The Complexity of Gradient Descent: CLS = PPAD ∩ PLS

ALEXANDROS HOLLENDER

JOINT WORK WITH JOHN FEARNLEY, PAUL GOLDBERG AND RAHUL SAVANI

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Find a mixed Nash equilibrium of a game.

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- Total: there is always a solution
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Not unless co-NP = NP…

The class TFNP [Megiddo-Papadimitriou, 1991]

Total NP search problems:

- “search” : looking for a solution, not just YES or NO
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TFNP lies between P and NP (search versions)

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$$\text{3-SAT} \leq \text{NASH} \quad \Rightarrow \quad \text{certificate for unsatisfiable 3-SAT formulas}$$

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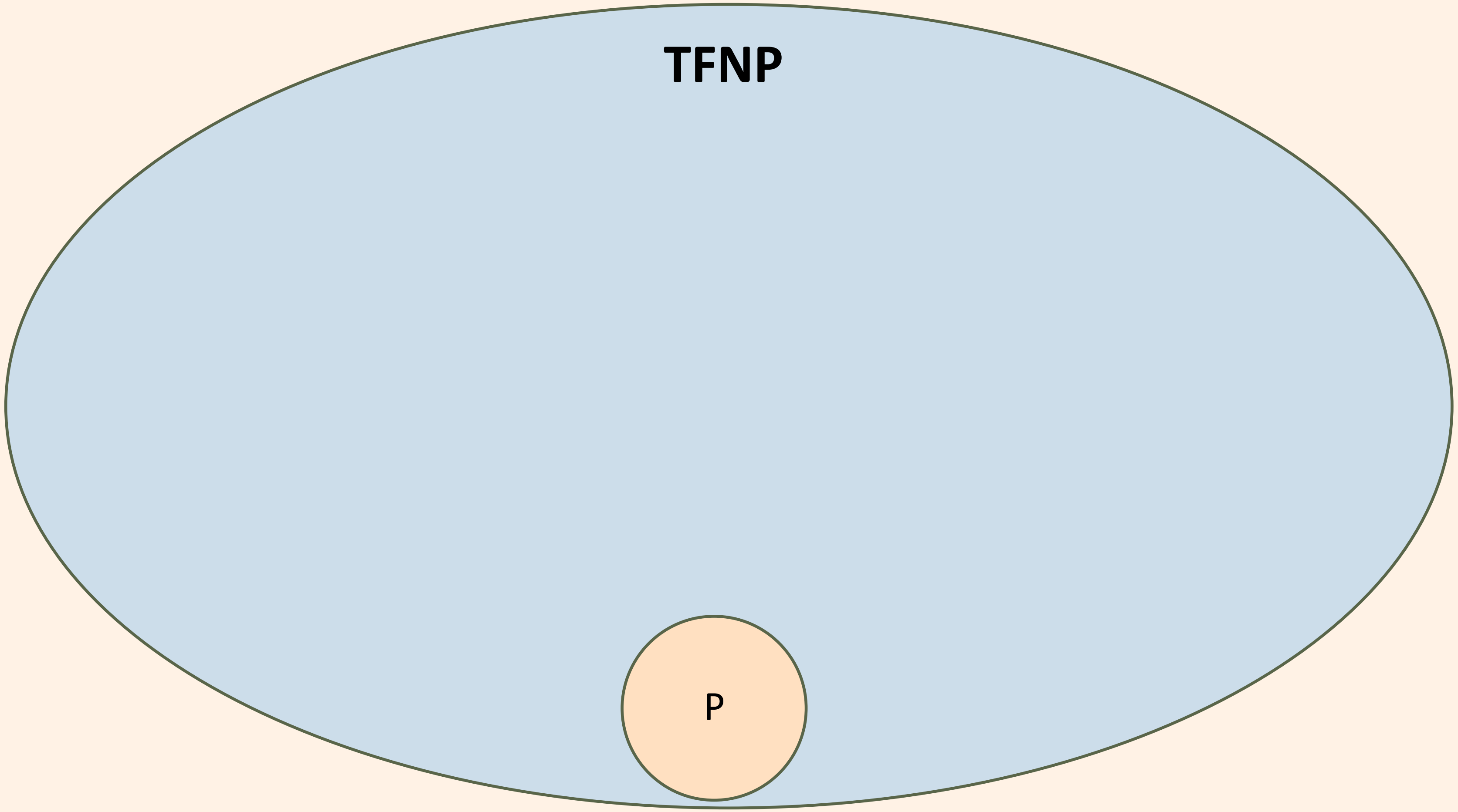
- Believed that no TFNP-complete problems exists…

The TFNP landscape



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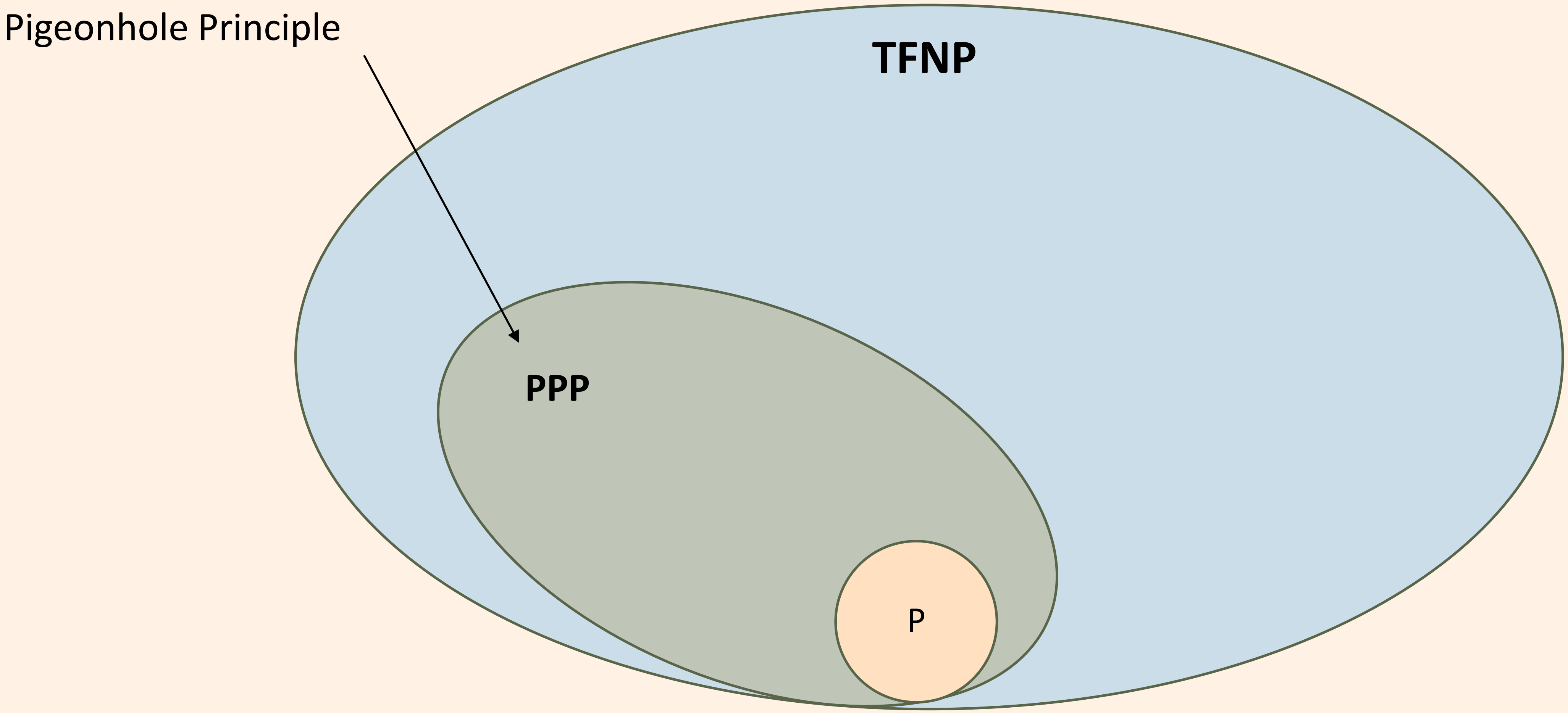

