Trading Probability for Fairness*

(Extended Abstract)

Marcin Jurdziński[1], Orna Kupferman[2], and Thomas A. Henzinger[1]

[1]EECS, University of California, Berkeley
[2]School of Computer Science and Engineering, Hebrew University

Abstract. Behavioral properties of open systems can be formalized as objectives in two-player games. Turn-based games model asynchronous interaction between the players (the system and its environment) by interleaving their moves. Concurrent games model synchronous interaction: the players always move simultaneously. Infinitary winning criteria are considered: Büchi, co-Büchi, and more general parity conditions. A generalization of determinacy for parity games to concurrent parity games demands probabilistic (mixed) strategies: either player 1 has a mixed strategy to win with probability 1 (almost-sure winning), or player 2 has a mixed strategy to win with positive probability.

This work provides efficient reductions of concurrent probabilistic Büchi and co-Büchi games to turn-based games with Büchi condition and parity winning condition with three priorities, respectively. From a theoretical point of view, the latter reduction shows that one can trade the probabilistic nature of almost-sure winning for a more general parity (fairness) condition. The reductions improve understanding of concurrent games and provide an alternative simple proof of determinacy of concurrent Büchi and co-Büchi games. From a practical point of view, the reductions turn solvers of turn-based games into solvers of concurrent probabilistic games. Thus improvements in the well-studied algorithms for the former carry over immediately to the latter. In particular, a recent improvement in the complexity of solving turn-based parity games yields an improvement in time complexity of solving concurrent probabilistic co-Büchi games from cubic to quadratic.

1 Introduction

In *formal verification*, a *closed system* is a system whose behavior is completely determined by the state of the system, while an *open system* is a system that interacts with its environment and whose behavior depends on this interaction [11]. While formal verification of closed systems uses models based on labeled transition systems, formal analysis of open systems, and the related problems of control and synthesis, use models based on *two-player games*, where

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one player represents the system, and the other player represents the environment [18, 19, 1, 7, 13, 14]. At each round of the game, player 1 (the system) and player 2 (the environment) choose moves, and the choices determine the next state of the game. Specifications of open systems can be expressed as objectives in such games, and deciding whether an open system satisfies a specification is reduced to deciding whether player 1 has a winning strategy in the game. The construction of winning strategies can also be used to *synthesize* correct systems and controllers from their specifications [18, 19].

In practice, games with finitary winning conditions, such as reachability and safety games play a prominent role. Games with infinitary winning conditions, such as Büchi, co-Büchi, and general parity games [21] are richer from the theoretical point of view. Apart from being a versatile tool in the theory of formal verification [21] they can also be used in practice to model liveness and fairness specifications [15].

While *turn-based* games have been heavily studied [15, 21, 12], *concurrent* games have been considered only recently [5, 4]. Modelling based on turn-based games assumes that interaction between the system and the environment is asynchronous and actions of the two players can be interleaved. Concurrent games are better suited for modeling synchronous interaction [2, 3]. In every round of a concurrent game the two players choose moves simultaneously and independently, and the pair of choices of both players determines the next state of the game. If a system exhibits a mix of synchronous and asynchronous interaction which depends on some external factors, one can attempt to reconcile the two by allowing in the game probabilistic moves assigning appropriate probabilities to each option.

Solving concurrent games requires new concepts and techniques when compared to turn-based games. For example, determinacy of turn-based parity games does not easily carry over to concurrent games. While deterministic and memoryless (pure) strategies suffice for turn-based games [9, 21, 23], probabilistic (mixed) strategies with possibly infinite memory are necessary for winning concurrent games [5, 4].

Theorem 1. [4] *In a concurrent parity game, in every vertex, either player 1 has a mixed strategy to win with probability 1, or player 2 has a mixed strategy to win with positive probability.*

We encourage the reader to refer to the papers by de Alfaro at al. [5] and de Alfaro and Henzinger [4] for small and lucid examples of games which exhibit some of the conceptual hurdles needed to be overcome in order to solve concurrent reachability, Büchi, and co-Büchi games.

This paper offers an alternative way to solve concurrent Büchi and co-Büchi games, by providing an efficient reduction of concurrent games to turn-based games. Specifically, we prove the following.

Theorem 2. *There are linear-time reductions from concurrent Büchi games to turn-based Büchi games, and from concurrent co-Büchi games to Parity(0,2) games.*[1]

From the theoretical point of view, interesting by-products of our proofs of the above fact are conceptually simple proofs of determinacy for concurrent Büchi and co-Büchi games that invoke the classical determinacy theorem for turn-based parity games [9, 21, 23]. On the practical side, our reductions turn solvers of non-probabilistic turn-based parity games into solvers of probabilistic concurrent games. Thus, improvements in the well-studied algorithms for the former [10, 15, 8, 20, 12, 22] will immediately carry over to the latter. In particular, a recent result [12] improving the complexity of parity games, together with our latter translation yields an improvement in the complexity of solving concurrent co-Büchi games from cubic [4] to quadratic.

