Computational Issues in Voting Power

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Weighted Voting

Example

- $[51; 50, 49, 1]$ for Labour, Lib Dem and the Green Party.
- Lib Dem and Green Party are *symmetric*.

Motivation

Motivation

- Application in Political Science (EU, UN, IMF)
- Relation to Neuorscience and Threshold Logic
- Application in Economics (Shareholders)
- Distributed Systems

Simple Voting Games

Simple Voting Game

Let $N = \{1, 2, ..., n\}$ be the set of players and W set of winning coalitions. Simple Voting Game (SVG), (N, W) is a game satisfying the following conditions:

- N is a winning coalition
- $\emptyset$ is a losing coalition W.
- *Monotonicity*: a super-set of a winning coalition is also winning

Weighted Voting Game

(Weighted Voting Game). A canonical *weighted voting game (WVG)* is denoted by $[q; w_1, w_2 ..., w_n]$ where w_i is the voting weight of player i
$W = \{X \subseteq N, \sum_{x \in X} w(x) \geq q\}$

Key Concepts

Key Concepts

- *Critical Player*: can change outcome of a coalition
- *Dummy player*: does not any power.
- *Minimal Winning Coalition*: does not include another winning coalition.

Banzhaf Index

Banzhaf Index

Banzhaf Value of a player is the number of times it is critical.
Banzhaf Index, β_i is the ratio of the number of times a player is critical to the number of times all players are critical.

Example

$[51; 50, 49, 1]$ for Labour, Lib Dem and the Green Party.
The winning coalitions are:

- $\{$Labour, Lib Dem$\}$: critical members are Labour and Lib Dem
- $\{$Labour, Green Party$\}$: critical members are Labour and Green Party
- $\{$Labour, Green Party, Lib Dem$\}$: critical member is Labour.

Banzhaf Index

Banzhaf Index

- Number of times Labour is critical: 3
- Number of times Green Party is critical:1
- Number of times Lib Dem is critical:1
- *Banzhaf index* of Labour is 3/5, *Banzhaf Index* of Green Party is 1/5 and the *Banzhaf Index* of Lib Dem is 1/5.

Shapley-Shubik Index

SS Index

The Shapley-Shubik (SS) Index of a player *i* can be summarised as the ratio of the number of permutations in which a player is ‘pivotal’ to the number of orderings all players are pivotal.

Example

$[51; 50, 49, 1]$ for Labour, Lib Dem and the Green Party
All the permutations are:
Labour, Lib Dem, Green Party
Labour, Green Party, Lib Dem
Lib Dem, Labour, Green Party
Lib Dem, Green Party, Labour
Green Party, Lib Dem, Labour
Green Party, Labour, Lib Dem,

Shapley-Shubik Index

SS Index

- Number of times Labour is pivotal: 4
- Number of times Lib Dem is pivotal: 1
- Number of times Green Party is pivotal: 1
- The SS index of Labour is $4/6$, the SS index of Lib Dem is $1/6$ and the SS index of the Green Party is $1/6$.

Other Indices

Let ω be the number of winning coalitions and η_i the number of times player i is critical.

- Power of the Body to Act = PTA = $\frac{\omega}{2^n}$
- Power of a member to prevent action = $PPA_i = \frac{\eta_i}{\omega}$
- Power of a member to initiate action = $PIA_i = \frac{\eta_i}{2^n - \omega}$
- Deegan-Packel index
- Holler index
- Johnston's index

Complexity

Complexity

It is NP-HARD to compute voting power indices.

Complexity

It is even NP-HARD to identify the dummy players or symmetric players in a WVG. [Matsui and Matsui, 1994]

Voting Power Algorithms

Computing Voting Power

- Direct Enumeration: exponential in n
- Dynamic programming: pseudo polynomial in n and q
- Generating Functions: pseudo polynomial in n and q
- Multi-linear Extensions Approximation
- Monte Carlo Method

Generating Function Method

GF for Banzhaf Index

- $B(x) = (1+x^{w_1})(1+x^{w_2})...(1+x^{w_n}) =$
  $1 + c_1x + c_2x^2 + c_3x^3 + ... + c_Wx^W$ where $W = \sum_{i \in N} w_i$
  Coefficient of x^i in $B(x)$ is the number of coalitions with the total weight, i.
- Number of times i is critical is:
  $\beta_i{}' = [x^q]\frac{B(x)}{1+x^{w_i}}(x + x^2 + ...x^{w_i})$

GF for SS Index

Similar method for *Shapley-Shubik index*.

Skewed WVS

Skewed WVS

Skewed WVS $[q; w_1, ..., w_n]$ in which $w_j > w_{j+1} + ... + w_n$ $\forall j$ $1 \leq j \leq n$

Problem $P(i)$

Problem $P(i)$: Is player i a dummy or not? This is equivalent to the finding whether i is *critical* for some coalition or not. This can be represented as a *subset sum problem*:
Maximise $\sum_{j \in N \setminus \{i\}} w_j x_j$
Condition: $\sum_{j \in N \setminus \{i\}} w_j x_j < q$, $x_j \in \{0, 1\}$
Player i is critical if the solution $+ w_i \geq q$.

