NP-Hardness of Constant-round Communication Complexity

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- I predict 1h15m will be enough, maybe a little more or a little less depending on questions.

Part 1

A fundamental problem

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- The question also depends on the kind of **computational problem** for which we want to find optimal algorithms. For example computing functions, solving search problems, sampling problems, streaming problems, *etc*.

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  - Maybe we only care to have an **average case** solution, e.g. solve the problem efficiently for *most* given functions.

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- I.e., known positive results show that the search version (find the algorithm) can be solved efficiently, whereas all the known negative results show that the *decision version* (distinguish complex from simple computational problems) is already hard.

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- One might optimistically ask: maybe it is *always* possible to find the best algorithm efficiently?

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- However, all of the above problems are in $\mathrm{NP} \cap \mathrm{coNP}$ (*), so one might still conjecture, for example, that MCSP is in $\mathrm{NP} \cap \mathrm{coNP}$.

The duality question

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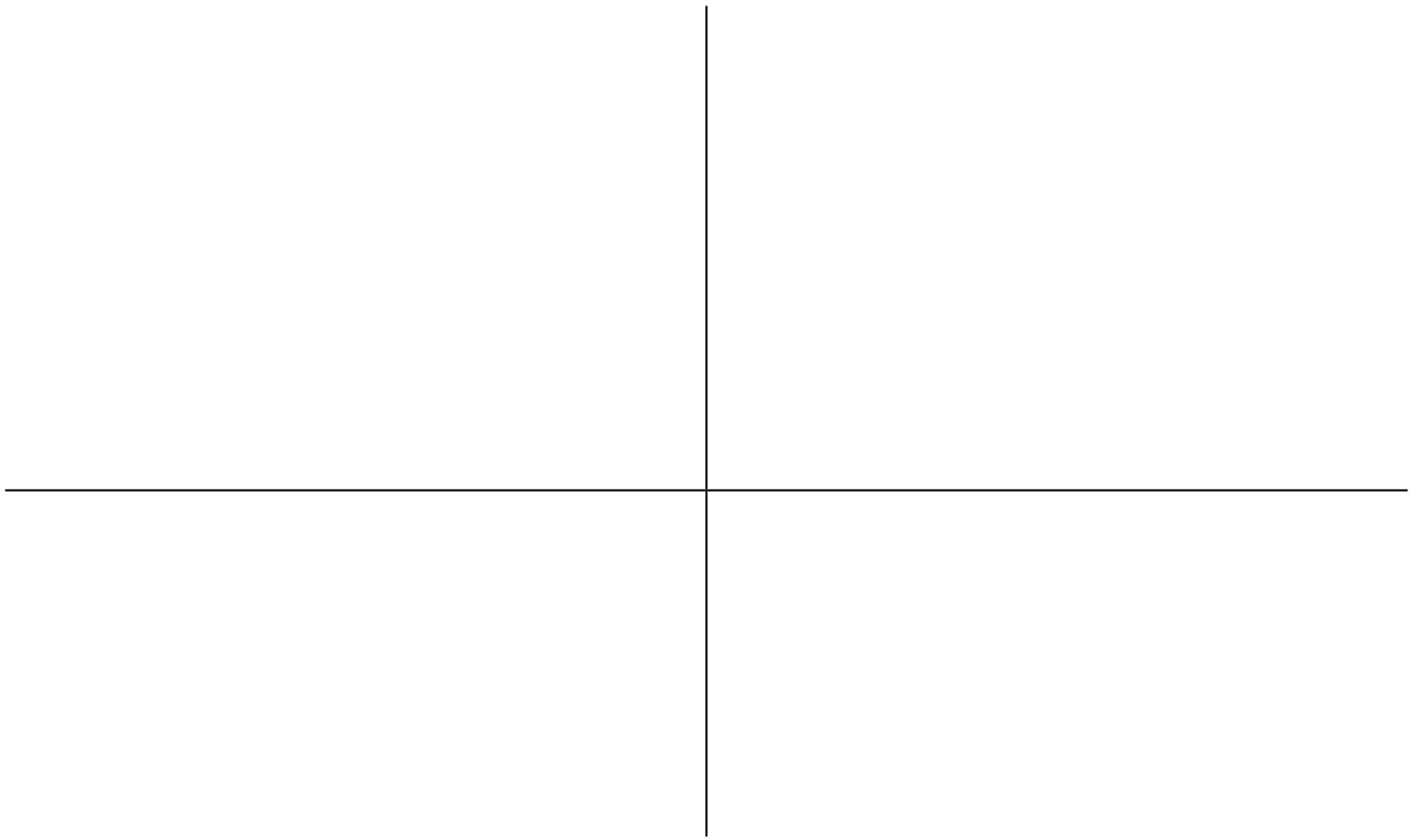

