A NEW BASIS SELECTION PARADIGM FOR WAVELET PACKET IMAGE CODING

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ABSTRACT

In this paper, work on a new wavelet packet basis selection paradigm is reported which emphasizes the crucial role of quantization strategy being used. This paradigm is coupled with a new Markov chain based estimation of the cost of zerotree quantization to develop a progressive wavelet packet image coder which gives better results than its wavelet counterpart.

1. INTRODUCTION

The last decade or so has seen a surge of interest in wavelet transform based image coding methods [1, 2]. These methods, however, perform much better on images consisting of smooth regions with well-defined boundaries than on more complex or textured images. Wavelet packets were invented [3] to pinpoint the signal phenomena occurring locally in the frequency domain, and wavelet packet image coding methods such as [4, 5] have established their superiority over wavelet based methods for complex images and particularly those containing oscillatory patterns.

Various basis selection methods [6, 7] have been proposed to select the best basis among a library of available wavelet packet bases. The use of different cost functions may result in different *best* bases which, in turn, may produce different coding results using the same quantization method. It is, therefore, important to take into account the quantization strategy at the time of basis selection to ensure that the basis chosen by employing certain criterion will actually result in better performance in terms of coding gains. In this paper, a new wavelet packet coding paradigm is advocated which unites both basis selection and quantization for compression purposes.

2. WAVELET PACKET IMAGE CODING

2.1. Wavelet Packets

In order to represent a signal f using the wavelet transform, the space V_j of its approximation at a resolution 2^{-j} is decomposed into a lower resolution space V_{j+1} and a detail space W_{j+1} by dividing the orthonormal basis $\{\phi_j(t-2^j n)\}_{n\in Z}$ of V_j into two new orthonormal bases $\{\phi_{j+1}(t-2^{j+1}n)\}_{n\in Z}$ of V_{j+1} and $\{\psi_{j+1}(t-2^{j+1}n)\}_{n\in Z}$ of W_{j+1} using lowpass and highpass filters $h[n]$ and $g[n]$ respectively. Coifman and Meyer [3] proved that if $\{\psi_j^i(t-2^j n)\}_{n\in Z}$ is an orthonormal basis of a general space W_j^i (with $\psi_j^0 = \phi_j$ and $\psi_j^1 = \psi_j$), then W_j^i can be decomposed into two orthogonal subspaces W_j^{2i} and W_j^{2i+1} using $h[n]$ and $g[n]$ filters respectively such that $\{\psi_{j+1}^{2i}(t-2^{j+1}n)\}$ and $\{\psi_{j+1}^{2i+1}(t-2^{j+1}n)\}_{n\in Z}$ are orthonormal bases of W_j^{2i} and W_j^{2i+1} respectively, and $W_j^i = W_j^{2i} \oplus W_j^{2i+1}$.

The discrete wavelet packet transform coefficients of a signal $\mathbf{x} = \{x_n\}_{0\leq n<N}$ of length $N = 2^{-J}$ can be computed as follows

$$
\begin{aligned}
w_{2n,j+1,l} &= \sum_k h_{k-2l}\, w_{n,j,k} & 0\leq l<N2^{J-j-1} \\
w_{2n+1,j+1,l} &= \sum_k g_{k-2l}\, w_{n,j,k} & 0\leq l<N2^{J-j-1} \\
w_{0,J,l} &= x_l & 0\leq l<N
\end{aligned}
$$

where $w_{n,j,l}$ is the transform coefficient corresponding to the wavelet packet having support size 2^{-j}, n and l denote the frequency and position indices respectively. The transform is invertible if appropriate dual filters $\tilde{h}[n]$, $\tilde{g}[n]$ are used on the synthesis side.