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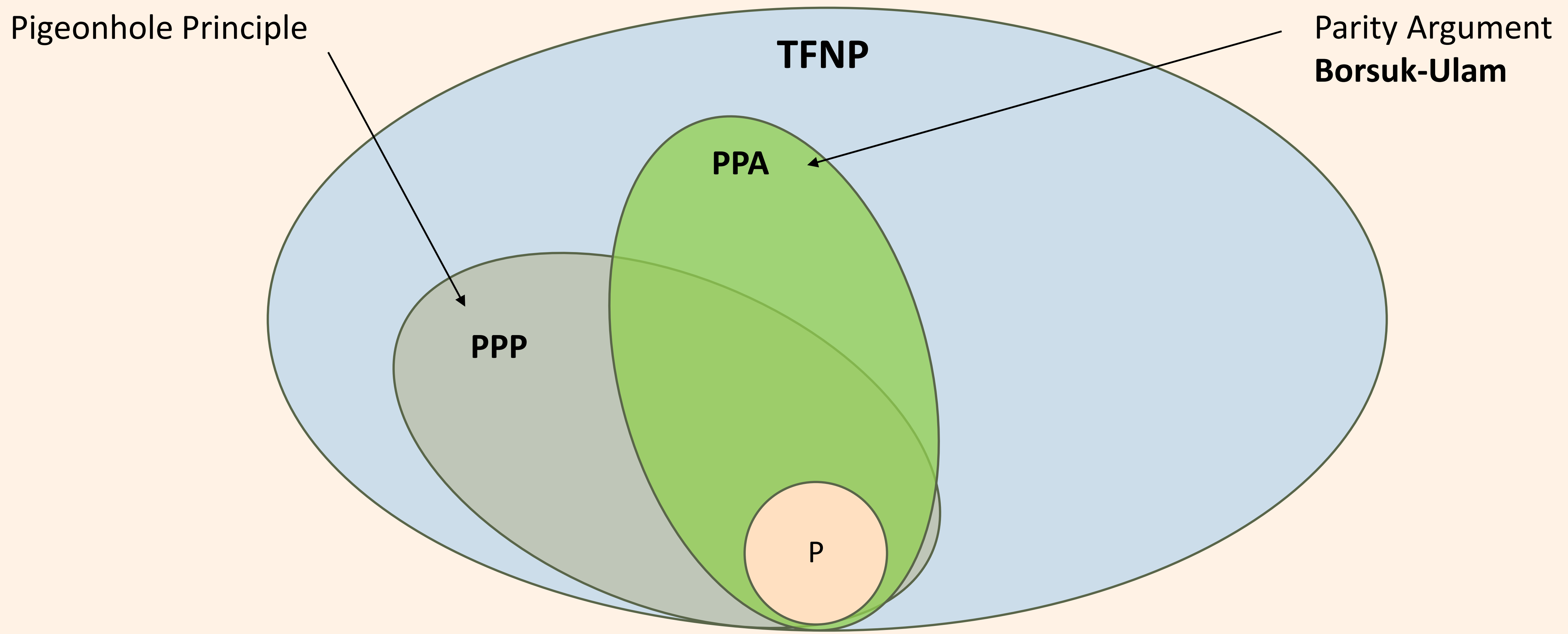

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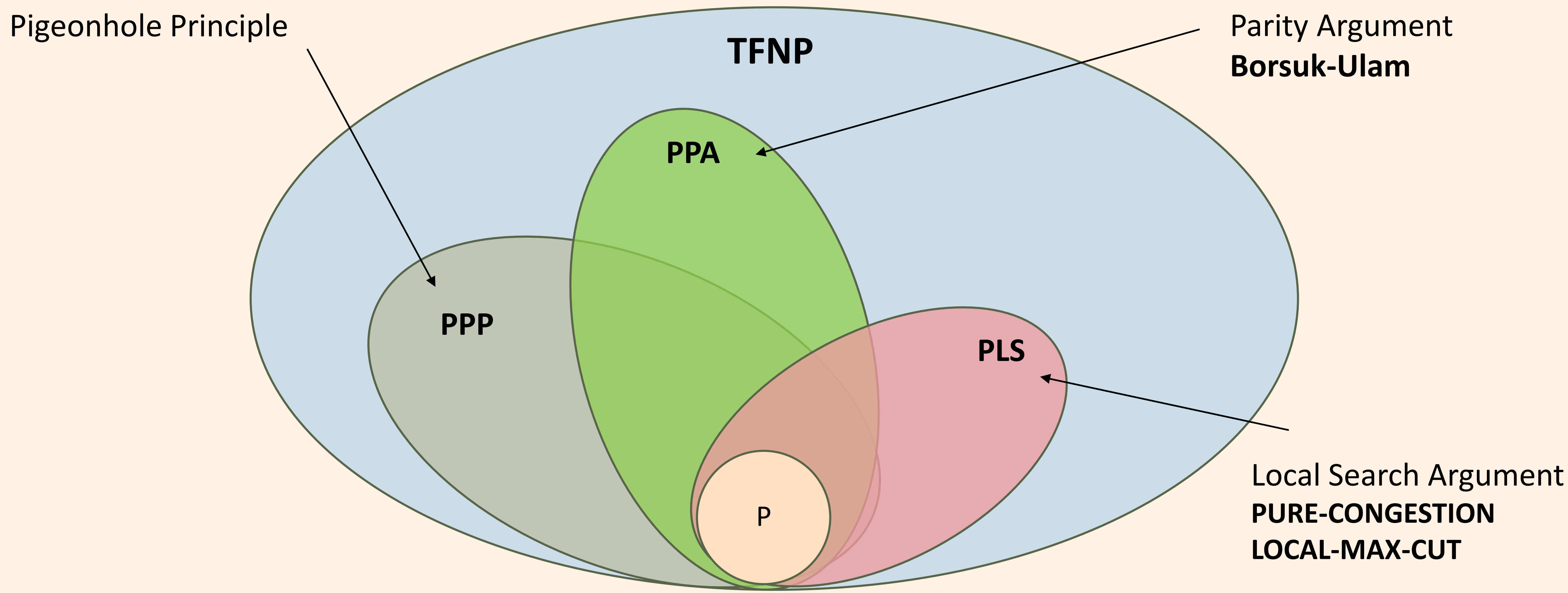