A key novel technical concept behind the correctness proofs of our reductions is that of *witness functions* for concurrent Büchi and co-Büchi games, generalizing signature assignments [9, 23] and progress measures [12] from turn-based games to concurrent games. Witness functions label the states of a concurrent game with (tuples of) numbers so that certain local conditions on edges of the game graph are satisfied. A technical advantage of witness functions is that it suffices to check the local conditions on a set of vertices in order to conclude that the respective player has a winning strategy in an infinite game from every state in the set. As in the article of de Alfaro and Henzinger [4] the local conditions are expressed in terms of probability distributions of moves (mixed moves) each player can take from a vertex. For our reductions from concurrent to turn-based games we establish "finitary" characterizations of those conditions in terms of pure moves. Then we show that these finitary characterizations can be modeled by small sub-games in which the two players follow a certain "protocol" of choosing pure moves.

Due to lack of space, this extended abstract omits many proofs, some key technical auxiliary results, and generalizations of the main results to limit-sure winning and general parity winning conditions. A full version of this paper will deal with those issues in more detail.

[1] A Parity(0,2) winning condition consists of a partition of the state space into three sets P_0, P_1, and P_2, and the objective of player 1 is either to visit P_0 infinitely often, or visit P_2 infinitely often and P_1 only finitely often.

2 Concurrent probabilistic games

For a finite set X, a *probability distribution* on X is a function $\xi : X \to [0,1]$ such that $\sum_{x \in X} \xi(x) = 1$. We denote the set of probability distributions on X by $\mathcal{D}(X)$. For a probability distribution $\xi \in \mathcal{D}(X)$ we define $||\xi||$, the *support* of ξ, by $||\xi|| = \{\, x \in X \,:\, \xi(x) > 0 \,\}$.

A two-player *concurrent probabilistic game structure* $G = (V, A, A_1, A_2, \delta)$ consists of the following components.

- A finite set V of vertices, and a finite set of actions A.
- Functions $A_1, A_2 : V \to 2^A$, such that for every vertex v, $A_1(v)$ and $A_2(v)$ are non-empty sets of actions available in vertex v to players 1 and 2, respectively.
- A probabilistic transition function $\delta : V \times A \times A \to \mathcal{D}(V)$, such that for every vertex v and actions $a \in A_1(v)$ and $b \in A_2(v)$, $\delta(v, a, b)$ is a probability distribution on the successor vertices.

At each step of the game, both players choose *moves* to proceed with. We consider two options here.

- *Pure action moves.* The set of moves is the set of actions $M = A$. The sets of moves available to players 1 and 2 in vertex v are $M_1(v) = A_1(v)$ and $M_2(v) = A_2(v)$, respectively.
- *Mixed (randomized) action moves.* The set of moves is the set of probability distributions on the set of actions $M = \mathcal{D}(A)$. The sets of moves available to players 1 and 2 in vertex v are $M_1(v) = \mathcal{D}\big(A_1(v)\big)$ and $M_2(v) = \mathcal{D}\big(A_2(v)\big)$, respectively. In this case we extend the transition function to $\delta : V \times M \times M \to \mathcal{D}(V)$, by $\delta(v, \alpha, \beta)(w) = \sum_{a \in A_1(v)} \sum_{b \in A_2(v)} \alpha(a) \cdot \beta(b) \cdot \delta(v, a, b)$.

We often write $\mathrm{Pr}_v^{\alpha,\beta}[w]$ for $\delta(v, \alpha, \beta)(w)$, and for a set $W \subseteq V$, we define $\mathrm{Pr}_v^{\alpha,\beta}[W] = \sum_{w \in W} \mathrm{Pr}_v^{\alpha,\beta}[w]$.

Thus, $\mathrm{Pr}_v^{\alpha,\beta}[w]$ is the probability that the successor vertex is w, given that the current vertex is v and the players chose to proceed with α and β. Similarly, $\mathrm{Pr}_v^{\alpha,\beta}[W]$ is the probability that the successor vertex is a member of W.

A concurrent probabilistic game is played in the following way. If v is the current vertex in a play then player 1 chooses a move $\alpha \in M_1(v)$, and simultaneously and independently player 2 chooses a move $\beta \in M_2(v)$. Then the play proceeds to a successor vertex w with probability $\mathrm{Pr}_v^{\alpha,\beta}[w]$.

A *path* in G is an infinite sequence $v_0, v_1, v_2, \ldots$ of vertices, such that for all $k \geq 0$, there are moves $\alpha \in M_1(v_k)$ and $\beta \in M_2(v_k)$, such that $\mathrm{Pr}_{v_k}^{\alpha,\beta}[v_{k+1}] > 0$. We denote by Ω the set of all paths.

We say that a concurrent game structure $G = (V, A, A_1, A_2, \delta)$ is:

- *Turn-based*, if for all $v \in V$, we have either $|A_1(v)| = 1$ or $|A_2(v)| = 1$; i.e., in every vertex only one player may have a non-trivial choice;
- *Deterministic*, if for all $v \in V$, $a \in A_1(v)$, and $b \in A_2(v)$, we have $||\delta(v, a, b)|| = 1$; i.e., in every move the next vertex is uniquely determined by the pure action moves chosen by the players. In this case we often write $\delta(v, a, b)$ for the unique $w \in V$, such that $\delta(v, a, b)(w) = 1$.

Strategies. A *strategy* for player 1 is a function $\pi_1 : V^+ \to M$, such that for a finite sequence $\overline{v} \in V^+$ of vertices, representing the history of the play so far, $\pi_1(\overline{v})$ is the next move to be chosen by player 1. A strategy must prescribe only available moves, i.e., $\pi_1(\overline{w} \cdot v) \in M_1(v)$, for all $\overline{w} \in V^*$, and $v \in V$. Strategies for player 2 are defined analogously. We write Π_1 and Π_2 for the sets of all strategies for players 1 and 2, respectively.

