Seminar 8: Wiener filter

Assume we have a grayscale image above, and the table denote the small region intensity. If we know that the image has been degraded by a constant power noise of 0.8, and the denote the grayscale at position \((i, j)\) by \(X(i, j)\), then

(i) Design a Wiener filter to clean the image

(ii) Find the grayscale value at position \((2, 2)\) after applying a 3 by 3 Wiener filter.

(iii) Use `wiener2` from MATLAB to experiment how \(N, M\) affect the final outcome, where \(N \times M\) is the local square area we used to
estimate the local mean and variance

Solution:

Basic idea of Wiener filter:

\[ y(m,n) = a(m,n)x(m,n), \]

where \( y \) is the filtered out signal, \( x \) is the given signal, and \( a \) is a constant dependent on the position \((m,n)\). Our goal is to find out the value of \( a(m,n) \) given signal \( x(m,n) \).

For a given signal \( x \), we assume it is composed of two independent parts: real signal \( s \) plus noise \( \xi \), i.e.

\[ x = s + \xi. \]

The objective of Wiener filter is to find filter \( y \) such that the filtered signal \( y \) is as close as possible to the true signal \( s \), i.e.

Objective: \[ \min_a E (s - y)^2 = \min_a E(s - ax)^2 \]

In order to find the value of \( a \), we take the derivative of the objective function and set it zero:

\[ \frac{dE(s - ax)^2}{da} = 0 \rightarrow -2E(xs) + 2aE(x^2) = 0. \]
So, we have

\[ a = \frac{E(xs)}{E(x^2)} = \frac{E(x(x - \xi))}{E(x^2)} = \frac{E(x^2) - E(x\xi)}{E(x^2)} = 1 - \frac{E(x\xi)}{E(x^2)} = 1 - \frac{E((s + \xi)\xi)}{E(x^2)} \]

because \(s\) and \(\xi\) are independent, i.e. \(E(s\xi) = 0\).

Now, since we know \(\xi\) is the random noise with mean zero (\(E(\xi) = 0\)) and variance given (\(E(\xi^2) = 0.8\)), the problem is how to get \(E(x^2)\).

From basic statistics property, \(E(x^2) = \text{var}(x) + E(x)^2\).

The final thing to notice is that Wiener filter has a basic requirement: both the signal \(s\) and noise \(\xi\) have zero means. However, from the data in the question given above, the signal does not have zeros mean. Thus, we need to do a trick to linear transform the signal to zero mean, by just subtracting the mean value out of each signal digits, i.e.

\[ E(x - E(x)) = 0, \]

even though \(E(x)\) may not be zero.

Therefore, we get our final result:

\[ y - E(x) = a(x - E(x)) \]

\[ \iff y - E(x) = \left(1 - \frac{E(\xi^2)}{\text{var}(x)}\right)(x - E(x)) \]

\[ \iff y = E(x) + \left(1 - \frac{E(\xi^2)}{\text{var}(x)}\right)(x - E(x)). \]

Since all the value in the formula is given or can be calculated from the given signal \(x(m,n)\), I'll leave the detailed calculation to you.

Hint:

\[ E(x) = \sum_{i,j=1,2,3} x(i,j) + E(x) \]

\[ \text{var}(x) = \frac{1}{9} \sum_{i,j=1,2,3} (x(i,j) - E(x))^2 \]