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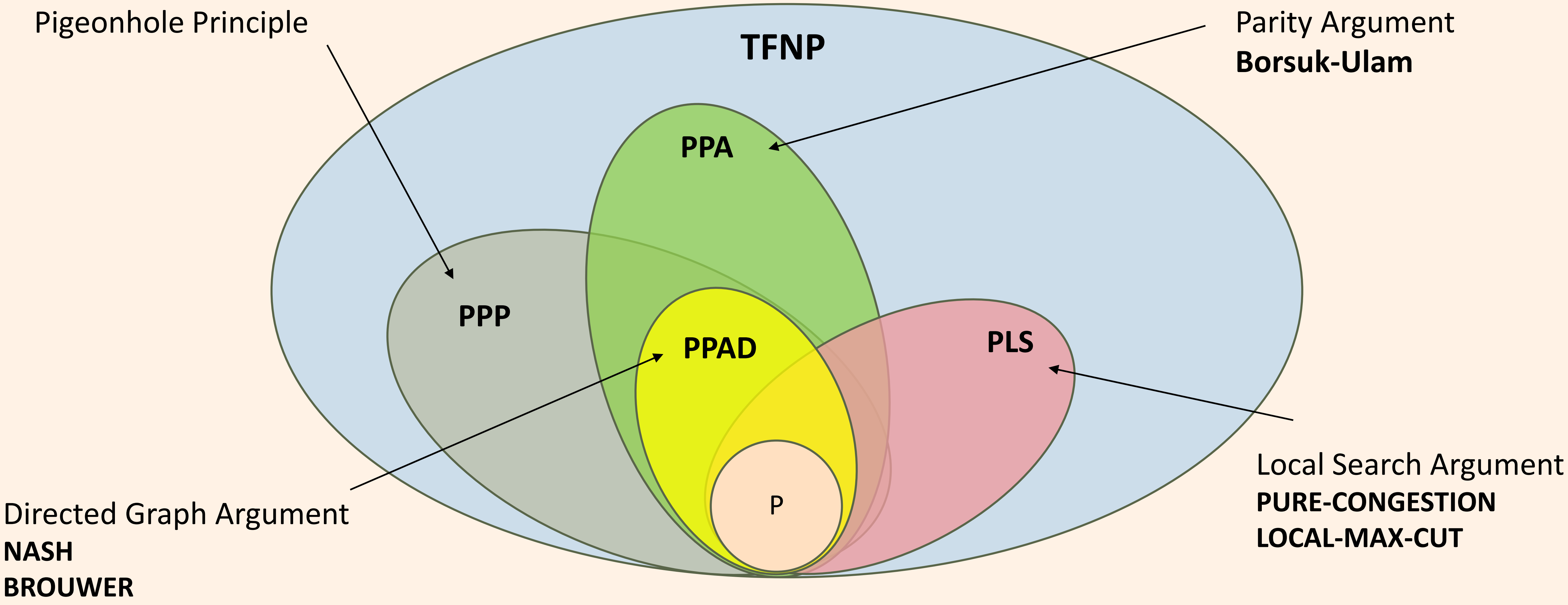

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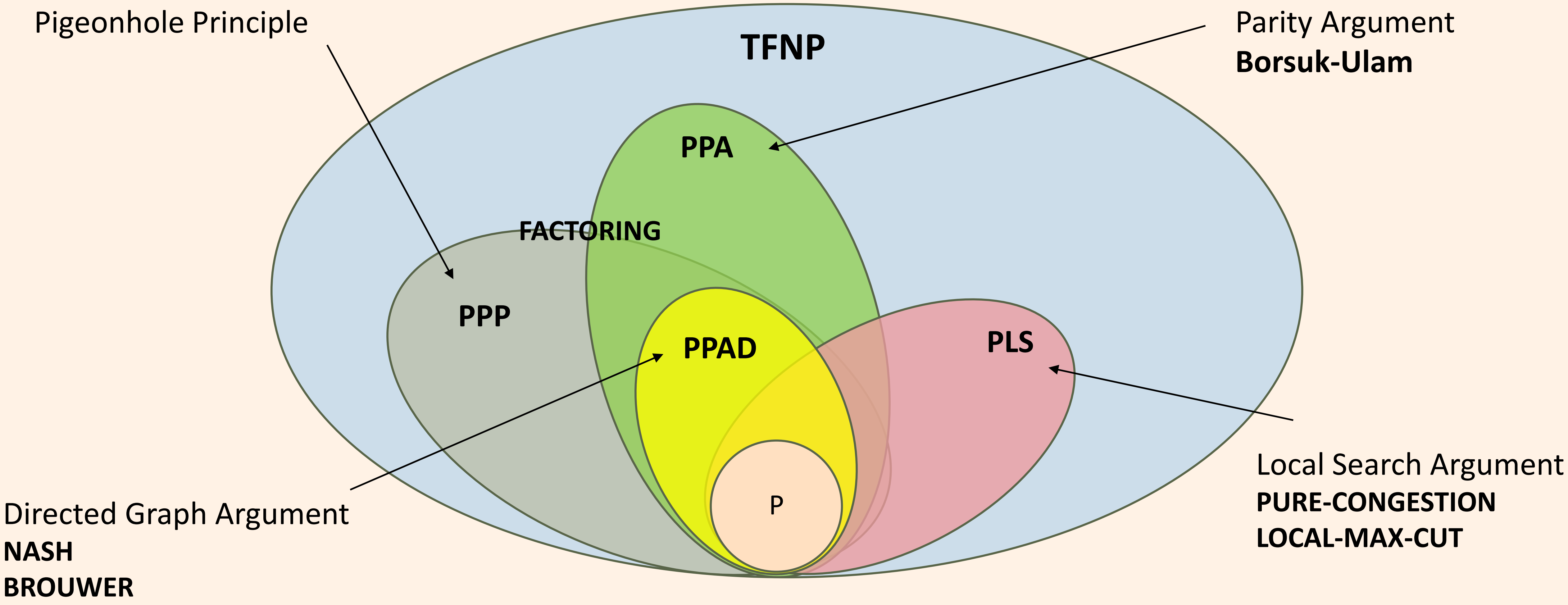

