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Frequency Response of an MA filter

We can formally represent the frequency response of the filter by substituting
\[ z = \exp(j\omega) \]
and obtain
\[ H(\omega) = K[(\exp(-j\omega) - \alpha_1)\cdots(\exp(-j\omega) - \alpha_N)] \]

Consider an example
What is a Filter?

• For a given power spectrum of a signal
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What is a Filter?

- For a given power spectrum of a signal
Ideal response function as plotted below

- Find coefficients $a$ and $b$

\[ y(n) = a_0 x(n) + a_1 x(n-1) + \ldots + a_N x(n-N) + b_1 y(n-1) + \ldots + b_N y(n-N) \]
Example

\[ H(\omega) = K[(\exp(-j\omega) - \alpha_1)(\exp(-j\omega) - \alpha_2)] \]

(sampling frequency \(F_s\), stop frequency \(F_0\), put zero = \((F_0/F_s/2)\pi = 2\pi F_0/F_s\))

with \(\alpha_1 = \exp(-j\pi/4)\), \(\alpha_2 = \exp(j\pi/4)\), for example, then

\[ Y(\omega) = H(\omega) X(\omega) \]

and \(Y(\omega) = 0\) whenever \(\omega = \pi/4\).

Any signal with a frequency of \(F_0\) will be stopped.
Transfer function

```matlab
h=0.01;
for i=1:314
    x(i)=i*h;
    f(i)=(exp(-j*x(i))-exp(-j*pi/4))*(exp(-j*x(i))-exp(j*pi/4));
end
plot(x,abs(f))
```
Moving $\alpha_1$ along the circle, we are able to stop a signal with a given frequency.
The original difference expression can be recovered by

\[ H(z) = k \left[ (z^{-1} - \alpha_1)(z^{-1} - \alpha_2) \right] \]
\[ = k \left[ z^{-2} - (\alpha_1 + \alpha_2) z^{-1} + (\alpha_1 \alpha_2) \right] \]

\[ y(n) = k \left[ x(n-2) - (\alpha_1 + \alpha_2) x(n-1) + (\alpha_1 \alpha_2)x(n) \right] \]

\[ a_0 = k \left( \alpha_1 \alpha_2 \right) = k, \quad a_1 = -k(\alpha_1 + \alpha_2), \quad a_2 = k \]
Recursive Filters

Of the many filter transfer function, the most commonly use in DSP are the recursive filters, so called because their current output depends not only on the last N inputs but also on the last N outputs.

\[ y(n) = a_0 x(n) + a_1 x(n-1) + \ldots + a_N x(n-N) \]

\[ + b_1 y(n-1) + \ldots + b_N y(n-N) \]
Transfer function

\[ H(z) = \frac{A(z)}{B(z)} \]

where

\[ A(z) = a_0 + a_1 z^{-1} + \ldots + a_N z^{-N} \]

\[ B(z) = 1 - b_1 z^{-1} - \ldots - b_N z^{-N} \]
Block diagram

\[ x(n) \]

\[ a_N \]

\[ a_{N-1} \]

\[ a_1 \]

\[ a_0 \]

\[ y(n) \]

\[ b_N \]

\[ b_{N-1} \]

\[ b_1 \]
Poles and zeros

We know that the roots \( \alpha_n \) are the zeros of the transfer function.

The roots of the equation \( B(z) = 0 \) are called the poles of the transfer function.

They have greater significance for the behaviour of \( H(z) \): it is singular at the points \( z = \beta_n \).

Poles are drawn on the z-plane as crosses, as shown.

\[
H(z) = \frac{A(z)}{B(z)} = \frac{1}{N} \sum_{i=1}^{N} \frac{(z - \alpha_i)}{(z - \beta_i)}
\]
Poles and zeros
Poles and zeros

ON this figure, the unit circle has been shown on the z-plane.

In order for a recursive filter to be *bounded input bounded output (BIBO) stable*, all of the poles of its transfer function must lie inside the unit circle.

A filter is BIBO stable if any bounded input sequence gives rise to a bounded output sequence.

Now if the pole of the transfer lie insider the unit circle,

A simple case (N=1)

\[
H(z) = k \frac{(z)}{(z - b)} = \frac{k}{(1 - bz^{-1})}
\]

\[
y(n) = y(n - 1) + kx(n)
\]
Poles and zeros

\( y(n) \) is stable if and only if the pole is inside the unit circle
\[ |\beta| < 1 \]

In general, \( y(n) \) is stable is stable if and only if all poles are inside the unit circle
\[ |\beta_i| < 1, \ i = 1, \ldots, N \]
If $|\beta_i| = 1$, the filter is said to be conditionally stable: some input sequence will lead to bounded output sequence and some will not.

