w_0^2 \sum_{m=1}^{\infty} \frac{t^{2m}}{m!m!} \tag{76}$$

So, we have proved (15) in the case $\kappa = 0$. Then, using Remark 3, we obtain (15) for any κ.

To prove (16) we use the analog of (65) which in the case of (13) has the form

$$\begin{aligned} x_1(t) &= \xi_1 \left(1 + \int_0^t ds\, \hat{S}_n^0(t-s)\right) + \sum_{j=2}^{n} \frac{W_{1j}}{n^{1/2}} \Big(d_j(t) \\ &\quad + \int_0^t ds\, \hat{S}_n^0(t-s) d_j(s)\Big) + r_n(t) \\ &= s_n(t)\xi_1 + z_n + r_n(t), \\ E\{r_n^2(t)\} &\leq C(t)n^{-1}, \end{aligned} \tag{77}$$

where $d_j(t)$ and $\hat{S}_n^0(t)$ are independent of $\{W_{1i}\}_{j=2}^n$, $\hat{S}_n^0(t)$ is bounded and $d_j(t)$ satisfy (45). Thus, according to the central limit theorem, z_n converges in distribution to a Gaussian random variable, independent of ξ_1. Besides, $s_n(t)$ is a self averaging quantity. To prove this it is enough to prove that $\hat{S}_n^0(t)$ is a self averaging quantity. The last statement follows from the representation (61), if we know that $S_n^0(t)$ is a self averaging quantity. But, by definitions (58) and (56) and the spectral theorem,

$$S_n^0(t) = \int_0^t d\tau n^{-1} \mathrm{Tr} e^{\tau \mathbf{A}^{(1)}} = \int_0^t d\tau \int e^{\lambda\tau} dN_n^*(\lambda)$$

where

$$N_n^*(\lambda) = n^{-1}\sum_{i=1}^n \theta(\lambda - \lambda_i^*)$$

is a normalized counting measure of eigenvalues of $\mathbf{A}^{(1)}$. So, the self averaging of $s_n(t)$ follows from the self averaging of $N_n^*(\lambda)$, which is a well known result (see, e.g. [14] or the review paper [10]).

Thus, to finish the proof of (16) we are left to find $E\{s_n(t)\}$ and the variance of z_n in (77). But, (77) implies that

$$\begin{aligned} E\{s_n(t)\} &= w_0^{-2}E\{x_1(t)\xi_1\} + O(n^{-1/2}), \\ E\{z_n^2\} &= E\{x_1^2(t)\} - E^2\{s_n(t)\}E\{\xi_1^2\} + O(n^{-1}) \end{aligned} \quad (78)$$

So, using the fact that $x_1(t)$ is a solution of (1) with the initial condition (13), we get

$$\begin{aligned} E\{x_1(t)\xi_1\} &= E\{\xi_1^2\}E\{(e^{t\mathbf{A}})_{11}\} + \sum_{j=2}^n E\{(e^{t\mathbf{A}})_{1j}\}E\{\xi_1\}E\{\xi_j\} \\ &= E\{\xi_1^2\}E\{(e^{t\mathbf{A}})_{11}\} + E^2\{\xi_1\}\sum_{j=2}^n E\{(e^{t\mathbf{A}})_{1j}\} \end{aligned} \quad (79)$$

No, using (72), we get

$$E\{s_n(t)\} = E\{(e^{t\mathbf{A}})_{11}\} + O(n^{-1/2}) = n^{-1}E\{\mathrm{Tr}\ e^{t\mathbf{A}}\} + O(n^{-1/2}).$$

Hence, according to the results of [14], we get

$$\lim_{n\to\infty} E\{s_n(t)\} = \frac{1}{2\pi w}\int_{-2w}^{2w} e^{\lambda t}\sqrt{4w^2-\lambda^2}d\lambda.$$