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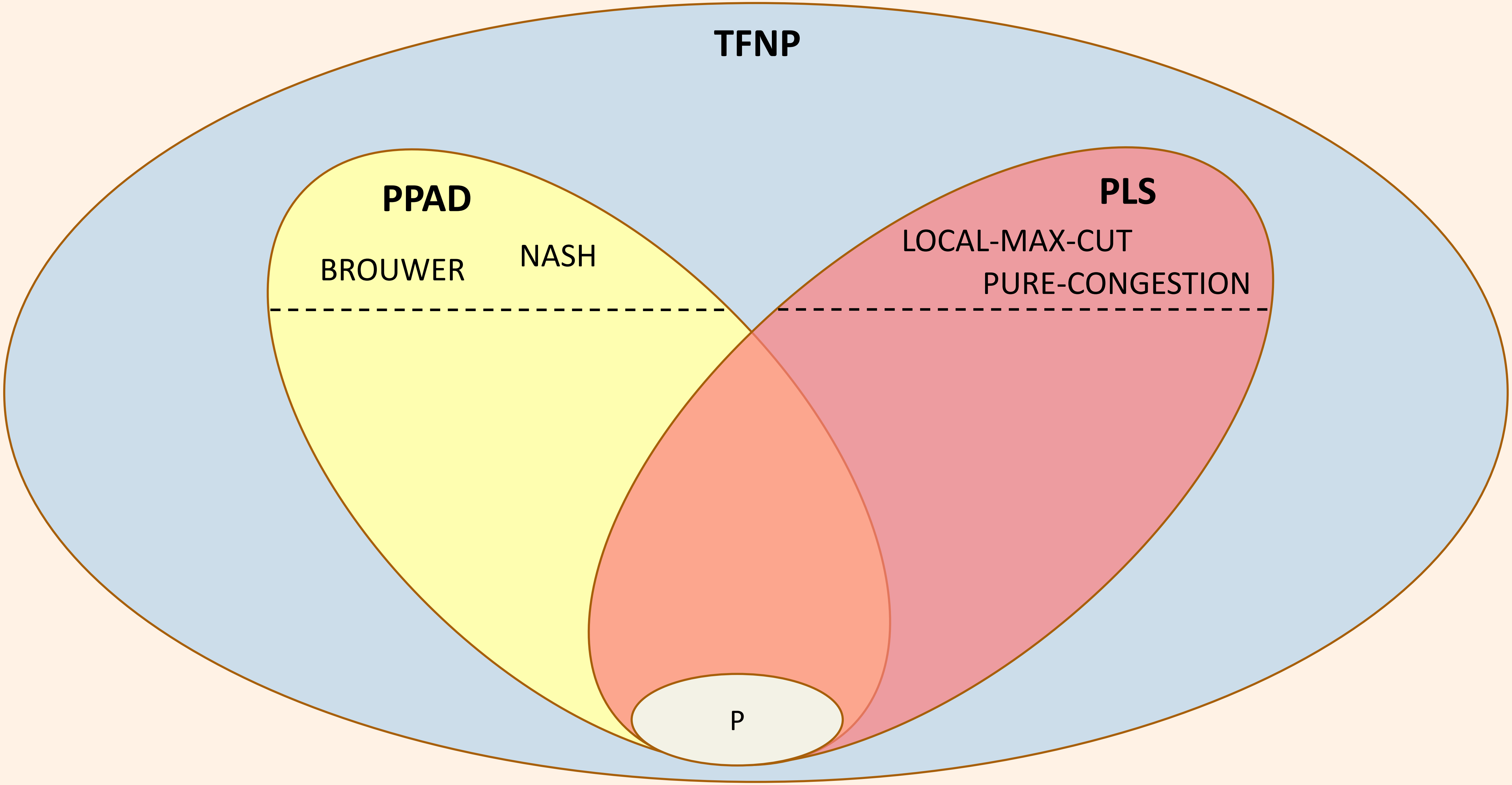
- oracle separations between the classes (in particular PPAD ≠ PLS)
- hard under cryptographic assumptions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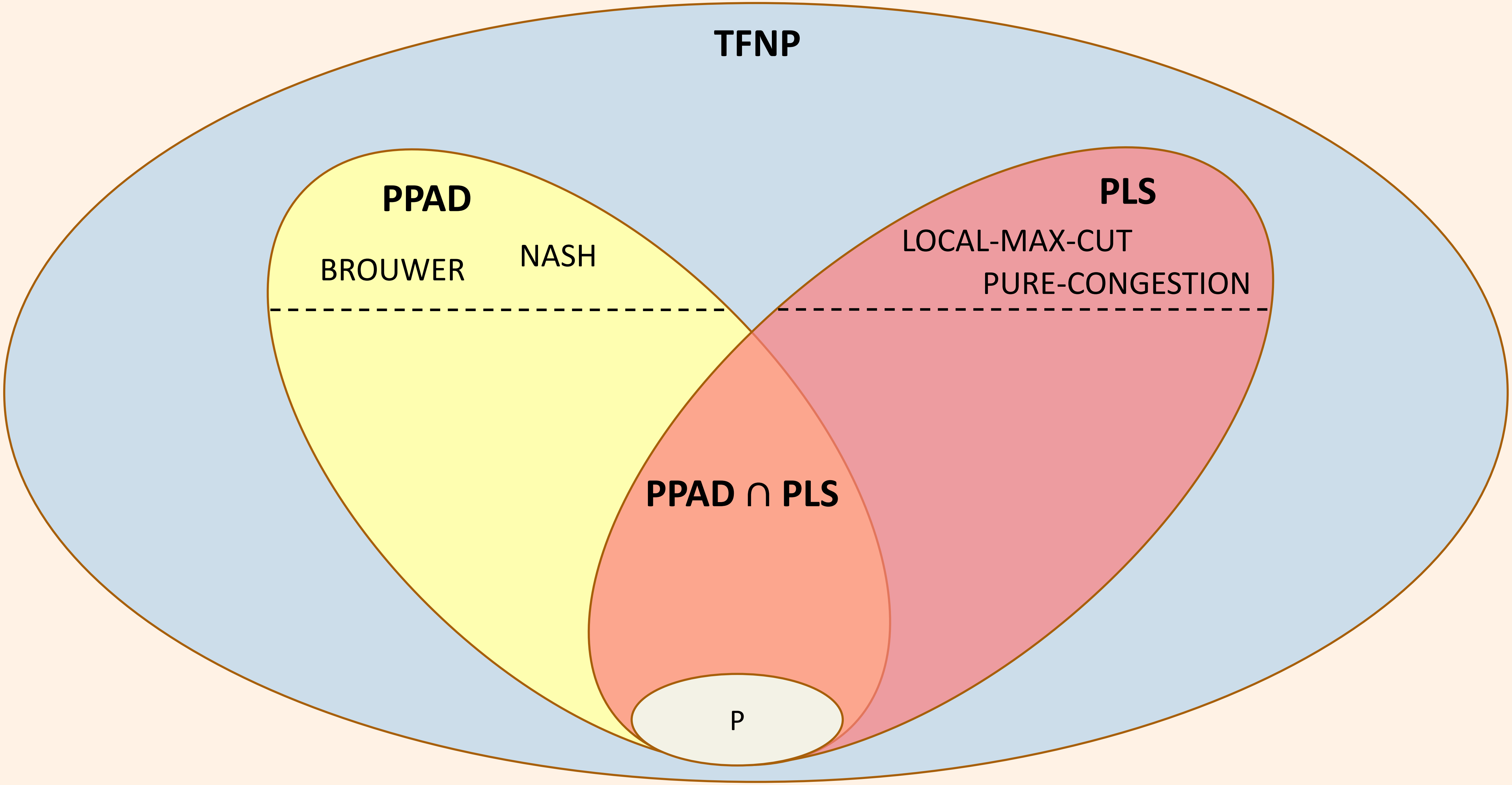