2.2. Basis Selection

Since this library of wavelet packet basis functions consists of a huge collection (more than 2^{2^D} possible bases for a tree of depth D), various algorithms [6, 7] have been proposed to compute the cost of a decomposition (split) and decide whether such a split should be retained or the decomposed coefficients should be merged back to the previous level. Dynamic programming offers an efficient solution to this optimization problem. Let $\mathcal{B} = \{b_m\}_{1\leq m\leq N}$ denote a basis used to represent signal f of length N, where $\{b_m\}$ is a set of basis vectors corresponding to N wavelet packets. Let I_M denote the set of indices of the M largest magnitude transform coefficients so that the remaining $N-M$ coefficients are ignored (due to quantization, for example). The

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resulting error is given by

$$\epsilon[M] = \parallel f \parallel^2 - \sum_{m \in I_M} |< f, b_m >|^2.$$

Let $\mathcal{C}(f, \mathcal{B})$ denote the cost of representing f in a basis $\mathcal{B}$. The best basis $\mathcal{B}^*$ from a collection $\mathcal{D}$ of different bases with respect to $\mathcal{C}$ is the one which satisfies the following relation,

$$\mathcal{C}(f, \mathcal{B}^*) \leq \mathcal{C}(f, \mathcal{B}) \qquad \forall \mathcal{B} \in \mathcal{D},\ \mathcal{B} \neq \mathcal{B}^*.$$

Selection of the best basis $\mathcal{B}^*$ from $\mathcal{D}$ ensures that there is no other basis in $\mathcal{D}$ which can represent f with a smaller cost (when using the cost function $\mathcal{C}$). It does not, however, ensure that there is no other basis from another library of bases that can represent f with a smaller cost. Neither does it guarantee that there is no other cost function $\mathcal{C}'$ which can be used to select another basis $\mathcal{B}'$ which is better than all other bases (including $\mathcal{B}^*$) in $\mathcal{D}$ (with respect to $\mathcal{C}'$) [8].

2.3. Zerotree Quantization

Zerotree quantization is an effective way of exploiting the self-similarities among high frequency subbands at various resolutions. Wavelet based zerotree image coding methods [1, 2] have proved their superiority over other wavelet based methods in terms of both computational complexity and compression performance. Moreover, enabling the embedded (progressive) transmission (reconstruction), which is required in many applications, using a zerotree method is quite straightforward. These motivating factors led to the development of a general zerotree structure, termed *compatible zerotrees*, for wavelet packet transforms [9]. If X and Y denote two random variables representing the significance of child and parent coefficients respectively, then the following relation

$$H(X|Y) = H(X) - I(X, Y),$$

tells us that encoding the conditioned significance is usually more efficient than just encoding the individual coefficients' significance. The performance of zerotree encoding depends largely upon the value of $I(X, Y)$ for a given basis. The larger the value of this measure, the more efficient the encoding is. Given a basis $\mathcal{B}$ for an image, this value determines how *friendly* the basis $\mathcal{B}$ would be to the zerotree quantization method.

3. NEW PARADIGM FOR BASIS SELECTION

An important issue that needs to be taken into account while selecting the best basis is that of the strategy used to encode the basis coefficients. Consider, for instance, the most commonly used encoding strategy which employs some sort of quantization $\mathcal{Q}$ (to suppress unimportant coefficients or some parts of them) followed by an entropy coding method. The encoding error $\epsilon[M]$ is now given by

$$(\epsilon[M])_{\mathcal{Q}} = ||f||^2 - \sum_{m \in I_M} |\mathcal{Q}(< f, b_m >)|^2$$

for $M = 1, \ldots, N$. Suppose two different quantizers $\mathcal{Q}^\alpha$ and $\mathcal{Q}^\gamma$, followed by an entropy coder, are used to encode the coefficients of same wavelet packet basis $\mathcal{B}$. Let $(\epsilon[M])_\alpha$ and $(\epsilon[M])_\gamma$ denote the encoding errors encountered due to using quantizers $\mathcal{Q}^\alpha$ and $\mathcal{Q}^\gamma$ (respectively) for encoding the coefficients of $\mathcal{B}$. In general, there exists $M \in [1, N]$ such that