For an initial vertex v, and strategies $\pi_1 \in \Pi_1$ and $\pi_2 \in \Pi_2$, we define $Outcome(v, \pi_1, \pi_2) \subseteq \Omega$ to be the set of paths that can be followed when a play starts from vertex v and the players use the strategies π_1 and π_2. Formally, $v_0, v_1, v_2, \ldots \in Outcome(v, \pi_1, \pi_2)$ if $v_0 = v$, and for all $k \geq 0$, we have that $\delta(v_k, \alpha_k, \beta_k)(v_{k+1}) > 0$, where $\alpha_k = \pi_1(v_0, \ldots, v_k)$ and $\beta_k = \pi_2(v_0, \ldots, v_k)$.

Once a starting vertex v and strategies π_1 and π_2 for the two players have been chosen, the probabilities of events are uniquely defined, where an *event* $\mathcal{A} \subseteq \Omega$ is a measurable set of paths. For a vertex v, and an event $\mathcal{A} \subseteq \Omega$, we write $\Pr_v^{\pi_1,\pi_2}(\mathcal{A})$ for the probability that a path belongs to $\mathcal{A}$ when the game starts from v, and the players use the strategies π_1 and π_2.

Winning criteria. A *game* $\mathcal{G} = (G, \mathcal{W})$ consists of a game structure G and a *winning criterion* $\mathcal{W} \subseteq \Omega$ (for player 1). In this paper we consider the following winning criteria.

- *Büchi criterion.* For a set B of vertices, the Büchi criterion is defined by:

  $$\text{Büchi}(B) = \{v_0, v_1, \cdots \in \Omega : \text{for infinitely many } k \geq 0, \text{ we have } v_k \in B\}.$$

- *Co-Büchi criterion.* For a set C of vertices, the co-Büchi criterion is defined by:

  $$\text{Co-Büchi}(C) = \{v_0, v_1, \cdots \in \Omega : \text{for finitely many } k \geq 0, \text{ we have } v_k \in C\}.$$

- *Parity criterion.* Let $P = (P_0, P_1, \ldots, P_d)$ be a partition of the set of vertices. The parity criterion is defined by:

  $$\text{Parity}(P) = \{\, \overline{v} \in \Omega \;:\; \min\big(\text{Inf}(\overline{v})\big) \text{ is even} \,\},$$

  where for a path $\overline{v} = v_0, v_1, v_2, \cdots \in \Omega$, we define

  $$\text{Inf}(\overline{v}) = \{\, i \in \mathbb{N} \;:\; \text{there are infinitely many } k \geq 0, \text{ such that } v_k \in P_i \,\}.$$

Note that a parity criterion Parity(P_0, P_1) is equivalent to the Büchi criterion Büchi(P_0), and a parity criterion Parity$(\emptyset, P_1, P_2)$ is equivalent to the co-Büchi criterion Co-Büchi(P_1).

For uniformity we phrase all the results below in terms of parity games. By $C(0, j)$ we denote concurrent probabilistic parity games with a parity criterion Parity$(P_0, P_1, \ldots, P_j)$, and by $C(1, j)$ we denote concurrent probabilistic parity games with a parity criterion Parity$(\emptyset, P_1, P_2, \ldots, P_j)$. By $D(i, j)$ we denote $C(i, j)$ games with turn-based deterministic game structures. Thus, we write $C(0, 1)$ for concurrent probabilistic Büchi games, $C(1, 2)$ for concurrent probabilistic co-Büchi games, $D(0, 1)$ for turn-based deterministic Büchi games, etc.

Winning modes. Let $\mathcal{G} = (G, \mathcal{W})$ be a game. We say that a strategy $\pi_1 \in \Pi_1$ for player 1 is:

- a *sure winning* strategy for player 1 from vertex v in the game $\mathcal{G}(G, \mathcal{W})$, if for all $\pi_2 \in \Pi_2$, we have $Outcome(v, \pi_1, \pi_2) \subseteq \mathcal{W}$,
- an *almost-sure winning* strategy for player 1 from vertex v in the game $\mathcal{G}(G, \mathcal{W})$, if for all $\pi_2 \in \Pi_2$, we have $\mathrm{Pr}_v^{\pi_1, \pi_2}[\mathcal{W}] = 1$,
- a *positive-probability winning* strategy for player 1 from vertex v in the game $\mathcal{G}(G, \mathcal{W})$, if for all $\pi_2 \in \Pi_2$, we have $\mathrm{Pr}_v^{\pi_1, \pi_2}[\mathcal{W}] > 0$.

The same notions are defined similarly for player 2, with the set $\mathcal{W}$ in the winning condition replaced by $\Omega \setminus \mathcal{W}$.

For a class $\mathcal{C}$ of games, and a winning mode $\mu \in \{\mathrm{s}, \mathrm{a}, \mathrm{p}\}$, we write $\mathcal{C}_\mu$ for the class of games in which the goal of player 1 is to win with the mode μ, where "s" stands for sure win, "a" stands for almost-sure win, and "p" stands for positive-probability win. For example, $C(0, 1)_{\mathrm{a}}$ are almost-sure win concurrent probabilistic Büchi games and $C(1, 2)_{\mathrm{p}}$ are positive-probability win concurrent probabilistic co-Büchi games.

