

SIMULATION OF THE RATE-DISTORTION BEHAVIOUR OF A MEMORYLESS LAPLACIAN SOURCE

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ABSTRACT

In the context of transform coding of still images, the distribution of high frequency transform coefficients has been approximated by a Laplacian distribution. In this short paper, an efficient way of simulating the rate-distortion behaviour of a memoryless Laplacian source is presented. Simulation results show that the estimated behaviour very closely matches the empirical one.

1. INTRODUCTION

In transform coding, it has been established that the statistics of individual high-frequency subbands (wavelet or wavelet packet) and non-DC discrete cosine transform (DCT) coefficients can best be approximated by a Laplacian distribution, for most natural images. Earlier research [5, 2] on the DCT transform coding schemes employed the intuitive assumption that the DC and the non-dc DCT coefficients for images are more likely to follow Gaussian and Laplacian distributions respectively. Later on, these assumptions were experimentally verified to be valid upto a reasonable extent by the goodness-of-fit tests using l_∞ -norm as a distance measure [3] between both the distributions: the actual one and the one under consideration. Some researchers have also proposed generalized Gaussian distribution (GGD) for both DCT and wavelet coefficients in image coding applications [4]. However, it was shown recently, in [1], that the GGD assumption performs better than the Laplacian one by no more than 0.08 bits/pixel using a mean square error (MSE) distortion measure.

In this short paper, analytical expressions are developed for rate (entropy) and distortion (MSE) resulting from the uniform scalar quantization of data generated by a memoryless Laplacian source. This is helpful in simulating the

rate-distortion behaviour of a memoryless Laplacian source without actually doing the quantization.

2. THE RATE-DISTORTION RELATIONSHIP

In this section, we will derive a relationship between rate and distortion, as defined earlier in the previous section, for a memoryless Laplacian source (with zero mean) which may provide the governing distribution for non-DC DCT or high-frequency wavelet transform coefficients. Let $p(x)$ denote the probability of occurrence, also known as the probability density function (pdf), of the source value x given by

$$p(x) = \frac{\lambda}{2} e^{-\lambda|x|} \quad (1)$$

where λ is the distribution parameter related directly to the source variance σ^2 by $\lambda = \sqrt{2}/\sigma^2$. In order to investigate the distortion-rate function for this distribution, let us suppose that the uniform step size Δ produces $2n + 1$ equally spaced points $(x_{-n}, \dots, x_{-1}, x_0, x_1, \dots, x_n)$; $x_i = i\Delta$, yielding the distortion D_δ and requiring R_δ bits/sample, at the least, for encoding. The quantities R_δ and D_δ can thus be expressed as follows,

$$R_\delta = - \sum_i p'(x_i) \log_2 p'(x_i) \quad (2)$$

and

$$D_\delta = \sum_i d_i, \quad (3)$$

where

$$p'(x_i) = \int_{a_i}^{b_i} p(x) dx \quad (4)$$

is a new pdf for the quantized source data, and

$$d_i = \int_{a_i}^{b_i} p(x)(x - x_i)^2 dx \quad (5)$$

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is the distortion due to mapping of the source data values $x \in (a_i, b_i)$ to x_i , with $a_i = x_i - \frac{\Delta}{2}$ and $b_i = x_i + \frac{\Delta}{2}$. The new pdf $p'(x_i)$ for quantized coefficients given in equation (4) can be simplified as follows,

$$p'(x_i) = \frac{2 \sinh \frac{\lambda \Delta}{2}}{\lambda} \cdot p(x_i) \quad (6)$$