$$(\epsilon[M])_{\mathcal{Q}^\alpha} \neq (\epsilon[M])_{\mathcal{Q}^\gamma}.$$

Now consider two different bases $\mathcal{B}^0$ and $\mathcal{B}^1$ used to represent f. Suppose the quantizers $\mathcal{Q}^\alpha$ and $\mathcal{Q}^\gamma$ are utilized to encode the coefficients of $\mathcal{B}^0$ and $\mathcal{B}^1$ (respectively). If there exist $M_1, M_2 \in [1, N]$ such that

$$(\epsilon^0[M_1])_{\mathcal{Q}^\alpha} < (\epsilon^1[M_1])_{\mathcal{Q}^\gamma}$$

and

$$(\epsilon^0[M_2])_{\mathcal{Q}^\alpha} > (\epsilon^1[M_2])_{\mathcal{Q}^\gamma},$$

then neither $\mathcal{B}^0$ nor $\mathcal{B}^1$ is a better basis than the other. In other words, if it happens that the basis coefficients of $\mathcal{B}^0$ when encoded by $\mathcal{Q}^\alpha$ cannot always produce less error than the coefficients of $\mathcal{B}^1$ encoded by $\mathcal{Q}^\gamma$, neither of $\mathcal{B}^0$ and $\mathcal{B}^1$ is a better basis than the other. To emphasize the crucial role of the quantization strategy at the time of basis selection, let us re-define the best basis as follows.

Definition: A basis $\mathcal{B}^i$ belonging to a collection of bases $\mathcal{D} = \{\mathcal{B}^k\}_{k=1,2,\ldots}$ is the best basis with respect to a cost function $\mathcal{C}$ and a quantization strategy $\mathcal{Q}$ if

$$\mathcal{C}(f, \mathcal{B}^i) \leq \mathcal{C}(f, \mathcal{B}^j)$$

and

$$(\epsilon^i[M])_{\mathcal{Q}} \leq (\epsilon^j[M])_{\mathcal{Q}}$$

for all $j \neq i$ and $M \in [1, N]$.

4. MARKOV CHAIN BASED COST ESTIMATION

One way of taking into account the mutual information between the parent and children subbands in this compatible zerotree organization is to estimate the cost of zerotree quantization without actually encoding the coefficients. The cost $\mathcal{C}(f, \mathcal{B})$ of encoding the coefficients of a D-level wavelet packet basis $\mathcal{B}$ can be written as

$$\mathcal{C}(f, \mathcal{B}) = \sum_{i=0}^{n-1} [\mathcal{C}_{sm}(T_i) + \mathcal{C}_{re}(T_i)]$$

where $T_i = T_0/2^i$ is the threshold value used at the ith iteration, n denotes the number of stages of encoding, and $\mathcal{C}_{sm}$ and $\mathcal{C}_{re}$ denote the costs of encoding the significance map and the refinement information. It was observed that the symbol used to encode the refinement information using Shapiro's method is nearly random, which leads to the conclusion that it would suffice to estimate the first term in the above expression. The cost $\mathcal{C}_{sm}$ can be estimated by computing the entropy of a discrete random variable whose value is drawn from the set of codewords used to encode the significance map. These codewords include two symbols (0 and 1) to represent whether a coefficient is significant or not, and a zerotree symbol whose probability can be computed as follows.

Due to the fact that the significance of child coefficients in a subband is related only to the significance of their parent coefficient, the subbands (nodes) belonging to each family of compatible zerotrees can be modelled as a non-homogeneous discrete-time Markov chain (MC) with time replaced by node depth in the tree (ie, the root compatible zerotree). Let X_j denote a random variable corresponding to the coefficients of all nodes at tree depth j. The sample space for these random variables $X_1, X_2, \ldots, X_D$ (where D denotes depth of the tree or the number of transform levels) is $\{0, 1\}$, where a value of 0 denotes that the coefficient is insignificant with respect to a threshold and 1 denotes its magnitude being larger than the threshold.