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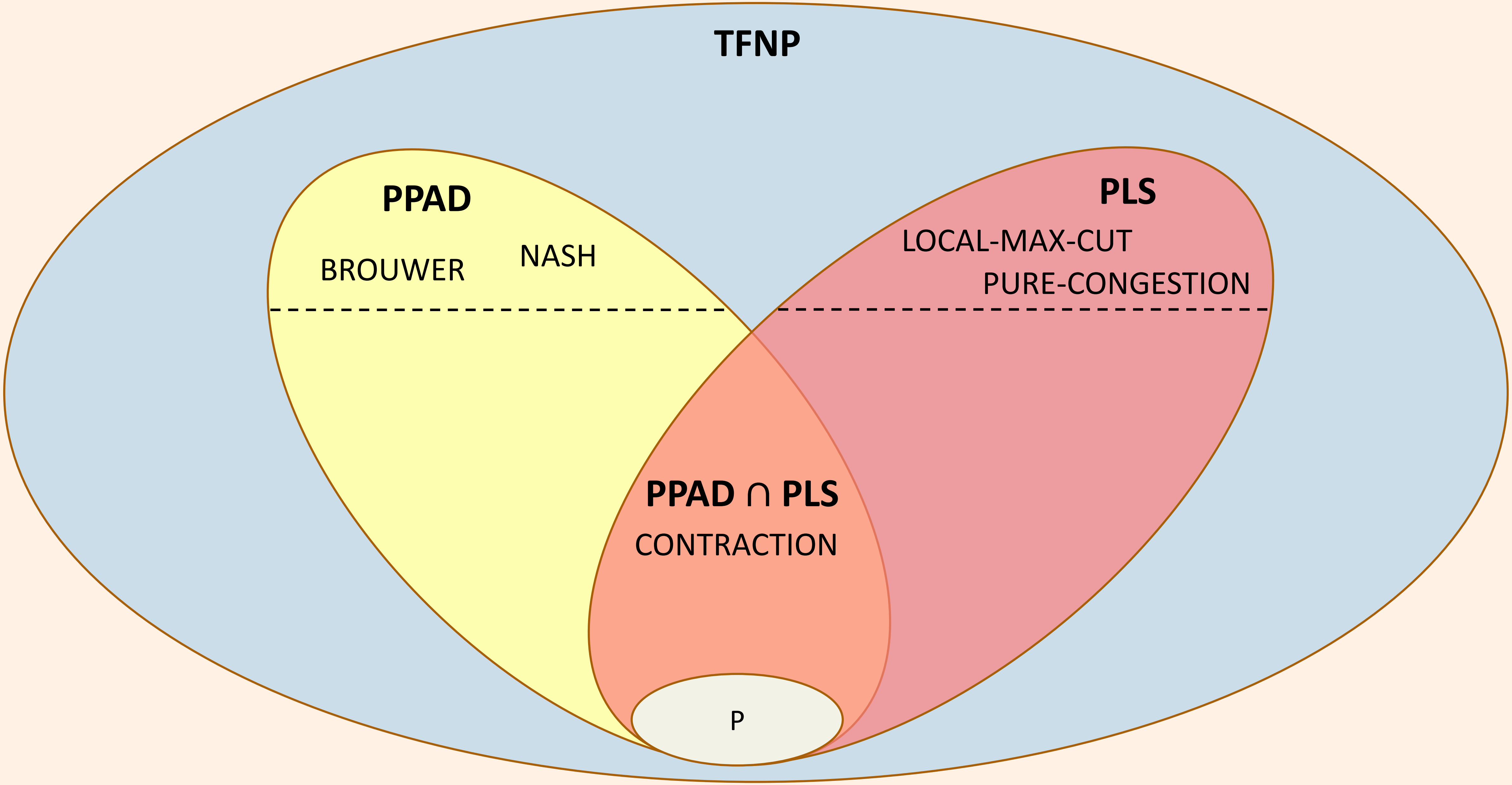
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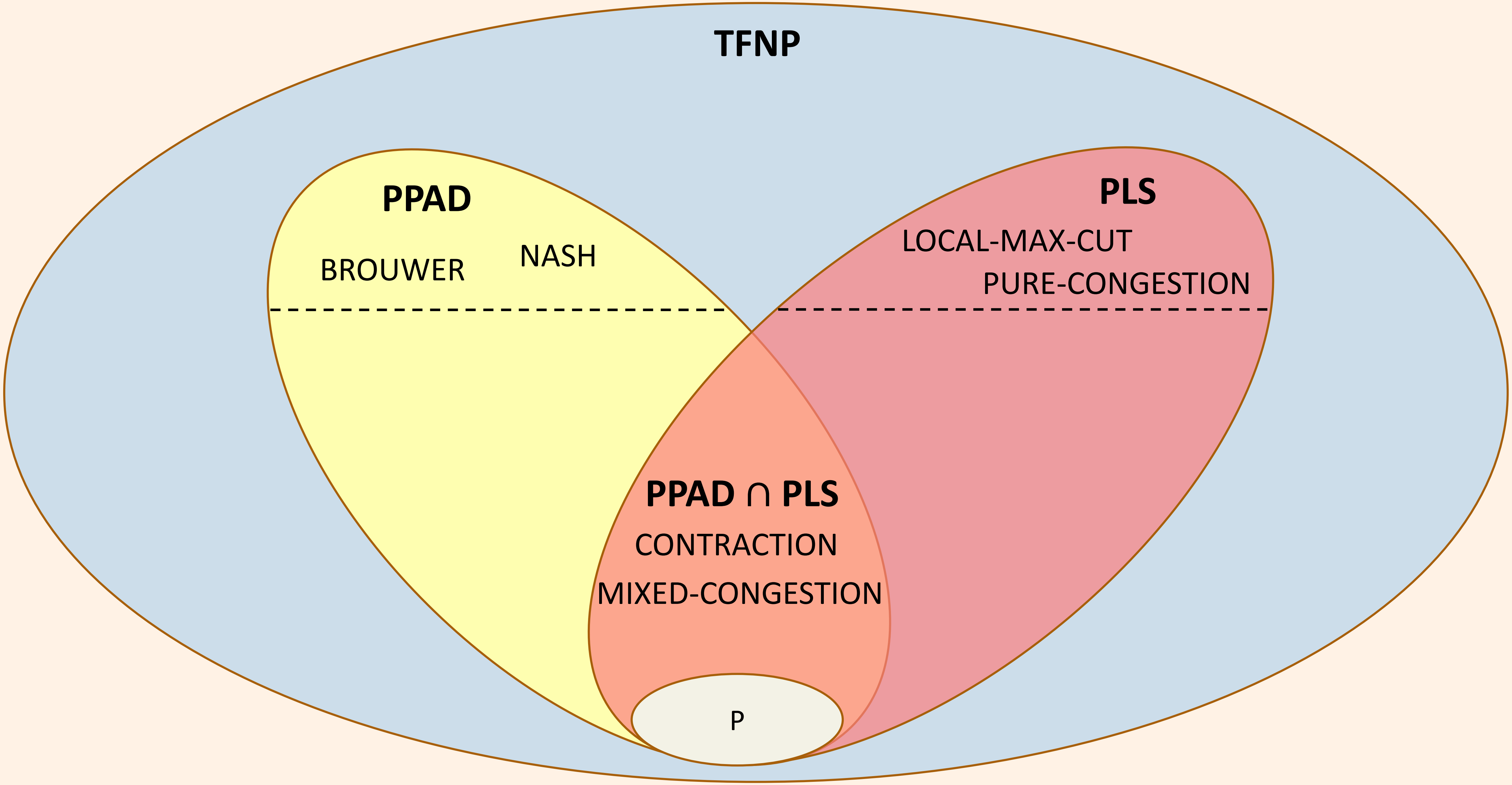
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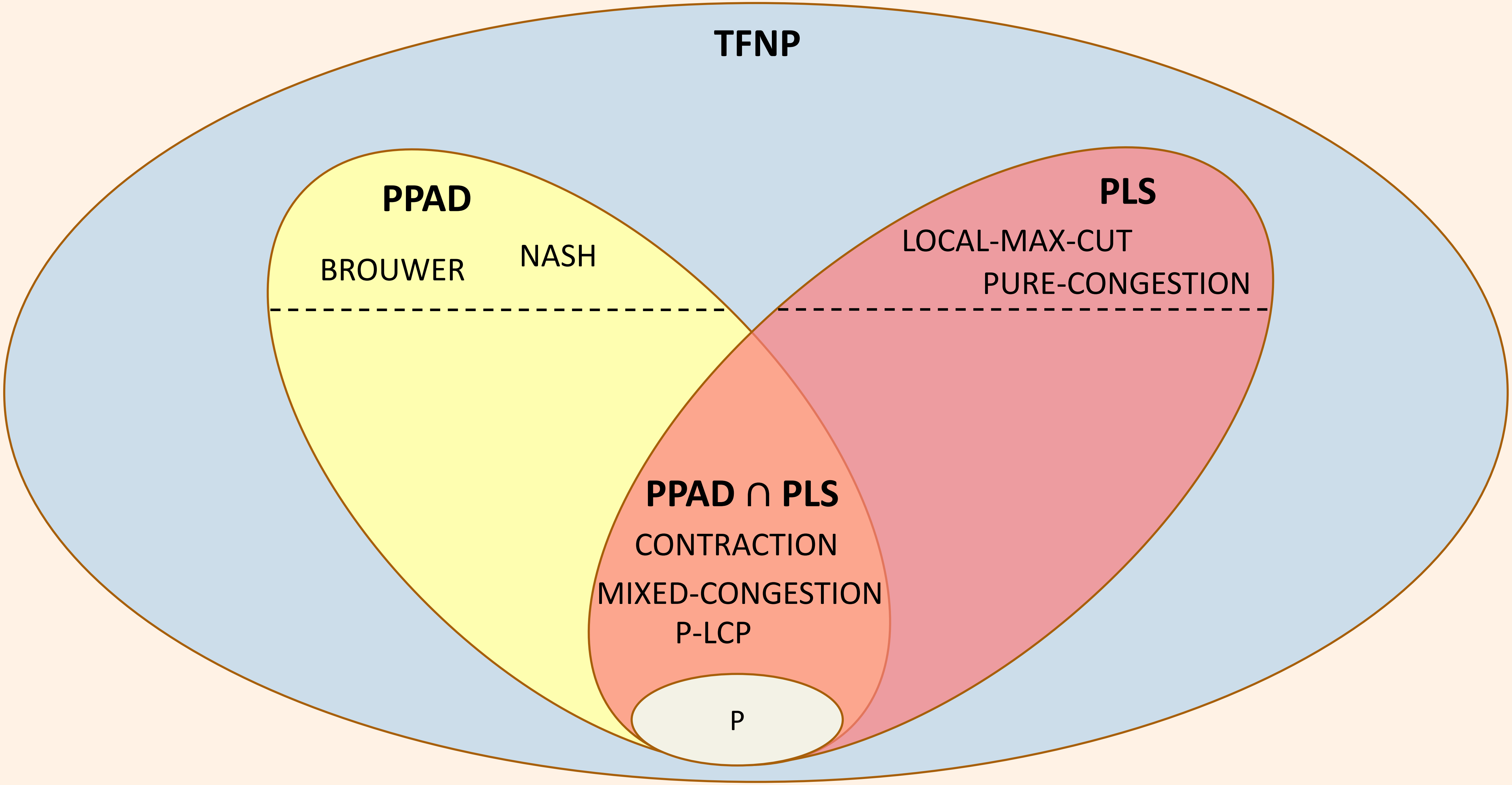