The above equation can be derived by integrating (4). It shows that $p'(x_i)$ is directly proportional to $p(x_i)$, with the value of constant increasing monotonically with the step size Δ . The obvious fact that $p(x_i) = p'(x_i)$ for small values of Δ is also clearly decipherable from this equation. Now putting the value of $p(x)$ from (1) in (5), we obtain

$$d_i = \frac{\lambda}{2} \int_{a_i}^{b_i} (x - x_i)^2 e^{-\lambda|x|} dx; \quad \forall i, i \neq 0$$

and

$$d_0 = \lambda \int_0^{\frac{\Delta}{2}} x^2 e^{-\lambda x} dx.$$

Integration of the right hand sides of above equations yields the following expression for d_i and d_0

$$d_i = \left\{ \frac{\Delta^2}{4} + \frac{2}{\lambda^2} - \frac{\Delta}{\lambda} \coth \frac{\lambda \Delta}{2} \right\} p'(x_i); \quad \forall i, i \neq 0 \quad (7)$$

and

$$d_0 = \frac{2}{\lambda^2} - \left(\frac{\Delta^2}{4} + \frac{2}{\lambda^2} + \frac{\Delta}{\lambda} \right) e^{-\frac{\lambda \Delta}{2}}. \quad (8)$$

We also know that $p'(x_0) = p'(0) = 2 \int_0^{\Delta/2} p(x) dx = 1 - e^{-\frac{\lambda \Delta}{2}}$ and so

$$\sum_{i; i \neq 0} p'(x_i) = 1 - p'(0) = e^{-\frac{\lambda \Delta}{2}}.$$

Replacing d_i in (3) with the expressions in (7) and (8), we obtain the following expression for total distortion

$$D_\delta = \frac{2}{\lambda^2} - \frac{\Delta}{\lambda} (1 + \coth \frac{\lambda \Delta}{2}) e^{-\frac{\lambda \Delta}{2}} \quad (9)$$

The above equation confirms the fact that the distortion D_δ approaches towards the source variance $\sigma^2 = 2/\lambda^2$ as the step Δ increases. As for the rate R_δ , (2) can be re-written as

$$R_\delta = -p'(0) \log_2 p'(0) - \sum_{i; i \neq 0} \log_2 \left(\sinh \frac{\lambda \Delta}{2} e^{-\lambda|x_i|} \right) p'(x_i)$$

or

$$R_\delta = -p'(0) \log_2 p'(0) - e^{-\frac{\lambda \Delta}{2}} \log_2 \sinh \frac{\lambda \Delta}{2} + \frac{\lambda}{\log 2} S \quad (10)$$

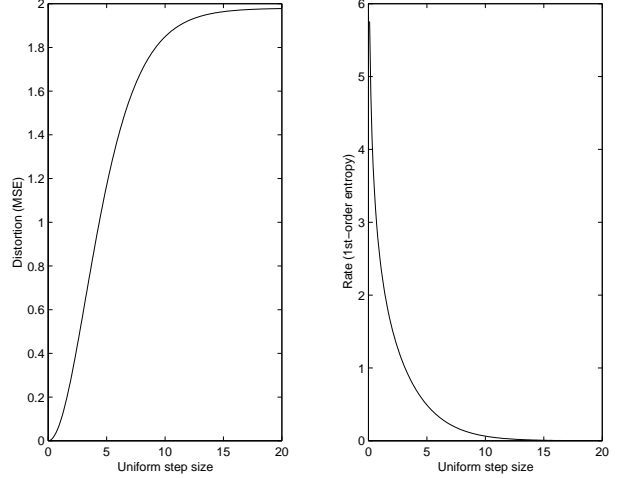


Figure 1: Typical plots of D_δ and R_δ versus Δ

where $p'(0) = 1 - e^{-\lambda \Delta/2}$ and S is given by

$$S = \sum_i |x_i| \cdot p'(x_i) = 2\Delta \cdot \sinh \frac{\lambda \Delta}{2} \sum_{i=0}^n i e^{-i\lambda \Delta} \quad (11)$$

From (10), it is clear that as the step size Δ increases, both $e^{-\lambda \Delta/2}$ and S approach towards zero and thus R_δ has more of the contribution to its value from the source values $x \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$, which are quantized to zero, than for smaller values of Δ . The typical curves for D_δ and R_δ when plotted against Δ (for $\lambda = 1$) are shown in Figure 1. As is clear from their corresponding expressions, $R_\delta \rightarrow 0$ and $D_\delta \rightarrow \sigma^2$ as the step size Δ becomes very large.