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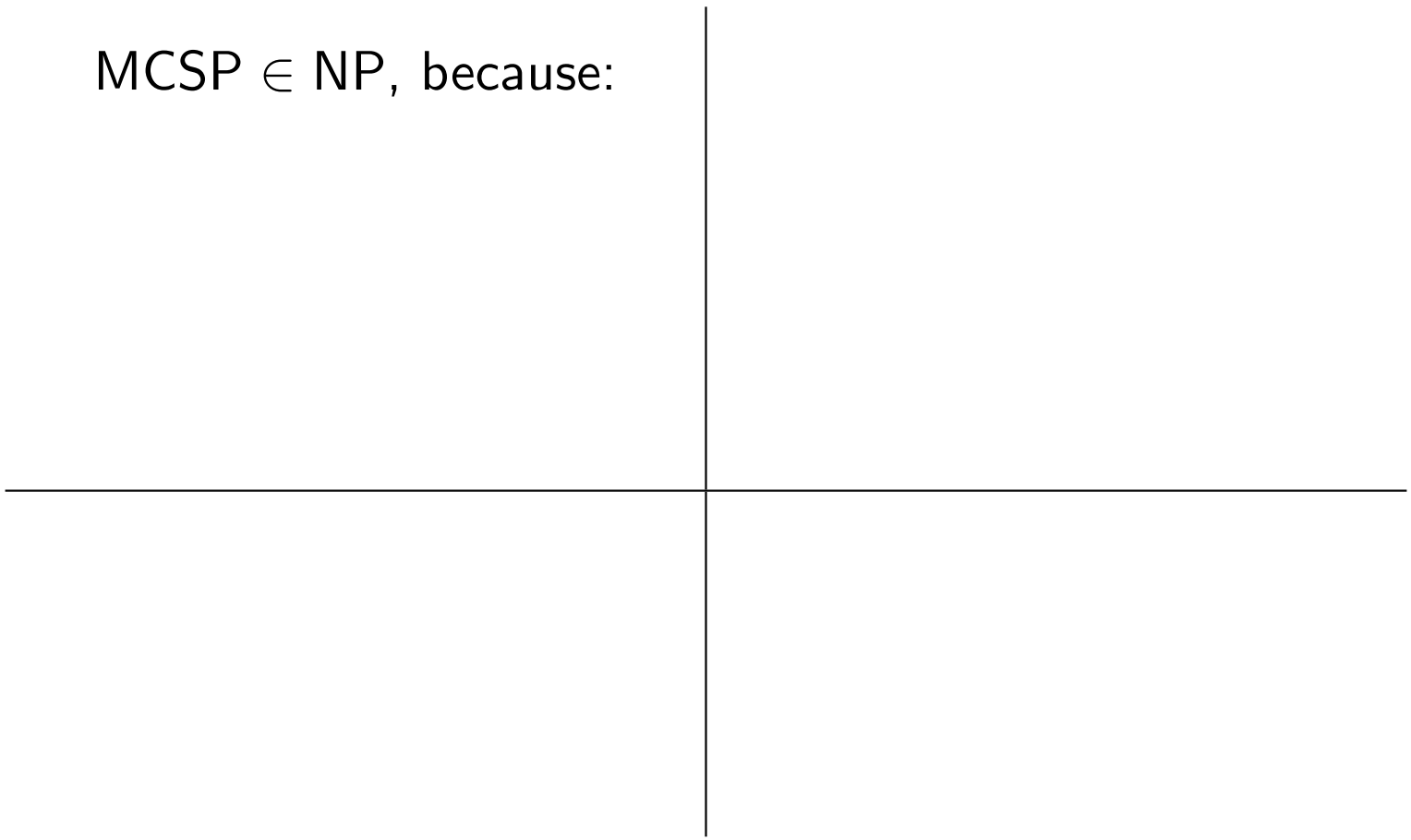