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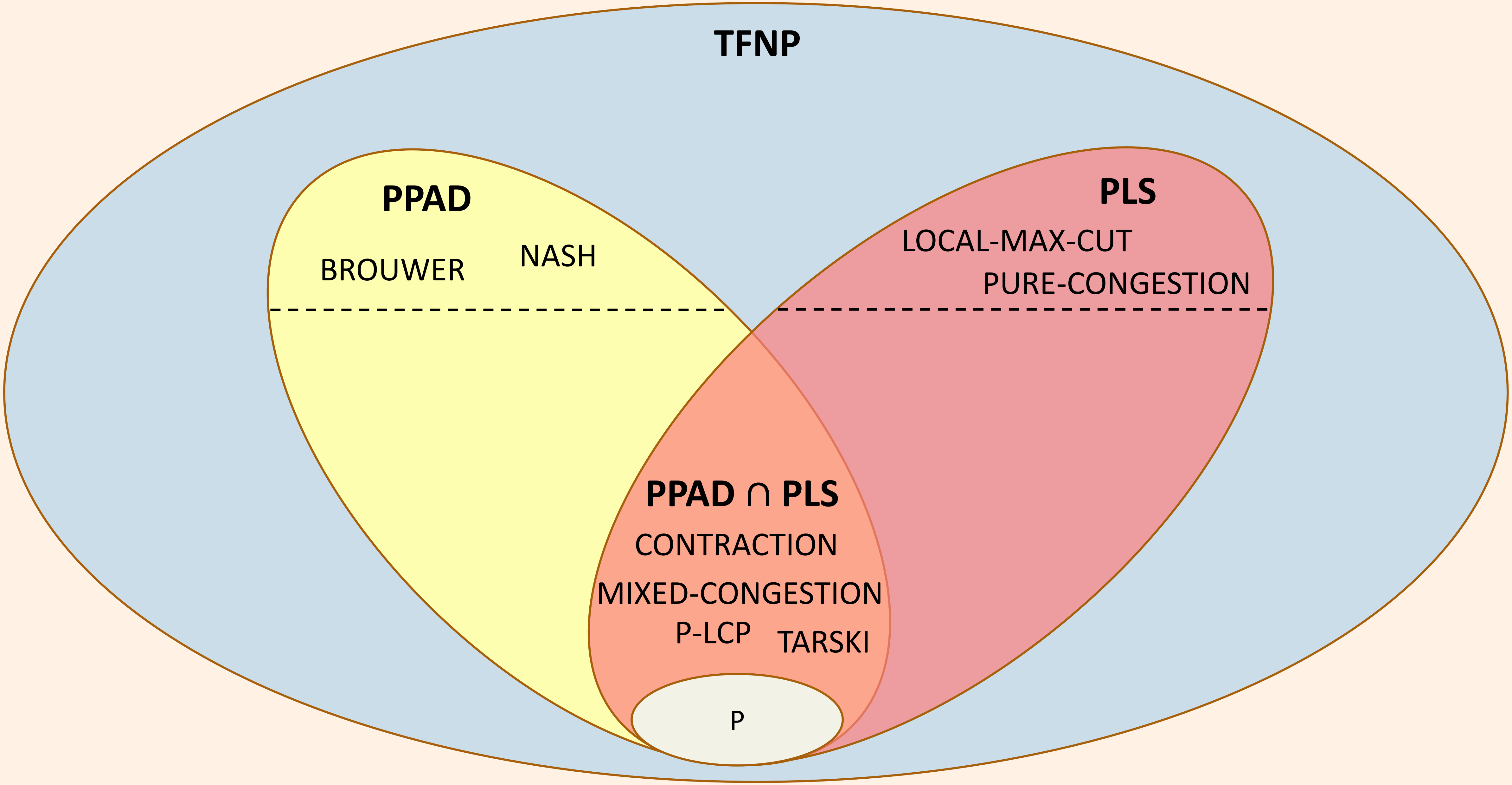
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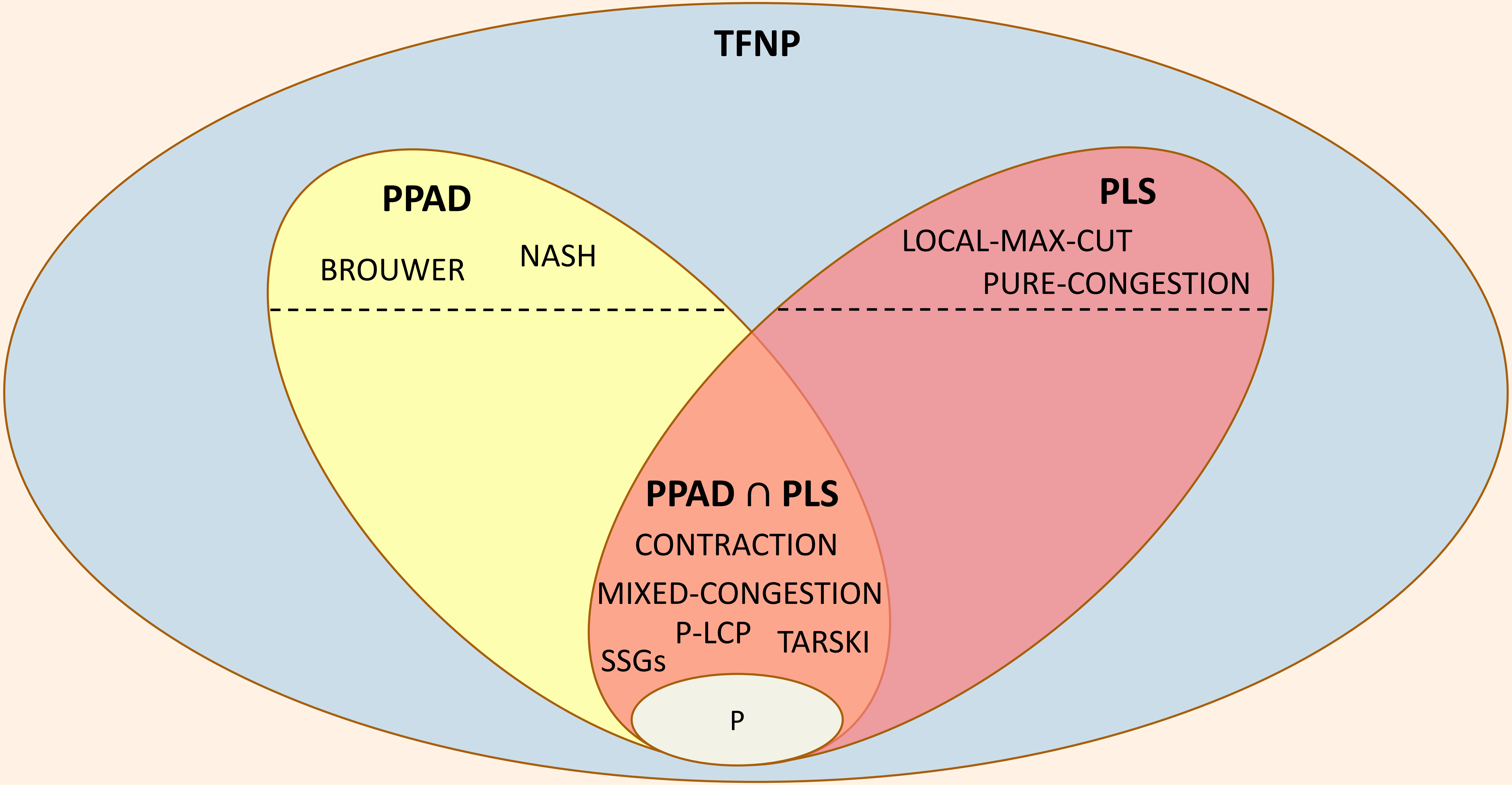
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Problem A : PPAD-complete