Let $P_k(0)$ denote $Pr(X_k = 0)$, the probability of a coefficient belonging to subband nodes at tree depth k being insignificant, and $P_{j,i}(0|0)$ denote the probability of all child coefficients at depth j to be insignificant given all of their corresponding parent coefficients at the previous depth i are insignificant. It was observed that given a coefficient and all its child coefficients are insignificant, it is very likely that its siblings and all their children are insignificant too. The probability $P_{j,i}(0|0)$ can, therefore, be approximated by the probability of all four child coefficients at depth j being insignificant given their parent coefficient at depth i is insignificant.

Let $P_k(\mathbf{0})$ denote the joint probability of all the coefficients originating from nodes at tree depth k and all their child coefficients being insignificant. In other words, it denotes the probability of a zerotree of length $D-k$, which consists of $(4^{D-k+1}-1)$ coefficients in a wavelet zerotree. According to the multidimensional pmf theorem of Markov chains $P_k(\mathbf{0})$ is given by

$$P_k(\mathbf{0}) = P_k(0) P_{k+1,k}(0|0) P_{k+2,k+1}(0|0) \cdots P_{D,D-1}(0|0) \tag{1}$$

or

$$P_k(\mathbf{0}) = P_k(0) \prod_{i=k}^{D-1} P_{i+1,i}(0|0) \tag{2}$$

Zerotrees of length l, however, contain all zerotrees of length $l-1$ and less. A recursive update of the following sort is, therefore, required to adjust the number of zerotrees of different lengths.

$$P_i(\mathbf{0}) \longleftarrow P_i(\mathbf{0}) - P_{i-1}(\mathbf{0}) \tag{3}$$

for $i = D-1, D-2, \ldots, 2$. The cost $\mathcal{C}_{sm}$ can now be computed as follows

$$\mathcal{C}_{sm} = H_s - \sum_{k=1}^{D} P_k(\mathbf{0}) \log P_k(\mathbf{0}) \tag{4}$$

where

$$H_s = -\sum_{k=1}^{D} \left(P_k(0) \log P_k(0) + P_k(1) \log P_k(1) \right) \tag{5}$$

gives the entropy of the one and zero codewords identifying respectively the position of significant coefficients and those insignificant coefficients which are not contained in any of the zerotrees.

Fig. 1 shows the graph of the cost of encoding the significance map (both estimated and real) plotted against various threshold values for the *Barbara* image. The MC based computation of the cost of encoding the significance map proves to be a good estimate, particularly at large threshold values which correspond to low bit rates.

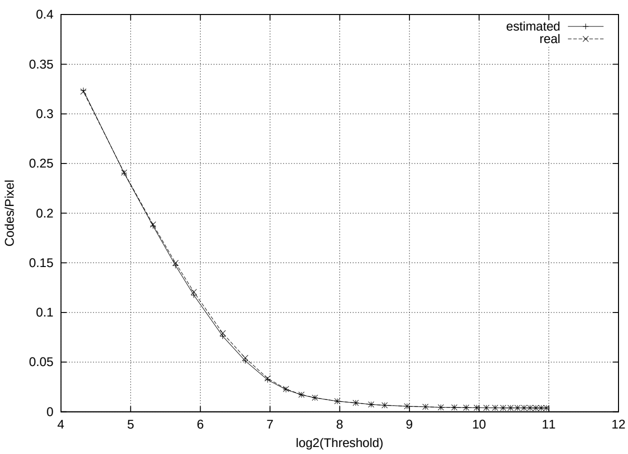


Fig. 1. Estimated and real cost of encoding the significance map vs. threshold for *Barbara*

5. EXPERIMENTAL RESULTS

The use of a bottom-up search method, along with a cost function that takes into account the quantization strategy, ensures the selection of a best basis for compressing a given image using that particular quantization method. Based on the cost estimate described in the previous section, a simple heuristic can be devised to select what can be termed as a *zerotree friendly* wavelet packet basis. The complexity of wavelet packet transform for an image of size $M \times N$ is $O(MND)$ where D is the number of transform levels or the depth of full wavelet packet tree. Moreover, the new basis selection algorithm needs to compute the estimated cost of quantization $(4^{D-1} - 1)/3$ times.