3. SIMULATION RESULTS

Equations (9) and (10) provide us a fast way of approximating the (R_δ, D_δ) points of the rate-distortion curve for a given data which is assumed to follow a Laplacian distribution. The only parameter that we need to know about this data is its variance σ^2 (with λ given by $\lambda = \sqrt{2/\sigma^2}$).

We generated data sets of various sizes coming from a memoryless Laplacian source with $\lambda = 1$. The operational rate-distortion curve was obtained by simply quantizing the data uniformly with different step size Δ values and then computing the corresponding values for distortion (MSE) and rate (order-1 entropy). To obtain a fast approximation to the operational rate-distortion curve, (9) and (10) were used to compute the (R_δ, D_δ) points. Both the empirical and the approximated operational curves for rate-distortion for the number of data points $N = 4K$ and $N = 16K$ are shown in Figures 2 and 3 respectively.

4. CONCLUSIONS

A fast method of approximating the rate-distortion curve of a memoryless Laplacian source is presented. The simulation results verify the validity of expressions (9) and (10). An important problem in data compression where this may be helpful is the following: Given fixed rate and/or distortion value, find a fixed step size (Δ) to quantize the transform coefficients such that the given rate and/or distortion value are achieved. However, finding an analytical solution for this problem does not appear to be straightforward and needs some attention.

5. REFERENCES

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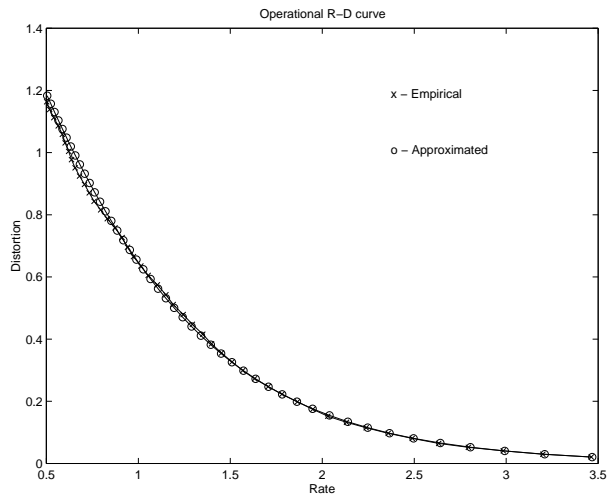


Figure 2: Empirical and approximated operational rate-distortion curves for a Laplacian data with $\lambda = 1$ and $N = 4K$

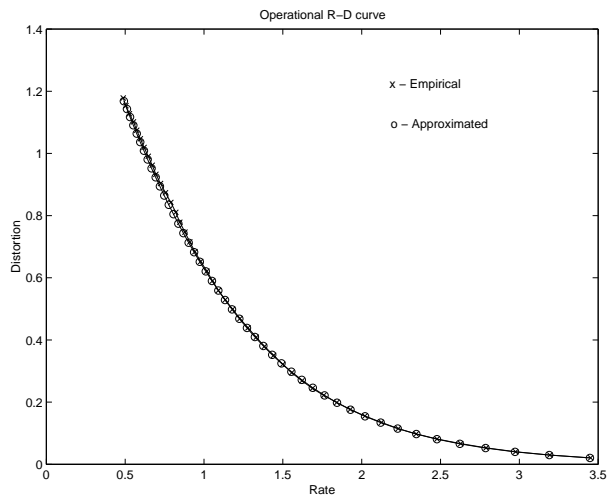


Figure 3: Empirical and approximated operational rate-distortion curves for a Laplacian data with $\lambda = 1$ and $N = 16K$