DCSP-4: Applications

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Recap

In general, a signal with $T = \frac{2\pi}{\omega}$ can be represented as follows

$$x(t) = A_0 + \sum_{n=1}^{N} A_n \cos(nt) + \sum_{n=1}^{N} B_n \sin(nt)$$

$$\sim A_0 + \sum_{n=1}^{N} A_n \cos(nt) + \sum_{n=1}^{N} B_n \sin(nt)$$
Recap

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$$X(t) \sim A_0 + A_1 \cos(\omega t)$$
Recap

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$$X(t) \sim A_0 + A_1 \cos(\omega t) + A_2 \cos(2\omega t)$$
Recap

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$$X(t) \sim A_0 + A_1 \cos(\omega t) + A_2 \cos(2\omega t) + A_3 \cos(3\omega t)$$
Recap

In general, a signal with $T=2\pi/\omega$ can be represented as follows:

$$X(t) \sim A_0 + A_1 \cos(\omega t) + A_2 \cos(2\omega t) + A_3 \cos(3\omega t) + A_4 \cos(4\omega t) + A_5 \cos(5\omega t) + A_6 \cos(6\omega t) + A_7 \cos(7\omega t) + A_8 \cos(8\omega t) + A_9 \cos(9\omega t)$$
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+ A_4 \cos(4\omega t) + A_5 \cos(5\omega t) + A_6 \cos(6\omega t) \\
+ A_7 \cos(7\omega t) + A_8 \cos(8\omega t) + A_9 \cos(9\omega t) \\
+ \ldots
\]
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Recap

In general, a signal with $T=2\pi/\omega$ can be represented as follows

$$x(t) = A_0 + \sum_{n=1}^{N} A_n \cos(nt) + \sum_{n=1}^{N} B_n \sin(nt)$$

$$\sim A_0 + \sum_{n=1}^{N} A_n \cos(nt) + \sum_{n=1}^{N} B_n \sin(nt)$$

- The more terms we have, the more accurate: compression
- The plot of $\{A_n, B_n\}$ against $\{n\}$ is called frequency spectrum
Today’s Summary

• A few applications

• General form of FT

• A few applications
Example I

\[ x(t) = 1, \quad 0 < t < \pi, \quad 2\pi < t < 3\pi, \quad 0 \text{ otherwise} \]

Hence \( x(t) \) is a signal with a period of \( 2\pi \)
Example 1

\[ A_0 = \frac{1}{2} \int_0^\frac{1}{2} dt = \frac{1}{2} \]

\[ A_n = \frac{2}{2} \int_0^\frac{1}{2} \cos(nt) \, dt = \frac{1}{n} \sin(n) = 0, n = 1, 2, \ldots \]

\[ B_n = \frac{2}{2} \int_0^\frac{1}{2} \sin(nt) \, dt = \frac{1}{n} (1 - \cos(n)), n = 1, 2, \ldots \]

Finally, we have

\[ x(t) = \frac{1}{2} + 2 \left[ \sin(t) + \frac{1}{3} \sin(3t) + \frac{1}{5} \sin(5t) + \ldots \right] \]
Example I

The description of a signal in terms of its constituent frequencies is called its **frequency (power) spectrum**.
Example I

Time domain

Frequency domain

\[ |X(F)| \]
Example I

- A periodic signal is uniquely determined by its coefficients \( \{A_n, B_n\} \).

- If we truncated the series into finite term, the signal can be approximated by a finite sines as shown below (compression, MP3, MP4, JPG, ... )
Example II (understanding music)
Example II

a. Pure tone:  

This confirms our earlier belief that it is a signal with a finite bandwidth (N-S sampling Thm)
Example II

b. Different waveforms

This confirms our earlier belief that it is a signal without bandlimit (N-S sampling Thm)
Bandwidth can be properly defined

- **Bandwidth** is the difference between the upper and lower frequencies in a set of frequencies. It is typically measured in **hertz**.
Bandwidth can be properly defined

- **Bandwidth** is the difference between the upper and lower frequencies in a continuous set of frequencies. It is typically measured in **hertz**.
Example II