Solving games. The algorithmic problem of solving $\mathcal{C}_\mu$ games is the following: given a game $\mathcal{G}$ from class $\mathcal{C}$ and a vertex v in the game graph as the input, decide whether player 1 has a μ-winning strategy in game $\mathcal{G}$ from vertex v.

3 Witnesses for turn-based deterministic games

In order to prove that a strategy is winning for a player in a parity game, one needs to argue that all infinite plays consistent with the strategy are winning for the player. A technically convenient notion of a witness has been used in [9, 23, 12] to establish existence of a winning strategy by verifying only some finitary local conditions. We recall here the definitions and basic facts about witnesses (also called signature assignments [9, 23], or progress measures [12]) for a relevant special case $D(0,2)$ games; we leave it as an exercise to the reader to provide similar notions of witnesses for the even simpler case of $D(0,1)$ games.

For $n \in \mathbb{N}$, we write $[n]$ for the set $\{0,1,2,\ldots,n\}$, and $[n]_\infty$ for the set $\{0,1,2,\ldots,n,\infty\}$, where the element ∞ is bigger than all the others. Let $G = (V, A, A_1, A_2, \delta)$ be a game structure and let $\varphi : V \to [n]_\infty$. We define $\varphi_\infty = \{\, w \in V \;:\; \varphi(w) = \infty \,\}$, and for a vertex $v \in V$, we define $\varphi_{<v} = \{\, w \in V \;:\; \varphi(w) < \varphi(v) \,\}$, and $\varphi_{>v} = \{\, w \in V \;:\; \varphi(w) > \varphi(v) \,\}$.

Let $\mathcal{G} = \big(G, \text{Parity}(P_0, P_1, P_2)\big)$ be a $D(0,2)$ game, where G is a concurrent game graph (V, A, A_1, A_2, δ), and $\delta : V \times A \times A \to V$.

Witness for player 1. For a function $\varphi : V \to [n]_\infty$, we say that a vertex $v \in V$ is *φ-progressive for player 1* if the following holds:

$$\begin{aligned} \exists a \in A_1(v).\, \forall b \in A_2(v).\; &\big(v \in P_0 \;\Rightarrow\; \delta(v,a,b) \notin \varphi_\infty\big) \wedge \\ &\big(v \in P_1 \;\Rightarrow\; \delta(v,a,b) \in \varphi_{<v}\big) \wedge \\ &\big(v \in P_2 \;\Rightarrow\; \delta(v,a,b) \notin \varphi_{>v}\big). \end{aligned} \tag{1}$$

We say that the function φ is a *(sure win) witness for player 1* if every vertex $v \in \varphi_{<\infty}$ is φ-progressive for player 2.

Witness for player 2. For a pair of functions $\psi = (\psi^0, \psi^2)$, such that $\psi^0 : V \to [n]_\infty$, and $\psi^2 : V \to [n]$, we say that a vertex $v \in V$ is *ψ-progressive for player 2* if the following holds:

$$\begin{aligned} \exists b \in A_2(v).\, \forall a \in A_1(v).\; &\big(v \in P_0 \;\Rightarrow\; \delta(v,a,b) \in \psi^0_{<v}\big) \wedge \\ &\big(v \in P_1 \;\Rightarrow\; \delta(v,a,b) \notin \psi^0_{>v}\big) \wedge \\ &\big(v \in P_2 \;\Rightarrow\; \delta(v,a,b) \in \psi_{<v}\big), \end{aligned} \tag{2}$$

where we define

$$\psi_{<v} = (\psi^0, \psi^2)_{<v} = \big\{\, w \in V \;:\; \big(\psi^0(w), \psi^2(w)\big) <_{\text{lex}} \big(\psi^0(v), \psi^2(v)\big) \,\big\},$$

and $<_{\text{lex}}$ is the lexicographic ordering. We define $\psi_{<\infty} = \psi^0_{<\infty}$. We say that the function ψ is a *(sure win) witness for player 2* if every vertex $v \in \psi_{<\infty}$ is ψ-progressive for player 2.

Lemma 1. *If φ is a witness for player 1 and ψ is a witness for player 2, then player 1 has a winning strategy from every vertex $v \in \varphi_{<\infty}$, and player 2 has a winning strategy from every vertex $v \in \psi_{<\infty}$.*

The following fact amounts to determinacy for turn-based parity games.

Theorem 3. [9, 23] *If $\mathcal{G}$ is a deterministic turn-based parity game, then there is a witness φ for player 1, and a witness ψ for player 2, such that $\varphi_{<\infty} \cup \psi_{<\infty} = V_{\mathcal{G}}$. Therefore, from every vertex one of the players has a winning strategy.*

In Section 4 we define witnesses for both players in concurrent almost-sure win Büchi games. Then in Section 5 we use them to give a reduction from concurrent probabilistic almost-sure win Büchi games to turn-based deterministic Büchi games. As a by-product we get the following as a corollary of Theorem 3.

Theorem 4. *If $\mathcal{G}$ is a $C(0,1)_a$ game, then there is a witness φ for player 1, and a witness ψ for player 2, such that $\varphi_{<\infty} \cup \psi_{<\infty} = V_{\mathcal{G}}$.*

In Section 6 we define witnesses for both players in concurrent almost-sure win co-Büchi games. Then in Section 7 we use them to give a reduction from concurrent probabilistic almost-sure win co-Büchi games to $D(0,2)$ games. As a by-product we get the following as a corollary of Theorem 3.