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Input: instance I_A of A, instance I_B of B
Goal: find a solution of I_A, or a solution of I_B

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BROUWER:
Input: a continuous function $f: [0,1]^n \to [0,1]^n$
Goal: find a fixpoint x

$$f(x) = x$$

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BROUWER:
Input: a continuous function $f\colon [0,1]^n \to [0,1]^n$, precision $\varepsilon > 0$
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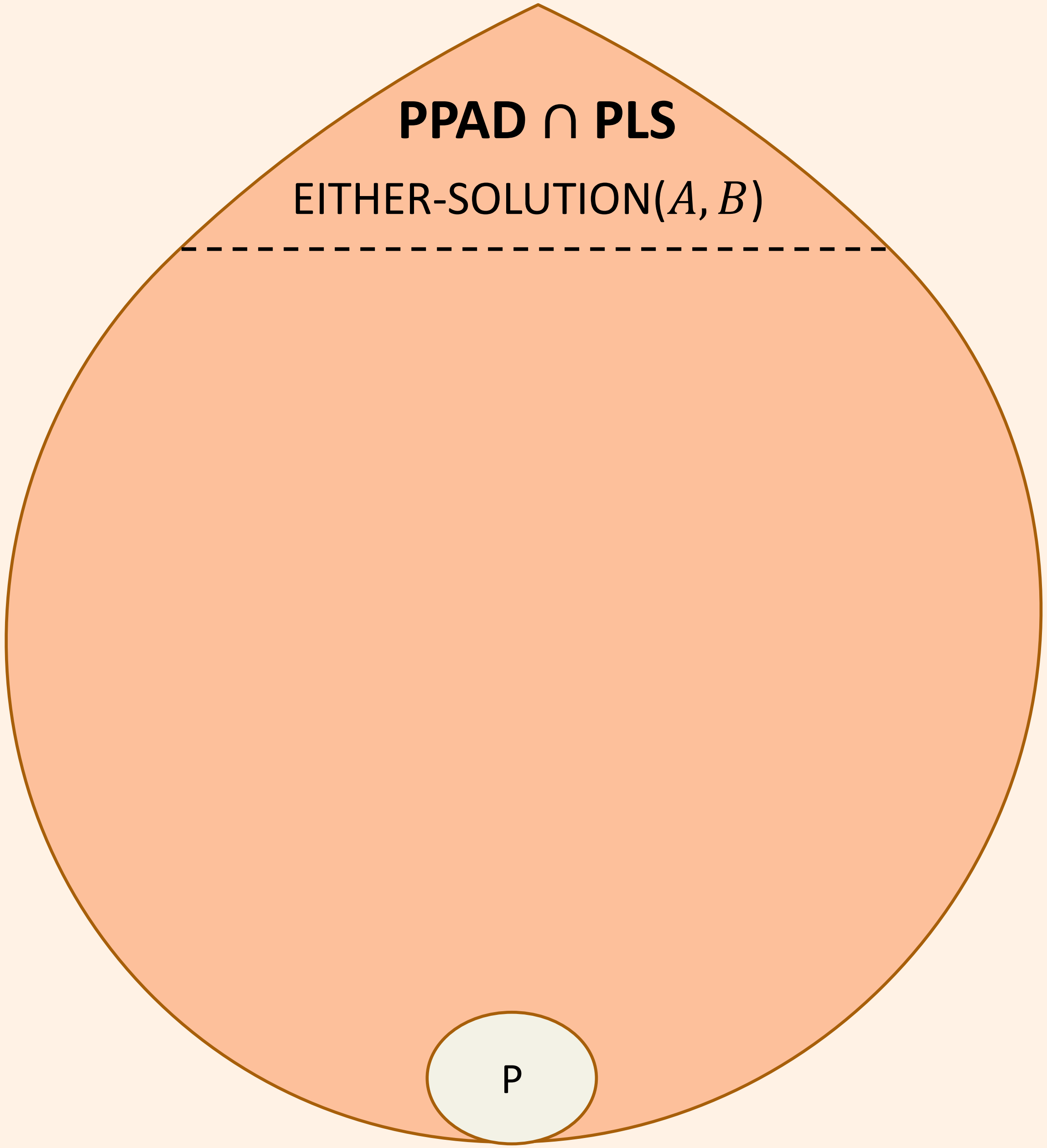
→ class **Continuous Local Search (CLS)** [Daskalakis-Papadimitriou, 2011]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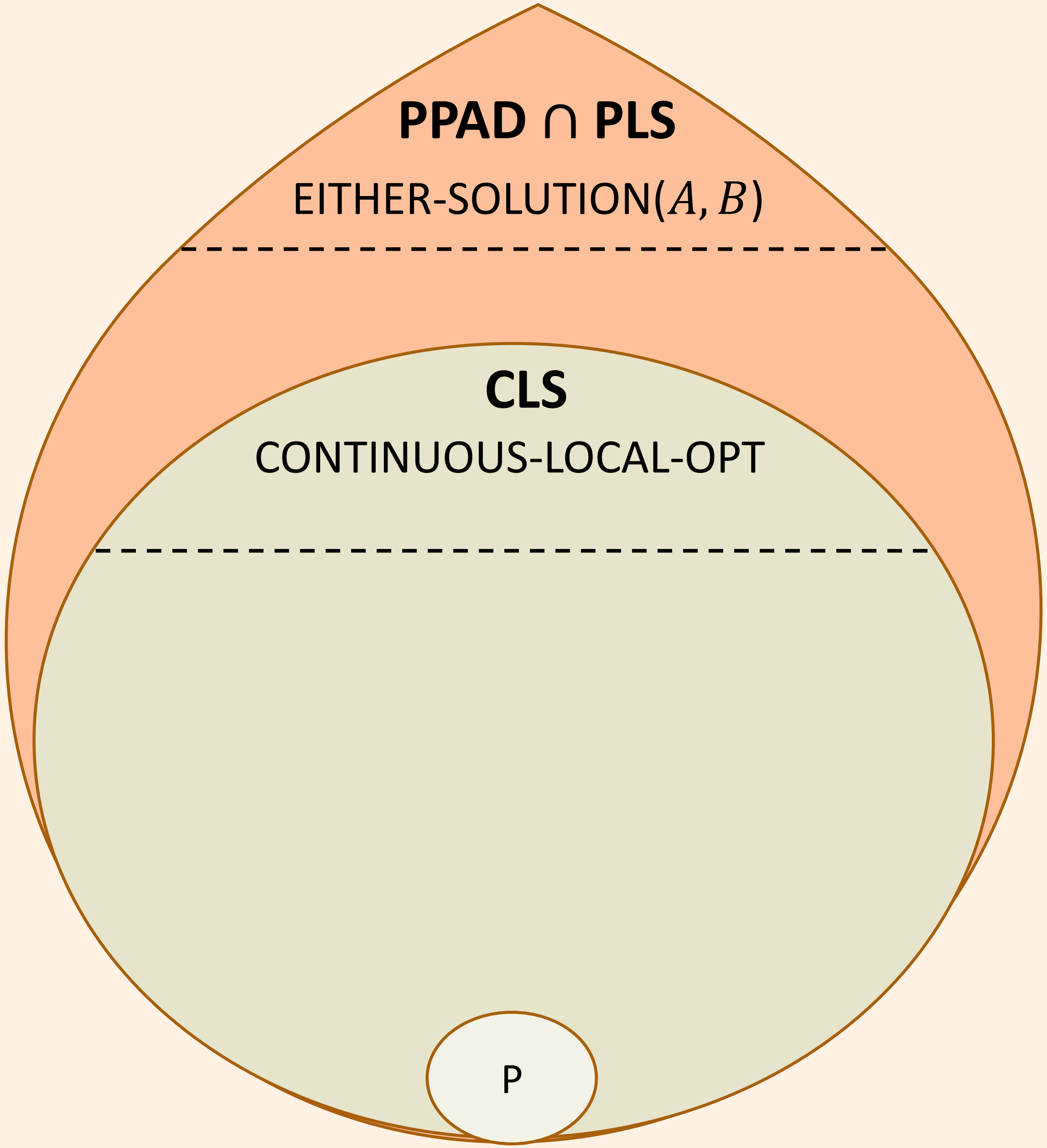