C. Approximation (compression)

Without doing much, we can compress the original data now, as in Example I.
In general, a signal can be represented as follows:

\[ x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(wn t) + \sum_{n=1}^{\infty} B_n \sin(wn t) \]

\[ = A_0 + \sum_{n=1}^{\infty} A_n \left[ \exp(jwn t) + \exp(-jwn t) \right] / 2 + \]

\[ + \sum_{n=1}^{\infty} B_n \left[ \exp(jwn t) \exp(-jwn t) \right] / (2j) \]

\[ = A_0 + \sum_{n=1}^{\infty} \left[ A_n / 2 + B_n / (2j) \right] \exp(jwn t) + \]

\[ + \sum_{n=1}^{\infty} \left[ A_n / 2 - B_n / (2j) \right] \exp(jwn t) \]
In general, a signal can be represented as follows

\[ x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\omega nt) + \sum_{n=1}^{\infty} B_n \sin(\omega nt) \]

\[ = A_0 + \sum_{n=1}^{\infty} A_n \left[ \exp(j\omega nt) + \exp(-j\omega nt) \right] / 2 + \]

\[ + \sum_{n=1}^{\infty} B_n \left[ \exp(j\omega nt) - \exp(-j\omega nt) \right] / (2j) \]

\[ = A_0 + \sum_{n=1}^{\infty} \left[ A_n / 2 - B_n / (2j) \right] \exp(-j\omega nt) + \]

\[ + \sum_{n=1}^{\infty} \left[ A_n / 2 + B_n / (2j) \right] \exp(j\omega nt) \]

\[ = \sum_{n=-\infty}^{\infty} c_n \exp(j\omega nt) \]
FT : Neat Form

- Which is the exponential form of the Fourier series.

- In this expression the values $C_n$ are complex number, we have

\[ x(t) \xrightarrow{\text{FT}} \{ C_n, n=\ldots,-1,0,1,2,\ldots, \} \]

\[
C_{-1}^2 = C_1^2 = \left( A_1^2 + B_1^2 \right) / 4
\]

\[
C_{-2}^2 = C_{-2}^2 = \left( A_2^2 + B_2^2 \right) / 4
\]
FT : Neat Form

- Which is the exponential form of the Fourier series.
- In this expression the values $C_n$ are complex number, we have

$$x(t) = \sum_{F} X(F) \exp(jFt)$$

\[ X(n\omega) = X(F) = c_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \exp(-jn\omega t) x(t) dt \]

where $\omega = (2\pi) / T$
FT in complex exponential

If the periodic signal is replace with an aperiodic signal (general case)

FT is given by

\[ X(F) = FT(x) = \int_{-\infty}^{\infty} x(t) e^{-2\pi j F t} dt \]

\[ x(t) = FT^{-1}(X) = \int_{-\infty}^{\infty} X(F) e^{2\pi j F t} dF \]

Is that deadly simple?
Fourier’s song

• Integrate your function times a complex exponential

• It's really not so hard you can do it with your pencil

yes, I agree
Consider the case of a rectangular pulse. In particular, let us define

\[ x(t) = \begin{cases} 
1, & \text{if } 0.5 < t < 0.5 \\
0, & \text{otherwise} 
\end{cases} \]

Its FT is given by

\[ X(F) = \frac{1}{F} \int_{0.5}^{0.5} x(t) \exp(-j2\pi F t) dt = \frac{\sin(F)}{F} \]

Which is called the Sinc function.
There are a number of features to note:

1. The bandwidth of the signal is only approximately finite.

Most of the energy is contained in a limited region called the main-lobe.

However, some energy is found at all frequencies.

2. The spectrum has positive and negative frequencies.

These are symmetric about the origin.

This may seem non-intuitive, but can be seen from equations in periodic case.

3. Note that the spectrum is continuous now:

\[ X(F) = \text{Sinc function} \]
Fourier's Song

Integrate your function times a complex exponential. It's really not so hard you can do it with your pencil. And when you're done with this calculation, you've got a brand new function - the Fourier Transformation. What a prism does to sunlight, what the ear does to sound, Fourier does to signals; it's the coolest trick around. Now filtering is easy, you don't need to convolve. All you do is multiply in order to solve.