Theorem 5. *If $\mathcal{G}$ is a $C(1,2)_a$ game, then there is a witness φ for player 1, and a witness ψ for player 2, such that $\varphi_{<\infty} \cup \psi_{<\infty} = V_{\mathcal{G}}$.*

Note that Theorems 4 and 5 together with Lemma 1 imply Theorem 1, i.e., determinacy of concurrent Büchi and co-Büchi games.

4 Witnesses for concurrent Büchi games

Proving that a player in a concurrent parity game has a winning strategy, in particular for non-sure winning modes, is often quite involved. Instead of proving from first principles that certain strategies are winning for a player, we introduce, for various winning modes and criteria, the notions of witnesses, which are functions that assign natural numbers to vertices so that the assignment satisfies certain "local" constraints. We then prove that a witness for a player gives rise to a winning strategy for him. Once we show that witnesses are sufficient conditions for existence of winning strategies, we only need to focus on constructing witnesses, which is easier than analyzing probabilities of sets of infinite probabilistic plays induced by strategies, since only local finitary constraints need to be verified.

Let $\mathcal{G} = \big(G, \text{Parity}(P_0, P_1)\big)$ be a $C(0,1)$ game with $G = (V, A, A_1, A_2, \delta)$.

Witness for player 1. For a function $\varphi : V \to [n]_\infty$, we say that a vertex $v \in V$ is *φ-progressive for player 1* if the following holds:

$$\begin{aligned} \exists_{\varepsilon>0} \cdot \exists_{\alpha \in \mathcal{D}(A_1(v))} \cdot \forall_{\beta \in \mathcal{D}(A_2(v))} \cdot \big(v \in P_0 \;\Rightarrow\; \mathrm{Pr}_v^{\alpha,\beta}[\varphi_\infty] = 0\big) \wedge \\ \big(v \in P_1 \;\Rightarrow\; \mathrm{Pr}_v^{\alpha,\beta}[\varphi_\infty] = 0 \;\wedge\; \mathrm{Pr}_v^{\alpha,\beta}[\varphi_{<v}] \geq \varepsilon\big). \end{aligned} \tag{3}$$

We say that the function φ is an *(almost-sure win) witness for player 1* if every vertex $v \in \varphi_{<\infty}$ is φ-progressive for player 1.

Lemma 2. *If $\varphi : V \to [n]_\infty$ is a witness for player 1 in the $C(0,1)_a$ game, then he has an (almost-sure) winning strategy from every vertex in $\varphi_{<\infty}$.*

Witness for player 2. For a function $\psi : V \to [n]_\infty$, we say that a vertex $v \in V$ is *ψ-progressive for player 2* if the following holds:

$$\begin{aligned} \forall_{\delta>1}.\ \exists_{\beta\in\mathcal{D}(A_2(v))}.\ \forall_{\alpha\in\mathcal{D}(A_1(v))}.\ & \big(v \in P_0 \ \Rightarrow\ \mathrm{Pr}_v^{\alpha,\beta}[\psi_{<v}] > 0\big) \wedge \\ & \big(v \in P_1 \ \Rightarrow\ \mathrm{Pr}_v^{\alpha,\beta}[\psi_{<v}] > 0 \ \vee\ \mathrm{Pr}_v^{\alpha,\beta}[\psi_{>v}] \le 1/\delta\big). \end{aligned} \tag{4}$$

We say that the function $\psi : V \to [n]_\infty$ is a *(positive win) witness for player 2* if every vertex $v \in \psi_{<\infty}$is ψ-progressive for player 2.

Lemma 3. *If $\psi : V \to [n]_\infty$ is a witness for player 2 in the $C(0,1)_a$ game, then he has a (positive-probability) winning strategy from every vertex in $\psi_{<\infty}$.*

5 Translation of $C(0,1)_\mathrm{a}$ games to $D(0,1)$ games

The following "finitary" characterizations of vertices that are φ- and ψ-progressive for player 1 and player 2, respectively, are the key to our reduction of concurrent probabilistic to turn-based non-probabilistic games.

Action progressive vertices. Let $\varphi, \psi : V \to [n]_\infty$. We say that a vertex $v \in V$ is *action φ-progressive for player 1* if the following holds:

$$\begin{aligned} & \big(v \in P_0 \ \Rightarrow\ \exists_{a\in A_1(v)}.\ \forall_{b\in A_2(v)}.\ \mathrm{Pr}_v^{a,b}[\varphi_\infty] = 0\big) \wedge \\ & \big(v \in P_1 \ \Rightarrow\ \forall_{b\in A_2(v)}.\ \exists_{a\in A_1(v)}.\ \mathrm{Pr}_v^{a,b}[\varphi_{<v}] > 0 \wedge \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad (\forall_{b'\in A_2(v)}.\ \mathrm{Pr}_v^{a,b'}[\varphi_\infty] = 0)\big). \end{aligned} \tag{5}$$