Problem P

Let Problem P be the related problem of
Max $\sum_{j \in N} w_j x_j$
Condition: $\sum_{j \in N} w_j x_j < q$, $x_j \in \{0, 1\}$

Skewed WVS

IS-WVS-SKEWED

INPUT: $[q; w_1, ...w_n]$
OUTPUT: YES if the WVS is skewed, NO if the WVS is not skewed.
SUM = w_n
Loop($i = n - 1, i > 0, i -$)
$\{$
IF($w_i <$ SUM) RETURN NO
SUM = SUM $+ w_i$
$\}$
RETURN YES
The complexity of the algorithm is O(n).

Skewed WVS

Greedy Algorithm

INPUT: $[q; w_1, ..., w_n]$
SUM =0, $j = 1$
Step 1:
IF (SUM $+w_j < q$) $\{g_j = 1$, SUM= SUM $+w_j, j = j + 1\}$
ELSE $\{g_j = 0\}$
Step 2: IF($j = n + 1$) OUTPUT: $\{g = (g_1, ...g_n)$, SUM $\}$
ELSE repeat *Step 1*

Skewed WVS

Theorem

The greedy algorithm solves Problem P if the WVS is skewed

Proof

Let x be another feasible solution of P such that $g_j = x_j$ for $j < i$
Then $x_i = 0$, $g_i = 1$ where i is the first index in which x and g differ.
This is because if $x_i = 1$ and x is a feasible solution then $g_i = 1$ because g is a greedy solution.
$\sum_{j \in N} w_j g_j \geq \sum_{j<i} w_j g_j + w_i \geq \sum_{j<i} w_j x_j + \sum_{j>i} w_i$
Therefore $\sum_{j \in N} w_j g_j \geq \sum_{j \in N} w_j x_j$ since $x_i = 0$

Skewed WVS

Theorem

$P(i)$ **for all** i **can be solved in linear time if the WVS is skewed.**

Proof

- INPUT: WVS $[q; w_1, ...w_n]$
  Run the greedy algorithm to solve P and output $g = (g_1, ..., g_n)$
- Now let's check the simulation of the greedy algorithm on set of all weights minus w_i. Let the output vector be v.
  For first $i-1$ steps, v and g are same.
  If $g_i = 0$ than the last $n-i$ indices of v are also same.
- But if $g_i = 1$ then SUM $+w_{i+1} + ... + w_n <$ SUM $+w_i \leq q$ and therefore the greedy algorithm puts $v_j = 1 \forall j > i$
- Therefore for every i we need to check g_i and get the corresponding vector v for that i. All this takes n steps. (The greedy algorithm can be used to solve P(i) for all i.)

Skewed WVS

Number of feasible solutions to P

If the WVS is skewed, the number of feasible solutions to P is $\sum_{j=1}^{n} g_j 2^{n-j} + 1$

Proof

- The set of feasible solutions = S = $\cup_{i=1}^{n} S_i + \{g\}$ where the 1 signifies the count for the greedy solution and S_i is a set of feasible feasible solutions which match the greedy solution in the first $i-1$ indices but differs on the ith.
- Now if $g_i = 0$ then $|S_i| = 0$
- Suppose $g_i = 1$. Then any solution in S_i matches with g in the first $i-1$ coordinates.
  $\sum_{j>i} w_j \leq w_i < q - \sum_{j<i} w_j$. So S_i has 2^{n-i} solutions. Therefore $|S| = \sum_{j=1}^{n} g_j 2^{n-j} + 1$.
- Number of feasible solutions of $P(i)$ can also be found in O(n).

Skewed WVS

Computing Banzhaf Indices

Banzhaf Index of all players can be found in O(n^2)

- A = Number of feasible solutions of $\sum_{j \in N \setminus \{i\}} w_j x_j < q$, $x_j \in \{0, 1\}$
- B = Number of feasible solutions of $\sum_{j \in N \setminus \{i\}} w_j x_j < q - w_i$, $x_j \in \{0, 1\}$
- Number of times player i is critical = $A - B$
- Similarly find how many times each player is critical.

Other Aspects

Aspects

- Ternary Voting (Absentation and Quorum)
- Spatial Voting (Considering Ideological preferences)
- Designing Voting Games

Future Work

Future Work

- Increased understanding of the axiomatic foundations of voting power.
- Improvement of algorithms
- Increased insight into designing Games
- Build up on the theory on ternary voting, spatial voting, multiple tier voting etc.
- Transform one index to another.
- Make a tool kit to compute and analyse voting power