From time into frequency - from frequency to time. Every operation in the time domain has a Fourier analog - that's what I claim. Think of a delay, a simple shift in time. It becomes a phase rotation - now that's truly sublime! And to differentiate, here's a simple trick. Just multiply by J omega, ain't that slick? Integration is the inverse, what you gonna do? Divide instead of multiply - you can do it too.

From time into frequency - from frequency to time. Let's do some examples... consider a sine. It's mapped to a delta, in frequency - not time. Now take that same delta as a function of time. Mapped into frequency - of course - it's a sine! Sine x on x is handy, let's call it a sinc. Its Fourier Transform is simpler than you think. You get a pulse that's shaped just like a top hat... Squeeze the pulse thin, and the sinc grows fat.
The world has changed

This module and a few following ones will make use of the frequency information.
This module is about practical applications

\[ \begin{align*}
X(F) &= FT(x) = x(t) \exp(-2jFt) dt \\
x(t) &= FT^1(X) = X(F) \exp(2jFt) dF
\end{align*} \]
\[ S_1(t) = 10 \cos(2\pi t) + \cos(10 \cdot 2\pi t) \]

for \( i = 1:10000 \)
\[ x(i) = 10 \cos(i \cdot 0.01) + \cos(i \cdot 10 \cdot 0.01); \]
End
Sound(x)
**App I**

\[ S_2(t) = \cos(2\pi t) + 10 \cos(10 \times 2\pi t) \]

for \( i = 1:10000 \)

\[ x(i) = 1 \times \cos(i \times 0.01) + 10 \times \cos(i \times 10 \times 0.01); \]

End

Sound(s)
In frequency domain, the height of the spectrum indicates the energy of the corresponding signal.

For example,

- for $S_1$, energy concentrating on the signal $\cos(2\pi t)$, a low frequency signal
- for $S_2$, energy concentrating on the signal $\cos(10*2\pi t)$, a high frequency signal
App II: Touch-tone dialing
App II: Touch-tone dialing

- Freqs 1209 Hz 1336 Hz 1477 Hz 1633 Hz
- 697 Hz 1 2 3 A
- 770 Hz 4 5 6 B
- 852 Hz 7 8 9 C
- 941 Hz * 0 # D

\[(1) = A_{n1} \cos(n1 \omega_0 t) + A_{n2}\cos(n2\omega_0 t)\]

1 Hz = \(\omega_0\)

Your phone is a section (two terms) of a FT
APP III: Spread spectrum techniques

• In Bluetooth, for example, to improve resistance to radio frequency interference by avoiding using crowded frequencies in the hopping sequence.

• In military use, it is a simple idea (radio guided torpedo, for example)
App IV: instruments

- We all know about pitch…
- It is really about frequency, or cycles per seconds
- How about harmonics?
- This is what gives the timbre of an instrument!
App IV: instruments

- The played note is the frequency of the first peak: 220Hz (note A) in this case.
- Other peaks are called harmonics: they define the typical sound of the instrument.
- Without Fourier, we could have been lost.
- When you play an instrument, it is a section of a FT.
How can an MP3 song sound so well, while being so compressed?

Compression in a sense introduces noise.

a combination of our hearing system + FT
App V: MP3+MP4

- Original noiseless music
- Original music + noise 2 → SNR = 13 dB
- Same amount of noise
- Impressive difference of sound
- What is the secret?

```matlab
>> r = audiorecorder(22050, 16, 1);
>> record(r)
>> stop(r)
>> mySpeech = getaudiodata(r, 'int16'); % get data as int16 array
>> audiowrite('compress_3.wav',mySpeech,Fs);
load handel.mat
filename = 'handel.wav';
audiowrite(filename,y,Fs);
clear Fs
```
App V: MP3+MP4
App V: MP3+MP4

- MP3: complex compression algorithm that introduces errors
- MP3 minimizes the perceived quality decay by shaping the compression errors
App VI: Mind Reading

• MRI is a medical imaging technique to visualize your internal body structure and/or activity in detail

• Using a physics term: it measures the oscillations of hydrogen nuclei when stimulated by different radio-frequency magnetic fields

• Reading you mind with FT

My brain activity during one scan
App VI: Mind Reading

- MRI is a medical imaging technique to visualize your internal body structure and/or activity in detail.