We say that a vertex $v \in V$ is *action ψ-progressive for player 2* if the following holds:

$$\begin{aligned} & \big(v \in P_0 \ \Rightarrow\ \forall_{a\in A_1(v)}.\ \exists_{b\in A_2(v)}.\ \mathrm{Pr}_v^{a,b}[\psi_{<v}] > 0\big) \wedge \\ & \big(v \in P_1 \ \Rightarrow\ \exists_{b\in A_2(v)}.\ \forall_{a\in A_1(v)}.\ \mathrm{Pr}_v^{a,b}[\psi_{>v}] = 0 \vee \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad (\exists_{b'\in A_2(v)}.\ \mathrm{Pr}_v^{a,b'}[\psi_{<v}] > 0)\big). \end{aligned} \tag{6}$$

Lemma 4. *Let $\varphi, \psi : V \to [n]_\infty$. 1. If a vertex $v \in V$ is action φ-progressive for player 1, then it is φ-progressive for him. 2. If a vertex $v \in V$ is action ψ-progressive for player 2, then it is ψ-progressive for him.*

We give a reduction of $C(0,1)_\mathrm{a}$ games to $D(0,1)$ games. The idea of the reduction is to replace each concurrent transition in the concurrent game by a small turn-based game in which each player aims at satisfying the condition in the definition of his action progressive vertex.

Let $G = (V, A, A_1, A_2, \delta)$ be a concurrent probabilistic game structure, and let $(G, \text{Parity}(P_0, P_1))_{\text{a}}$ be a $C(0,1)_{\text{a}}$ game. We define a $D(0,1)$ game $(G', (P_0', P_1'))$ in the following way. The set of vertices of G' includes the set V of vertices of G. We describe the transition function of G' from every vertex $v \in V$, and from the extra vertices that a few-step play in G' from v to another vertex $w \in V$ can go through.

First, for every $v \in V$, $a \in A_1(v)$, $b \in A_2(v)$ we define one-step games $\mathcal{H}_0(v, a, b)$ and $\mathcal{H}_1(v, a, b)$ as follows: the unique initial vertex of game $\mathcal{H}_i(v, a, b)$ has priority i, and the following hold.

- In the initial vertex of game $\mathcal{H}_0(v, a, b)$ player 2 chooses a successor $w \in V$, such that $\delta(v, a, b)(w) > 0$;
- In the initial vertex of game $\mathcal{H}_1(v, a, b)$ player 1 chooses a successor $w \in V$, such that $\delta(v, a, b)(w) > 0$.

In the correctness proof of the reduction given below, the games $\mathcal{H}_0(v, a, b)$ and $\mathcal{H}_1(v, a, b)$ act as gadgets that allow:

- in game $\mathcal{H}_0(v, a, b)$: player 1 to "verify" the condition $\text{Pr}_v^{a,b}[\varphi_\infty] = 0$, and player 2 to "verify" the condition $\text{Pr}_v^{a,b}[\psi_{<v}] > 0$; and
- in game $\mathcal{H}_1(v, a, b)$: player 1 to "verify" the condition $\text{Pr}_v^{a,b}[\varphi_{<v}] > 0$, and player 2 to "verify" the condition $\text{Pr}_v^{a,b}[\psi_{>v}] = 0$.

Next, we define the transition function of $\mathcal{G}'$ from every $v \in V$. Note that this transition function is simply a translation of the formula (5) from the definition of an action progressive vertex into a "formula evaluation" game, where player 1 is the "existential" player, and player 2 is the "universal" player.

- If $v \in P_0$ then the following game is played:
  1. in vertex $v \in P_0$, player 1 chooses a successor (v, a), where $a \in A_1(v)$;
  2. in vertex (v, a), player 2 chooses a one-step game $\mathcal{H}_0(v, a, b)$, where $b \in A_2(v)$.
- If $v \in P_1$ then the following game is played:
  1. in vertex $v \in P_1$, player 2 chooses a successor (v, b), where $b \in A_2(v)$;
  2. in vertex (v, b), player 1 chooses a successor (v, b, a), where $a \in A_1(v)$;
  3. in vertex (v, b, a), player 2 chooses either: the one-step game $\mathcal{H}_1(v, a, b)$, or the successor $(v, b, a, *)$;
  4. in vertex $(v, b, a, *)$ player 2 chooses a one-step game $\mathcal{H}_0(v, a, b')$, where $b' \in A_2(v)$.

Clearly, the game graph G' is turn-based and deterministic. The set P_0' contains P_0 and all the initial vertices of the one-step games $\mathcal{H}_0(v, a, b)$. All the other vertices belong to P_1'.

Theorem 6. *Let $\mathcal{G}$ be a $C(0,1)_a$ game. For every vertex $v \in V_{\mathcal{G}}$, player 1 has an (almost-sure) winning strategy from v in $\mathcal{G}$ if and only if player 1 has a (sure) winning strategy from v in the $D(0,1)$ game $\mathcal{G}'$.*

Proof idea: The idea of the proof is to argue that witnesses for either of the players in game $\mathcal{G}'$ give rise to witnesses for the same player in game $\mathcal{G}$. Then by the determinacy theorem for turn-based deterministic games (Theorem 3) and Lemma 3 we get Theorems 4 and 6. More precisely, it suffices to establish the following.

1. If $\varphi' : V' \to [n]_\infty$ is a witness for player 1 in the $D(0,1)$ game $\mathcal{G}'$ then the restriction φ of φ' to V is a witness for player 1 in the $C(0,1)_{\mathrm{a}}$ game $\mathcal{G}$.
2. If $\psi' : V' \to [n]_\infty$ is a witness for player 2 in the $D(0,1)$ game $\mathcal{G}'$ then the restriction ψ of ψ' to V is a witness for player 2 in the $C(0,1)_{\mathrm{a}}$ game $\mathcal{G}$.