To compute $E\{z_n^2\}$, we write, using that $x_1(t)$ is a solution of (1) with (13) and taking into account the symmetry of the problem,

$$\begin{aligned} E\{x_1(t)^2\} &= \sum_{i=1}^n E\{\xi_i^2\}E\{(e^{t\mathbf{A}})_{1i}^2\} + \sum_{i,j=1,i\neq j}^n E\{(e^{t\mathbf{A}})_{1i}(e^{t\mathbf{A}})_{1j}\}E\{\xi_i\}E\{\xi_j\} \\ &= E\{\xi_1^2\}E\{(e^{2t\mathbf{A}})_{11}\} + E^2\{\xi_1\}n^{-1}\sum_{i,j=1,i\neq j}^n\sum_{k=1}^n E\{(e^{t\mathbf{A}})_{ik}(e^{t\mathbf{A}})_{kj}\} \\ &= w_0^2E\{(e^{2t\mathbf{A}})_{11}\} + E^2\{\xi_1\}n^{-1}\sum_{i,j=1,i\neq j}^n E\{(e^{2t\mathbf{A}})_{ij}\}. \end{aligned} \quad (80)$$

But, according to (72) the second sum in the r.h.s of(80) is $O(n^{-1/2})$. And so, using the above consideration, we have

$$\lim_{n\to\infty} E\{x_1^2(t)\} = E\{\xi_1^2\}\lim_{n\to\infty} n^{-1}E\{\mathrm{Tr}\ e^{2t\mathbf{A}}\} = \frac{w_0^2}{2\pi w}\int_{-2w}^{2w} e^{2\lambda t}\sqrt{4w^2-\lambda^2}d\lambda.$$

Finally, we obtain

$$\begin{aligned} \lim_{n\to\infty} E\{z_n^2\} &= \frac{w_0^2}{2\pi w}\int_{-2w}^{2w} e^{2\lambda t}\sqrt{4w^2-\lambda^2}d\lambda - w_0^2\left(\frac{1}{2\pi w}\int_{-2w}^{2w} e^{\lambda t}\sqrt{4w^2-\lambda^2}d\lambda\right)^2 \\ &= w_0^2\sigma_s(t). \end{aligned} \quad (81)$$

Now, relations (77)-(81) imply (16) for $\kappa = 0$. Then, using Remark 3, we obtain (16) for any κ. □

Acknowledgements. J. F. was partially supported by grants from UK EPSRC (GR/R54569), (GR/S20574), and (GR/S30443).

REFERENCES

[1] A. Boutet de Monvel and M. Shcherbina, *On the norm of random matrices*, Mat. Zametki, **57** (1995), 688–698.

[2] J. E. Cohen and C. M. Newman, *The stability of large random matrices and their products*, Ann. Prob., **12** (1984), 283–310

[3] Y. C. Deng, M. Z. Ding and J. F. Feng, *Synchronization in stochastic coupled systems*, Jour. Phys. A., **37** (2004), 2163–2173.

[4] I. J. Farkas, I. Derenyi, A. L. Barabasi and T. Vicsek, *Spectra of "real-world" graphs: beyond the semicircle law*, Phys. Rev. E., **64** (2001), 026740.

[5] J. F. Feng and B. Tirozzi, *The SLLN for the free-energy of a class of neural networks*, Helv. Phys. Acta, **68** (1995), 365–379.

[6] J. F. Feng, V. Jirsa and M. Z. Ding, *Synchronization in Networks with Random Interactions: Theory and Applications*, Chaos, **16** (2006), 015109.

[7] J. F. Feng, M. Shcherbina and B. Tirozzi, *On the critical capacity of the Hopfield model*, Commun. Math. Phys., **216** (2001), 139–177.

[8] A. Guionnet and O. Zeitouni, *Concentration of the spectral measure for large matrices*, Electronic J. Prob., **5** (2000), 119–136.

[9] R. May, *Will a large complex system be stable?*, Nature, **238** (1972), 413–414.

[10] L. Pastur, *Random Matrices as Paradigm*, Mathematical Physics, A.Grigoryan, T.Kibble, B. Zegarlinskii (Eds.), World Scientific, Singapore, (2000), 216–266.

[11] L. Pastur and M. Shcherbina, *The absence of the selfaverageness of the order parameter in the Sherrington-Kirkpatrick model*, J. Stat. Phys., **62** (1991), 1–19.

[12] A. Soshnikov, *A note on universality of the dirtribution of the largest eigenvalue in certain sample covariance matrices*, J. Statistical Physics, **108** (2002), 1033–1056.

[13] M. Shcherbina and B. Tirozzi, *The free energy of the class of Hopfield models*, Jour. Stat. Phys., **72** (1993), 113–125.

[14] E. Wigner, *On the distribution of the roots of certain symmetric matrices*, Ann. of. Math., **67** (1958), 325–327.

Received January 2007; revised June 2007.

E-mail address: brunello.tirozzi@roma1.infn.it