- Using a physics term: it measures the oscillations of hydrogen nuclei when stimulated by different radio-frequency magnetic fields.

- Reading you mind with FT.

It is not about sex.

My brain activity during one scan.
App VI: Mind Reading

• Using a DCSP term: it measures the FT of the internal structure and/or activity of interest

• How does the collected data looks like?

```matlab
img=(T1_Image(:,:,50));
img = fftshift(img(:,:,1));
F = fft2(img);
figure;
imagesc(100*log(1+abs(fftshift(F))));
colormap(gray)
title('magnitude spectrum');
figure;
imagesc(angle(F));
colormap(gray);
title('phase spectrum');
```
Little information about the subject can be gathered in the original picture.

An inverse Fourier transform of the data reveals the slice of a human head activity.
App VI: Mind Reading

- Human Brain Project was one of the two projects with a budget of 1B Euros for ten years each (another is graphene) in 2013.

- The main purpose is to simulate the Brain.

- USA matched with 4.5 B US Dollars in 2014.

- China will do as well.
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Assignment: 2016

• Q1: you should be able to do it after last week seminar

• Q2: need a bit reading (my lecture notes)

• Q3: standard

• Q4: standard
Assignment: 2016

• Q5: standard

• Q6: standard

• Q7: after today’s lecture

• Q8: load jazz, plot soundsc
  load Tunejazz plot
  load NoiseJazz plot
Recap

• Fourier Transform for *a periodic signal*
  \[ \{ \sin(n \omega t), \cos(n \omega t) \} \]

• For *general function case*,

\[
X(F) = \int_{-\infty}^{\infty} x(t) \exp(-2\pi jFt) \, dt \\
x(t) = \int_{-\infty}^{\infty} X(F) \exp(2\pi jFt) \, dF
\]
Recap: this is all you have to remember (know)?

• For *general function case*,

\[
X(F) = x(t)\exp(-2\pi jFt)dt
\]

\[
x(t) = \int X(F)\exp(2\pi jFt)dF
\]
Can you do FT for \( \cos(2 \pi t) \)?

\[
X(F) = x(t) \exp(-2\pi jFt) \, dt
\]

\[
x(t) = X(F) \exp(2\pi jFt) \, dF
\]
Dirac delta function

\[ X(F) = \cos(2\ t)\exp(2\ jtF)dt \]
\[ = \frac{1}{2} \left[ \exp(2\ jt) + \exp(2\ jt) \right] \exp(2\ jtF)dt \]
\[ = \frac{1}{2} \left[ \exp(2\ jt(1+F)) + \exp(2\ jt(1-F)) \right]dt \]
\[ = \frac{1}{2} \left[ \frac{1}{2j(1+F)} \exp(2\ jt(1+F)) + \frac{1}{2j(1+F)} \exp(2\ jt(1-F)) \right] \]
\[ = ?????? \]

For example, take \( F=0 \) in the equation above, we have

\[ X(0) = \frac{1}{2} \left[ \frac{1}{2j} \exp(2\ jt) + \frac{1}{2j} \exp(2\ jt) \right] \]
\[ = \frac{1}{2} \sin(2\ t) \]

It makes no sense !!!!
Dirac delta function: A photo with the highest IQ (15 NL)
Dirac delta function: A photo with the highest IQ (15 NL)
Dirac delta function

The (digital) delta function, for a given $n_0$

\[ [n \quad n_0]f[n] = f(n_0) \]

\[ [n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \]

\[ d[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \]
Dirac delta function

The (digital) delta function, for a given $n_0$

$$\sum_{n=0}^{\infty} [n - n_0] f[n] = f(n_0)$$

$$[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Dirac delta function $\delta(x)$ (you could find a nice movie in Wiki);

$$\int_{-\infty}^{\infty} (t) f(t) \, dt = f(0)$$

$$\begin{cases} , t = 0 \\ 0, & t \neq 0 \end{cases}$$
Dirac delta function

Dirac delta function \( \delta(x) \);

\[
FT^{-1}(X(F)) = \left[ \frac{(F - 1) + (F + 1)}{2} \right] \exp(2 \pi Fjt) dF
\]

\[
= \left[ \frac{\exp(2 \pi jt) + \exp(-2 \pi jt)}{2} \right]
\]

\[= \cos(2\pi t)\]