Remark 1. Observe that the game graph G' we construct above contains vertices of the form (v, a, b), where v is a vertex of the original game graph G, and a and b are moves of players 1 and 2 in v, respectively. Thus formally, in order to claim that our reduction is linear, we need to assume, e.g., that the numbers of moves available to a player in every vertex are $O(1)$. The same applies to our reduction from Section 7.

6 Witnesses for concurrent co-Büchi games

Let $\mathcal{G} = \big(G, \mathrm{Parity}(\emptyset, P_1, P_2)\big)$ be a $C(1,2)$ game, with $G = (V, A, A_1, A_2, \delta)$.

Witness for player 1. For a function $\varphi : V \to [n]_\infty$, we say that a vertex $v \in V$ is *φ-progressive for player 1* if the following holds:

$$\begin{aligned}&\exists_{\varepsilon>0} \cdot \exists_{\alpha \in \mathcal{D}(A_1(v))} \cdot \forall_{\beta \in \mathcal{D}(A_2(v))} \cdot \\ &\quad \big(v \in P_1 \Rightarrow \mathrm{Pr}_v^{\alpha,\beta}[\varphi_\infty] = 0 \wedge \mathrm{Pr}_v^{\alpha,\beta}[\varphi_{<v}] \geq \varepsilon\big) \wedge \\ &\quad \big(v \in P_2 \Rightarrow \mathrm{Pr}_v^{\alpha,\beta}[\varphi_\infty] = 0 \wedge \mathrm{Pr}_v^{\alpha,\beta}[\varphi_{<v}] \geq \varepsilon \cdot \mathrm{Pr}_v^{\alpha,\beta}[\varphi_{>v}]\big).\end{aligned} \tag{7}$$

We say that the function φ is an *(almost-sure win) witness for player 1* if every vertex $v \in \varphi_{<\infty}$ is φ-progressive for player 1.

Lemma 5. *If $\varphi : V \to [n]_\infty$ is an (almost-sure win) witness for player 1, then he has an (almost-sure) winning strategy from every vertex in $\varphi_{<\infty}$.*

Witness for player 2. For a pair of functions $\psi = (\psi^0, \psi^2)$, such that $\psi^0 : V \to [n]_\infty$, and $\psi^2 : V \to [n]$, we say that a vertex $v \in V$ is *ψ-progressive for* *player 2* if the following holds:

$$\begin{aligned}
&\exists_{m \in \mathbb{N}} .\ \forall_{\delta > 1} .\ \exists_{\beta \in \mathcal{D}(A_2(v))} .\ \forall_{\alpha \in \mathcal{D}(A_1(v))} . \\
&\quad \big(v \in P_1 \ \Rightarrow\ \mathrm{Pr}_v^{\alpha,\beta}[\psi^0_{<v}] > 0 \ \vee\ \mathrm{Pr}_v^{\alpha,\beta}[\psi^0_{>v}] \le 1/\delta\big) \ \wedge \\
&\quad \big(v \in P_2 \ \Rightarrow\ \mathrm{Pr}_v^{\alpha,\beta}[\psi^0_{<v}] > 0 \ \vee \\
&\qquad\qquad (\mathrm{Pr}_v^{\alpha,\beta}[\psi_{<v}] \ge 1/\delta^m \ \wedge\ \mathrm{Pr}_v^{\alpha,\beta}[\psi^0_{>v}] \le (1/\delta) \cdot \mathrm{Pr}_v^{\alpha,\beta}\big[\psi_{<v}\big])\big).
\end{aligned} \tag{8}$$

We say that the function $\psi : V \to [n]_\infty$ is a *(positive win) witness for player 2* if every vertex $v \in \psi_{<\infty}$ is ψ-progressive for player 2.

Lemma 6. *If $\psi = (\psi^0, \psi^2)$ is a (positive win) witness for player 2, then he has a (positive-probability) winning strategy from every vertex in $\psi_{<\infty}$.*

7 Translation of $C(1,2)_{\mathrm{a}}$ games to $D(0,2)$ games

The following “finitary” characterization of vertices that are φ-progressive for player 1 is the starting point of the idea behind our reduction of concurrent probabilistic co-Büchi games to turn-based non-probabilistic parity games.

Action progressive vertices. Let $\varphi : V \to [n]_\infty$. We say that a vertex $v \in V$ is *action φ-progressive for player 1* if the following holds:

$$\begin{aligned}
&\big(v \in P_1 \ \Rightarrow\ \forall_{b \in A_2(v)} .\ \exists_{a \in A_1(v)} .\ \mathrm{Pr}_v^{a,b}[\varphi_{<v}] > 0 \ \wedge \\
&\qquad\qquad\qquad\qquad (\forall_{b' \in A_2(v)} .\ \mathrm{Pr}_v^{a,b'}[\varphi_\infty] = 0)\big) \ \wedge \\
&\Big(v \in P_2 \ \Rightarrow\ \exists_{\emptyset \ne X \subseteq A_1(v)} .\ (\forall_{a \in X} .\ \forall_{b \in A_2(v)} .\ \mathrm{Pr}_v^{a,b}[\varphi_\infty] = 0) \ \wedge \\
&\qquad \big((\forall_{a \in X} .\ \forall_{b \in A_2(v)} .\ \mathrm{Pr}_v^{a,b}[\varphi_{>v}] = 0) \ \vee\ (\exists_{a' \in X} .\ \mathrm{Pr}_v^{a',b}[\varphi_{<v}] > 0)\big)\Big).
\end{aligned} \tag{9}$$