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$$x \notin P \iff \exists (w_1, \ldots, w_n)\ \langle x, w\rangle > 0 \quad \forall i \langle v_i, w\rangle < 0$$

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- **if the answer is Yes:** then we have a duality for circuit size, and we know what a lower-bound proof "looks like". 
- **if the answer is No:** Then there is no duality for circuit size, which roughly means that the generic tight lower-bound cannot be proven by a constructive proof. (We wouldn't be able to recognize a proof if we saw one.)

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- **if the answer is Yes:** then we have a duality for circuit size, and we know what a lower-bound proof "looks like". 
- **if the answer is No, and MCSP is** NP-**hard:** There is no exact duality unless NP $=$ coNP. But we may still have an *approximate* duality (for example, if $\mathrm{SIZE}(f) \geq s$ then there exists some efficiently-recognizable witness proving that $\mathrm{SIZE}(f) \geq s^{1/10}$).

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- **if the answer is No, and MCSP is** NP**-hard to approximate, even in the average case, even with a very high approximation ratio!:** This would give us a new barrier in complexity theory, similar to Razborov and Rudich's natural proofs barrier, but based on the assumption that NP $\neq$ coNP instead of based on cryptographic assumptions. It would tell us that for most functions, proving a circuit-size lower-bound, even a lower-bound which is only remotely close to optimum, cannot be done constructively (assuming NP $\neq$ coNP).

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- **if the answer is No, and MCSP is** NP**-hard to approximate, even in the average case, even with a very high approximation ratio!:** This would give us a new barrier in complexity theory, similar to Razborov and Rudich's natural proofs barrier, but based on the assumption that NP $\neq$ coNP instead of based on cryptographic assumptions. It would tell us that for most functions, proving a circuit-size lower-bound, even a lower-bound which is only remotely close to optimum, cannot be done constructively (assuming NP $\neq$ coNP).

(I.e., a complexity theorist's nightmare)
In this precise sense it could be said that NP $\neq$ coNP, if true, would "fight" against its own proof.

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- Sometimes this connection can be made formal. There are several results showing that proving NP-hardness of MCSP under various types of reductions would require proving new separations, e.g. EXP $\not\subseteq$ ZPP. (See surveys by Eric Allender)

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- There are some unconditional hardness results for this family of problems, e.g., MCSP cannot be solved by subcubic formulas, by subexponential AC_0 circuits, or by polysize $AC_0[p]$ circuits (Allender, Buhrman, Koucký, Melkebeek, and Ronneburger, 2006, Hirahara and Santhanam, 2017, Cheraghchi, Kabanets, Lu, and Myrisiotis, 2020, Golovnev, Ilango, Impagliazzo, Kabanets, Kolokolova, and Tal, 2019).

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- Remarkably, it can be formally shown that proving lower-bounds for certain variants of our problem, in some cases *even lower-bounds that we already know how to prove for other problems*, will lead to breakthrough results in computational complexity.
- Meaning, even weak lower-bounds on our family of problems can sometimes be converted into strong lower-bounds which we do not know how to prove.