The FT of \( \cos(2\pi t) \) is

\[
FT(\cos(2\pi t)) = \frac{(F - 1) + (F + 1)}{2}
\]
A final note (in exam or future)

• Fourier Transform for a periodic signal
  \{ \sin(n \omega t), \cos(n \omega t) \}

• For general function case (it is true, but need a bit further work),

$$X(F) = x(t) \exp(-2 \pi j Ft) dt$$

$$x(t) = X(F) \exp(2 \pi j Ft) dF$$
Summary

Will come back to it soon (numerical)

This trick (FT) has changed our life

and

will continue to do so
This Week’s Summary

• Noise

• Information Theory
Noise in communication systems:

- Probability
- Random signals

```matlab
I = imread('peppers.png');
imshow(I);
noise = 1*randn(size(I));
Noisy = imadd(I,im2uint8(noise));
imshow(Noisy);
```
Noise

I = imread('peppers.png');
imshow(I);
noise = 1*randn(size(I));
Noisy = imadd(I,im2uint8(noise));
imshow(Noisy);
Noise

Noise is a random signal (in general).

By this we mean that we cannot predict its value.

We can only make statements about the probability of it taking a particular value.
The **probability density function** (pdf) $p(x)$ of a random variable $x$ is the probability that $x$ takes a value between $x_0$ and $x_0 + \delta x$.

We write this as follows:

$$p(x_0) \delta x = P(x_0 < x < x_0 + \delta x)$$
Probability that $x$ will take a value lying between $x_1$ and $x_2$ is

$$P(x_1 < x < x_2) = \int_{x_1}^{x_2} p(x) \, dx$$

The probability is unity. Thus

$$p(x) \, dx = 1$$
IQ distribution

- Mentally Inadequate: 23%
- Low Intelligence: 13.6%
- Average Intelligence: 34.1%
- Above Average Intelligence: 34.1%
- High Intelligence: 13.6%
- Superior Intelligence: 2.1%
- Exceptionally Gifted: 0.13%
A density satisfying the equation is termed normalized.

The cumulative distribution function (CDF) $F(x)$ is the probability $x$ is less than $x_0$

$$F(x_0) = P(x < x_0) = \int_{-\infty}^{x_0} p(x) \, dx$$

My IQ is above 85% ($F$(my IQ)=85%).
From the rules of integration:

\[ P(x_1 < x < x_2) = P(x_2) - P(x_1) \]

density function has two classes: continuous and discrete.
Continuous distribution

An example of a continuous distribution is the Normal, or Gaussian distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - \mu)^2}{\sigma^2}\right)$$

where $\mu$, $\sigma$ is the mean and standard variation value of $p(x)$.

The constant term ensures that the distribution is normalized.
Continuous distribution.

This expression is important as many actually occurring noise source can be described by it, i.e. **white noise** or **coloured noise**.
Generating $f(x)$ from matlab

```matlab
X = randn(1,1000); Plot(x);
```

- $X[1], x[2], \ldots, X[1000]$
- Each $x[i]$ is independent
- Histogram
If a random variable can only take discrete value, its pdf takes the forms of lines.

An example of a discrete distribution is the Poisson distribution

\[ P(n) = \frac{\lambda^n}{n!} \exp(-\lambda) \]
Discrete distribution.
Mean and variance

We can introduce measures that summarise what we expect to happen on average.

The two most important measures are the mean (or expectation) and the standard deviation.

The mean of a random variable $x$ is defined to be
Mean and variance

\[ EX = \int xp(x) \, dx \]

or

\[ EX = np(n) \]

• In the examples above we have assumed that the mean of the Gaussian distribution to be 0, the mean of the Poisson distribution is found to be \( \lambda \).
Mean and variance

- The mean of a distribution is, in common sense, the average value.
- Can be estimated from data
- Assume that \( \{x_1, x_2, x_3, \ldots, x_N\} \) are sampled from a distribution
- Law of Large Numbers: \( E(X) \sim (x_1+x_2+\ldots+x_N)/N \)
Mean and variance

- The more data we have, the more accurate we can estimate the mean
- \( \frac{x_1 + x_2 + \ldots + x_N}{N} \) against \( N \) for \( \text{randn}(1,N) \)
Mean and variance

- The variance is defined as

\[ s^2 = E(X - EX)^2 = (x - Ex)^2 p(x) \, dx \]

- The square root of the variance is called standard deviation.