Lemma 7. *Let $\varphi : V \to [n]_\infty$. If a vertex $v \in V$ is action φ-progressive for player 1, then it is φ-progressive for him.*

We present a reduction of $C(1,2)_{\mathrm{a}}$ games to $D(0,2)$ games. Let G be a concurrent probabilistic game graph (V, A, A_1, A_2, δ), and let $\big(G, \mathrm{Parity}(\emptyset, P_1, P_2)\big)$ be a $C(1,2)_{\mathrm{a}}$ game. We define a $D(0,2)$ game (G', P'_0, P'_1, P'_2) in the following way. The set of vertices of G' includes the set V of vertices of G. We describe the transition function of G' from every vertex $v \in V$, and the extra vertices that a game from v can go through.

We are going to use one-step games $\mathcal{H}_0(v, a, b)$ and $\mathcal{H}_1(v, a, b)$ defined in Section 5. Moreover we define a very similar one-step game $\mathcal{H}_2(v, a, b)$, such that its unique initial vertex has priority 2, and in the unique initial vertex of $\mathcal{H}_2(v, a, b)$ player 2 chooses a successor $w \in V$, such that $\delta(v, a, b)(w) > 0$.

As in the case of Büchi games in Section 5, the one-step games $\mathcal{H}_i(v, a, b)$, for $i \in \{0, 1, 2\}$, serve as gadgets that allow the players to “verify” certain conditions occurring in the definition of an action progressive vertex.

Next, we define the transition relation from every $v \in V$.

- If $v \in P_1$ then the same game is played as for $v \in P_1$ in the reduction of $C(0,1)_{\mathrm{a}}$ games to $D(0,1)$ games described in Section 5.
- If $v \in P_2$ then the following game is played:
  1. in vertex $v \in P_2$, player 1 chooses a successor (v, a), where $a \in A_1(v)$;
  2. in vertex (v, a), player 2 chooses a successor (v, a, b), where $b \in A_2(v)$;
  3. in vertex (v, a, b), player 2 chooses either: the one-step game $\mathcal{H}_0(v, a, b)$, or the successor $(v, a, b, *)$;
  4. in vertex $(v, a, b, *)$, player 1 chooses either: the one-step game $\mathcal{H}_2(v, a, b)$, or the successor (v, b);
  5. in vertex (v, b), player 1 chooses a successor (v, b, a'), where $a' \in A_1(v)$;
  6. in vertex (v, b, a'), player 2 chooses either: the one-step game $\mathcal{H}_1(v, a', b)$, or the vertex (v, a').

The vertices in V keep their priority, i.e., P'_1 includes P_1, and P'_2 includes P_2. All the other new vertices different from the initial vertices of games $\mathcal{H}_i(v, a, b)$ have priority 2.

Theorem 7. *Let $\mathcal{G}$ be a $C(1,2)_a$ game. For every vertex $v \in V_{\mathcal{G}}$, player 1 has an (almost-sure) winning strategy from v in $\mathcal{G}$ if and only if player 1 has a (sure) winning strategy from v in the $D(0,2)$ game $\mathcal{G}'$.*

Since $D(0,2)$ games can be solved in quadratic time [12] and $\mathcal{G}'$ is linear in $\mathcal{G}$, we have the following.

Theorem 8. *$C(1,2)_a$ games can be solved in quadratic time.*

8 Discussion

Algorithms for solving concurrent probabilistic games that have been known so far [5, 4] are fairly complicated. On the other hand, the problem of solving turn-based games has been heavily studied and there are many algorithms available [10, 15, 8, 20, 12, 22]. So, from a practical point of view, our reductions allow to directly apply this work, and future related work, to solving concurrent probabilistic games. We note that even though the proofs of correctness of our translations are involved, the translations themselves are fairly simple, so at a very low cost, one can turn a solver for turn-based parity games into a solver for almost-sure winning concurrent reachability, Büchi, and co-Büchi games.

We demonstrated the reductions for the reachability, Büchi, and co-Buchi winning criteria. For turn-based Büchi games, special cases are known to be solvable in linear time. This includes *weak* games [16], and games whose transitions form a tree with back edges [17]. Using our reductions, one can define classes of concurrent probabilistic games for which the game can be decided in linear time.

We conjecture that our translations can be generalized to all almost-sure concurrent parity games and to limit-sure winning [4].

Conjecture 1. There is a linear-time reduction from the problem of solving $C(1,d)$ games to the problem of solving $D(0,d)$ games.

Since $D(0,d)$ games can be solved in time $O(n^{\lfloor d/2 \rfloor+1})$ [12], this would imply improving the asymptotic time complexity of solving almost-sure concurrent parity games with d priorities from $O(n^{d+1})$ [4] to $O(n^{\lfloor d/2 \rfloor+1})$.

Finally, let us note that the ability to reduce concurrent games to turn-based games does not mean that concurrent games are a superfluous model. Concurrent games are appropriate for modeling concurrent systems in which the underlying components interact synchronously [2, 3]. While our reductions are convenient for algorithmic analysis of such systems, the turn-based systems we construct no longer model the original system in any natural sense.

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