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- Almost there? These problems seem to require more refined lower-bound techniques than the ones we currently have (see *Hardness Magnification and Locality*, by Chen, Hirahara, Oliveira, Pich, Rajgopal, and Santhanam).

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- There is a connection to learning. The existence of certain learning algorithms can also be shown to be (roughly) equivalent to the hardness of certain average-case variants of our problem (Carmosino, Impagliazzo, Kabanets, and Kolokolova, 2016, Ilango, Loff and Oliveira 2020).

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End of Part 1

Part 2

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Main question:
Suppose we are given the communication matrix of f as input.
What is the complexity of computing $\mathrm{L}_r^{\mathrm{Alice}}(f)$?

Graph Coloring and non-binary 3-rounds

Definition

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For every $\varepsilon > 0$, there is a deterministic polynomial time algorithm that on an input $x \in \{0,1\}^*$, outputs a graph G on n vertices such that

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For every $\varepsilon > 0$, there is a deterministic polynomial time algorithm that on an input $x \in \{0,1\}^*$, outputs a graph G on n vertices such that

- *if x is a YES instance of SAT, then $\chi(G) \leq n^{\varepsilon}$, and*
- *if x is a NO instance of SAT, then $\chi(G) \geq n^{1-\varepsilon}$.*

$$A, B, K \in \mathbb{N}$$
$$f : [A] \times [B] \to \{0, 1, \ldots, K\}$$

Graph Coloring and non-binary 3-rounds

Definition

Given an undirected graph $G = ([n], E)$ over n vertices, a *coloring* is an assignment of colors to the vertices so that no edge exists in G between vertices having the same color.
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Let us try to understand $\mathrm{L}_1^{\mathrm{Bob}}$, $\mathrm{L}_2^{\mathrm{Alice}}$ and $\mathrm{L}_3^{\mathrm{Bob}}(f)$.

How to get the 3-round binary case?

(An idealized gadget. Won't work because Bob can break up the columns. But should work with a random matrix, and ultimately can be made to work with pseudorandom matrix.)

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Theorem

It is NP*-hard to distinguish, for a given* $f : [N] \times [N] \to \{0, 1\}$*, whether* $L_3^A(f) \leq N^{\varepsilon}$ *or whether* $L_3^A(f) \geq N^{\frac{1}{8}-\varepsilon}$.

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Definition

Let $d \geq 3$. Let $f : [a] \times [b] \to \{0,1\}$. Let
$F : ([k] \times [a]) \times ([k] \times [b] \times \{0,1\} \times \{0,1\}) \to \{0,1\}$ be given by

$$F(x_0, x_1; y_0, y_1, z, i) = \begin{cases} z & , \text{ if } i = 0,\ x_0 \neq y_0 \\ 1 - z & , \text{ if } i = 0,\ x_0 = y_0 \\ f(x_1; y_1) & , \text{ if } i = 1. \end{cases}$$

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Theorem

We always have

$$\min\{2k, \mathrm{L}_d^{\mathrm{Bob}}(f) - O(1)\} \leq \mathrm{L}_{d+1}^{\mathrm{Alice}}(F) - 2k \leq \mathrm{L}_d^{\mathrm{Bob}}(f).$$

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Given f and k, we can construct F such that

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Corollary

Let $0 < \epsilon < \frac{1}{8}$. Given an oracle computing a $(1+\epsilon)$-approximation of $\mathrm{L}_{d+1}^{\mathrm{Alice}}(\cdot)$ and a function $f : [a] \times [b] \to \{0,1\}$, one can deterministically compute a $(1+4\epsilon)$-approximation of $\mathrm{L}_d^{\mathrm{Bob}}(f)$ in time $(ab)^{O(1)}$

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Corollary

$\mathrm{L}_d^{\mathrm{Alice}}(\cdot)$ *is hard to approx. for all $d \geq 4$, to within a factor $1 + \varepsilon_d$.*

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End of Part 2