Since MA filters have no poles, they are always BIBO (bound input bound output) stable: zeros have no effect on stability.
Now suppose we have the transfer function of a filter

\[ H(z) = \frac{A(z)}{B(z)} = k \prod_{i=1}^{N} \frac{(z - a_i)}{(z - b_i)} \]

We can formally represent the frequency response of the filter by substituting

\[ z = \exp(j \omega) \]
Response

Obviously, \( H(\omega) \) depends on the locations of the poles and zeros of the transfer function, a fact which can be made more explicit by factoring the numerator and denominator polynomials to write

\[
H(\omega) = \frac{A(\omega)}{B(\omega)} = k \prod_{i=1}^{N} \left( \exp(j \omega) - a_i \right) \prod_{i=1}^{N} \left( \exp(j \omega) - b_i \right)
\]
Each factor in the numerator or denominator is a complex function of frequency, which has a graphical interpretation in the terms of the location of the corresponding roots in relation to the unit circle. Thus we can make a reasonable guess about the filter frequency response imply by looking at the pole-zero diagram.
There are four main classes of filter in widespread use:

- Lowpass
- highpass
- bandpass
- bandstop

The name are self-explanatory.
This four types are shown in the next figure.
Filter Types

(i) $H(\omega)$

(ii) $H(\omega)$

(iii) $H(\omega)$

(iv) $H(\omega)$
DCSP-11: Filter

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Aim

• To find the coefficient a’s and b’s for certain purposes:

for example, filter out noise, stop certain band signals, allow certain band signal to pass etc.
A Trick

\[\{y(n)\} = a_0\{x(n)\} + a_1\{x(n-1)\} + \ldots + a_N\{x(n-N)\} \]
\[+ b_1\{y(n-1)\} + \ldots + b_N\{y(n-N)\}\]

Multiplying \(z^{-n}\) on both size of the equation above where \(z\) is a complex number and summing over \(n\)

\[Y(z) = a_0X(z) + a_1z^{-1}X(z) + \ldots + a_Nz^{-N}X(z)\]
\[+ b_1z^{-1}Y(z) + \ldots + b_Nz^{-N}Y(z)\]

\[Y(z) - (b_1 z^{-1} Y(z) + \ldots + b_N z^{-N} Y(z)) = a_0 X(z) + a_1 z^{-1} X(z) + \ldots + a_N z^{-N} X(z)\]
A Trick

\[ [1-b_1z^{-1}-\ldots-b_Nz^{-N}] \ Y(z) \]
\[ = [ a_0+a_1z^{-1}+\ldots+a_Nz^{-N}] \ X(z) \]

\[ \frac{Y(z) = \{ [a_0+a_1z^{-1}+\ldots+a_Nz^{-N}] / [1-b_1z^{-1}-\ldots-b_Nz^{-N}] \} \ X(z) \]
\[ = H(z) \ X(z) \]

\(H(z)\) is usually called **transfer function**: it characterizes the input output relationship of a filter
A Trick

• \( Y(z) = \left\{ \frac{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}{1 - b_1 z^{-1} - \ldots - b_N z^{-N}} \right\} \)
  \[ X(z) \]
  \[ = H(z) X(z) \]

• For example, when \( z = \exp(j \omega) \)
  \[ Y(\omega) = H(\omega) X(\omega) \]
  \( X \) and \( Y \) are actually the DTFT of \( x \) and \( y \)

• In particular, look at a specific frequency \( \omega_0 \), for example
  \[ Y(\omega_0) = H(\omega_0) X(\omega_0) \]

• The magic of FT !!!!!
A Trick

- \( Y(\omega_0) = H(\omega_0) \times X(\omega_0) \)

- \( H(\omega_0) = 1 \) the signal will not be affected at all

- \( H(\omega_0) = 0 \), the signal is stopped

- your ideas for developing filter?
Nonrecursive Filters

• When a filter is nonrecursive, its difference equation can be written

\[ y(n) = a_0 x(n) + a_1 x(n-1) + \ldots + a_N x(n-N) \]

• Such filters are also called finite impulse response filters, for the obvious reason that their IR contain only finitely many nonzero terms.

• Correspondingly, the TF of a nonrecursive filter can be written as

\[ H(z) = \sum_{m=0}^{N} a_m z^{-m} \]
Block diagram
Three types of representation

- Linear difference equation
- Block diagram
- Transfer function
Trying to figure out how an MA filter will behave, is not always so simple.

Another way of looking at it is through its frequency domain behaviors.

We can make a start on this by examining the zeros of its transfer function $H(z)$, i.e. those values of $z$ for which

$$H(z) = 0$$

since $H(z)$ is a polynomial of order $N$ with real coefficients,
Zeros

Trying to figure out how an FIR filter will behave, is not always so simple.

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$H(z)$ is a polynomial of order $N$ with real coefficients,

- the **complex conjugate root theorem** states that if $P$ is a polynomial in one variable with real coefficients, and $a + bi$ is a root of $P$ with $a$ and $b$ real numbers, then its complex conjugate $a - bi$ is also a root of $P$
Zeros

• We can express $H(z)$ in terms of the roots by writing

$$z^N H(z) = \prod_{m=1}^{N} (z - z_m)$$

where in general

$$z_m = |z_m| \exp(j \arg[z_m])$$

is the mth root, or zero of the transfer function.

• The zeros of a transfer function are usually denoted graphically in the complex z-plane by circles, as shown in the following Fig.
Zeros
Recursive Filters

Of the many filter transfer function which are not MA, the most commonly use in DSP are the recursive filters, so called because their current output depends not only on the last N+1 inputs but also on the last N outputs.