- Again, it can be estimated from data

\[ s^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - EX)^2 \]
Mean and variance

- The standard deviation is a measure of the spread of the probability distribution around the mean.

- A small standard deviation means the distribution are close to the mean.

- A large value indicates a wide range of possible outcomes.

- The Gaussian distribution contains the standard deviation within its definition ($\mu, \sigma$)
Mean and variance

• Communication signals can be modelled as a zero-mean, Gaussian random variable.

• This means that its amplitude at a particular time has a PDF given by Eq. above.

• The statement that noise is zero mean says that, on average, the noise signal takes the values zero.
Mean and variance

Einstein’s IQ

Einstein’s IQ = 160+
What about yours?

Mentally Inadequate 23%
Low Intelligence 13.6%
Average 34.1%
Above Average 34.1%
High Intelligence 13.6%
Superior Intelligence 2.1%
Exceptionally Gifted 0.13%
Signal to noise ratio is an important quantity in determining the performance of a communication channel.

The noise power referred to in the definition is the mean noise power.

It can therefore be rewritten as

\[ SNR = 10 \log_{10} \left( \frac{S}{\sigma^2} \right) \]
Correlation or covariance

- \( \text{Cov}(X,Y) = \mathbb{E}(X-\mathbb{E}X)(Y-\mathbb{E}Y) \)

- correlation coefficient is normalized covariance
  \( \text{Coef}(X,Y) = \frac{\mathbb{E}(X-\mathbb{E}X)(Y-\mathbb{E}Y)}{\sigma(X)\sigma(Y)} \)

- Positive correlation, Negative correlation

- No correlation (independent)
Stochastic process = signal

• A stochastic process is a collection of random variables $x[n]$, for each fixed $[n]$, it is a random variable

• Signal is a typical stochastic process

• To understand how $x[n]$ evolves with $n$, we will look at auto-correlation function (ACF)

$$g(k) = E(x[n+k] - Ex[n+k]) (x[n] - Ex[n]) (x[n+k]) (x[n])$$

• ACF is the correlation between $k$ steps
Stochastic process

\[
g(k) = \frac{E(x[n+k] \cdot Ex[n+k]) (x[n] \cdot Ex[n])}{(x[n+k]) (x[n])}
\]

- two signals are generated: y (red) is simply \(\text{randn}(1,200)\)
  - x (blue) is generated \(x[i+10] = 0.8x[i] + y[i+10]\)

- For y, we have \(\gamma(0) = 1, \gamma(n) = 0\), if \(n\) is not 0: having no memory
- For x, we have \(\gamma(0) = 1, \) and \(\gamma(n)\) is not zero, for some \(n\): having memory
white noise \( w[n] \)

- **White noise** is a random process we can not predict at all (independent of history)

\[
\gamma(k) = \frac{E(x[n + k] - Ex[n + k])(x[n] - Ex[n])}{\sigma(x[n + k])\sigma(x[n])} = 0, \text{ if } k \neq 0
\]

- In other words, it is the most ‘violent’ noise

- White noise draws its name from white light which will become clear in the next few lectures
white noise $w[n]$

- The most ‘noisy’ noise is a white noise since its autocorrelation is zero, i.e.
  \[
  \text{corr}(w[n], w[m]) = 0 \text{ when } n \neq m
  \]

- Otherwise, we called it colour noise since we can predict some outcome of $w[n]$, given $w[m]$, $m<n$
Why do we love Gaussian?

Sweety Gaussian
A linear combination of two Gaussian random variables is Gaussian again.

For example, given two independent Gaussian variables X and Y with mean zero:

- \(aX + bY\) is a Gaussian variable with mean zero and variance \(a^2 \sigma(X) + b^2 \sigma(Y)\).

This is very rare (the only one in continuous distribution) but extremely useful: panda in the family of all distributions.