The idea of compatible zerotree quantization was combined with the wavelet packet basis for progressive image coding and its performance was tested on two standard 8-bit greyscale 512×512 images - *Barbara* and *Fingerprints* - using both a wavelet basis and a zerotree friendly wavelet packet basis. The latter of these bases was selected using the cost estimate discussed earlier in Section 4 in order to maximize the mutual information $I(X, Y)$. For all the experiments, the factorized 9-7 biorthogonal filters [5] were used for efficiently computing the wavelet packet transform. Results for the performance of both variations of the CZQ coder - that is, with the wavelet basis CZQ-$\mathcal{WV}$ and with the wavelet packet basis CZQ-$\mathcal{WP}$ - for both the test images are presented in Tables 1-2. The measure used to describe the performance of each coder is the peak-signal-to-noise-ratio (PSNR).

Bitrate (bpp.)	Compression (:1)	PSNR (dB)	
		CZQ-$\mathcal{WV}$	CZQ-$\mathcal{WP}$
1.0	8	35.14	36.15
0.8	10	32.64	33.91
0.5	16	30.28	31.60
0.4	20	28.41	29.85
0.25	32	27.12	28.12
0.2	40	25.25	26.64
0.1	80	23.64	24.27

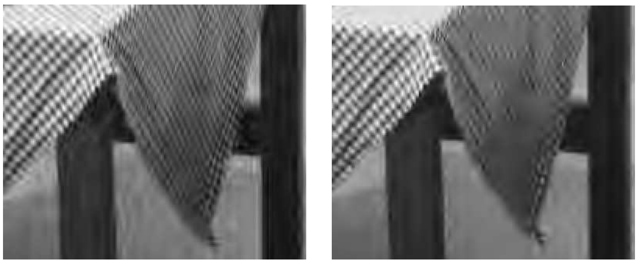
Table 1. Coding results for *Barbara*

Bitrate (bpp.)	Compression (:1)	PSNR (dB)	
		CZQ-$\mathcal{WV}$	CZQ-$\mathcal{WP}$
1.0	8	35.24	35.82
0.8	10	32.71	33.38
0.5	16	30.65	31.15
0.4	20	28.39	29.09
0.25	32	26.53	27.07
0.2	40	25.08	25.64
0.1	80	22.84	23.24

Table 2. Coding results for *Fingerprints*

While being capable of progressively reconstructing the encoded image and being relatively faster than other wavelet packet coders [5], the CZQ-$\mathcal{WP}$ coder performs comparably well. The coding gains achieved by it on top of CZQ-$\mathcal{WV}$ are 0.6-1.5dB for *Barbara*, and 0.4-0.7dB for *Fingerprints*.

A closer look at the reconstructed images encoded at 0.25 bpp. reveals that CZQ-$\mathcal{WP}$ yields better visual quality than the state-of-the-art SPIHT [2] image coder. Note, for instance, the quality of a portion (table cloth) of the reconstructed *Barbara* image encoded by CZQ-$\mathcal{WP}$ and SPIHT as shown in Fig. 2.



(*a*) CZQ-$\mathcal{WP}$ (28.12dB) (*b*) SPIHT (27.58dB)

Fig. 2. Portion of *Barbara* encoded at 0.25 bpp.

6. CONCLUSIONS

A new paradigm for the wavelet packet basis selection was presented which emphasized that the role of quantization strategy should be taken into account at the time of basis selection. An analysis of the popular zerotree quantization method based on the Markov chains was presented. The experimental results show that the unification of this paradigm of basis selection with quantization yields better coder performance.

7. REFERENCES

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