Seminar II

January 27, 2016

Summary of this week
1. Fourier Transform in terms of sin and cos: holds for periodic function only

\[ x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\omega nt) + \sum_{n=1}^{\infty} B_n \sin(\omega nt) \] (1)

2. Fourier Transform in terms of complex exponentials: holds for any reasonable function.

\[ X(F) = FT(x) = \int_{-\infty}^{\infty} x(t) \exp(-2\pi j F t) dt \] (2)

\[ x(t) = FT^{-1}(X) = \int_{-\infty}^{\infty} X(F) \exp(2\pi j F t) dF \] (3)

Exercises:

Question 1: Can you plot out the power spectrum for \( \cos(\omega t + \pi) \) using the two different FTs above?

Question 2: Assume that \( x(t) \) is a real signal (our voice for example). To prove that

\[ |X(F)| = |X(-F)| \]

Hence the negative frequency makes sense and we can only pay our attention to the positive part since it is symmetric (someone’s question, great).

Question 3: Work through our Fourier song (italic parts will be dealt with later in our module).

Integrate your function times a complex exponential
It’s really not so hard you can do it with your pencil
And when you’re done with this calculation
You’ve got a brand new function - the Fourier Transformation What a prism does to sunlight, what the ear does to sound
Fourier does to signals, it’s the coolest trick around
Now filtering is easy, you don’t need to convolve
All you do is multiply in order to solve.
From time into frequency - from frequency to time Every operation in the time domain
Has a Fourier analog - that’s what I claim
Think of a delay, a simple shift in time

\[
FT(\tilde{x}(t - \theta)) = \int x(t - \theta) e^{-2\pi jFt}dt
= \int x(s) e^{-2\pi j(s + \theta)F}ds
= \exp(-2\pi j(\theta)F) FT(x)
\]

It becomes a phase rotation - now that’s truly sublime! And to differentiate, here’s a simple trick

\[
FT(x'(t)) = \int x'(t) e^{-2\pi jFt}dt
= \int e^{-2\pi jFt}dx(t)
= \exp(-2\pi jFt)x(t)|_{-\infty}^{\infty} - \int x(t) e^{-2\pi jFt} dt
= 0 + 2\pi jF \int x(t) e^{-2\pi jFt} dt = 2\pi jF \times FT(x)
\]

Just multiply by J omega, ain’t that slick?
Integration is the inverse, what you gonna do?
Divide instead of multiply - you can do it too.
From time into frequency - from frequency to time
Let’s do some examples... consider a sine
It’s mapped to a delta, in frequency - not time
Now take that same delta as a function of time
Mapped into frequency - of course - it’s a sine!
Sine x on x is handy, let’s call it a sinc.
Its Fourier Transform is simpler than you think.
You get a pulse that’s shaped just like a top hat...
Squeeze the pulse thin, and the sinc grows fat.

\[
\int_{-L}^{L} \exp(-2\pi Ftj)dt = (4)
\]

Following lecture notes.
Question 4: For an image (two dimensional data), how can we do FT with it? Will talk in our module later on.

Question 5:

\[ f(x) = \begin{cases} 
-k & \text{if } -\pi < x < 0 \\
k & \text{if } 0 < x < \pi 
\end{cases} \]

\[ f(x + 2\pi) = f(x) \]

Question 6:

\[ f(x) = \begin{cases} 
0 & \text{if } -2 < x < -1 \\
k & \text{if } -1 < x < 1 \\
0 & \text{if } 1 < x < 2 
\end{cases} \]

where \( T = 2L = 4 \).

Question 7:

\[ f(x) = \begin{cases} 
1 & \text{if } |x| < 1 \\
0 & \text{if } |x| > 1 
\end{cases} \]
Periodic function with \( T = 2\pi \)

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0 \quad (\text{odd function}) \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \]

\[ = \frac{1}{\pi} \left( \int_{-\pi}^{0} -k \sin nx \, dx + \int_{0}^{\pi} k \sin nx \, dx \right) \]

\[ = \frac{1}{\pi} \left[ \left. \frac{k}{n} \cos nx \right|_{-\pi}^{0} + \left. -\frac{k}{n} \cos nx \right|_{0}^{\pi} \right] \]

\[ = \frac{1}{\pi} \left[ \frac{k}{n} - \frac{k}{n} \cos(-n\pi) + \left( -\frac{k}{n} \cos(n\pi) + \frac{k}{n} \right) \right] \]

\[ = \frac{1}{\pi} \left( \frac{k}{n} - \frac{k}{n} \cos(n\pi) - \frac{k}{n} \cos(n\pi) + \frac{k}{n} \right) \]

\[ = \begin{cases} \frac{4k}{n\pi} & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even} \end{cases} \]

\[ f(x) = \frac{4k}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \ldots \right) \]
Q6.

Periodic function with $T = 2L = 4$

\[ a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx = \frac{1}{4} \int_{-2}^{2} f(x) \, dx = \frac{1}{4} \int_{-1}^{1} k \, dx \]

\[ = \frac{1}{4} \left[ kx \right]_{-1}^{1} = \frac{k}{2} \]

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n \pi x}{L} \, dx \]

\[ = \frac{1}{2} \int_{-2}^{2} f(x) \cos \frac{n \pi x}{2} \, dx = \frac{1}{2} \int_{-1}^{1} k \cos \frac{n \pi x}{2} \, dx \]

\[ = \frac{k}{2} \left[ \frac{2k}{n \pi} \sin \frac{n \pi x}{2} \right]_{-1}^{1} = \frac{2k}{n \pi} \sin \frac{n \pi}{2} \]

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n \pi x}{L} \, dx \]

\[ = \frac{1}{2} \int_{-2}^{2} f(x) \sin \frac{n \pi x}{2} \, dx = \frac{1}{2} \int_{-1}^{1} k \sin \frac{n \pi x}{2} \, dx \]

\[ = 0 \]

\[ \Rightarrow f(x) = \frac{k}{2} + \frac{2k}{\pi} \left( \cos \frac{\pi x}{2} - \frac{1}{3} \cos \frac{3 \pi x}{2} + \frac{1}{5} \cos \frac{5 \pi x}{2} + \ldots \right) \]

(Remark: FT for function of period $2L$:

\[ f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n \pi x}{L} \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{n \pi x}{L} \right) \])
\[ F(w) = \int_{-\infty}^{\infty} f(x) e^{-i\omega t} \, dx \]

\[ = \int_{-1}^{1} e^{-i\omega t} \, dt = \left[ \frac{e^{-i\omega t}}{-i\omega} \right]_{-1}^{1} \]

\[ = \frac{e^{-i\omega} - e^{i\omega}}{-i\omega} = \frac{e^{i\omega} - e^{-i\omega}}{i\omega} \]

1. When \( w \neq 0 \), \[ F(w) = \frac{2}{w \sin w} \]

Since \( \sin w = \frac{e^{i\omega} - e^{-i\omega}}{2i} \)

2. When \( w = 0 \), \[ F(w) = \int_{-1}^{1} dt = 2 \]

Also \( \lim_{w \to 0} \frac{2}{w \sin w} = 2 \)

Here, I used FT of form:

\[ F(w) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx, \quad \text{and Inverse FT would be:} \]

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{i\omega x} \, dw \]