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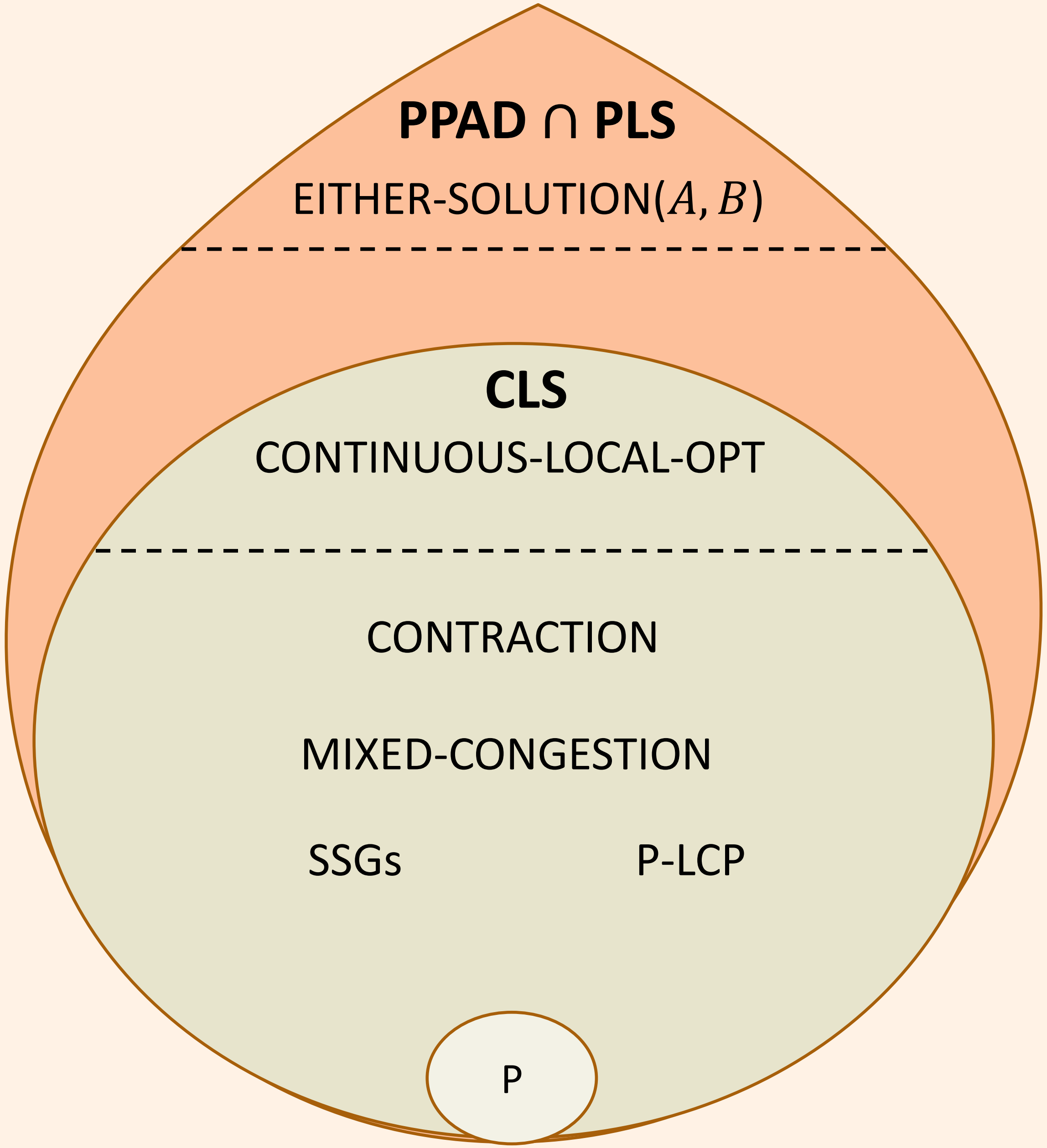

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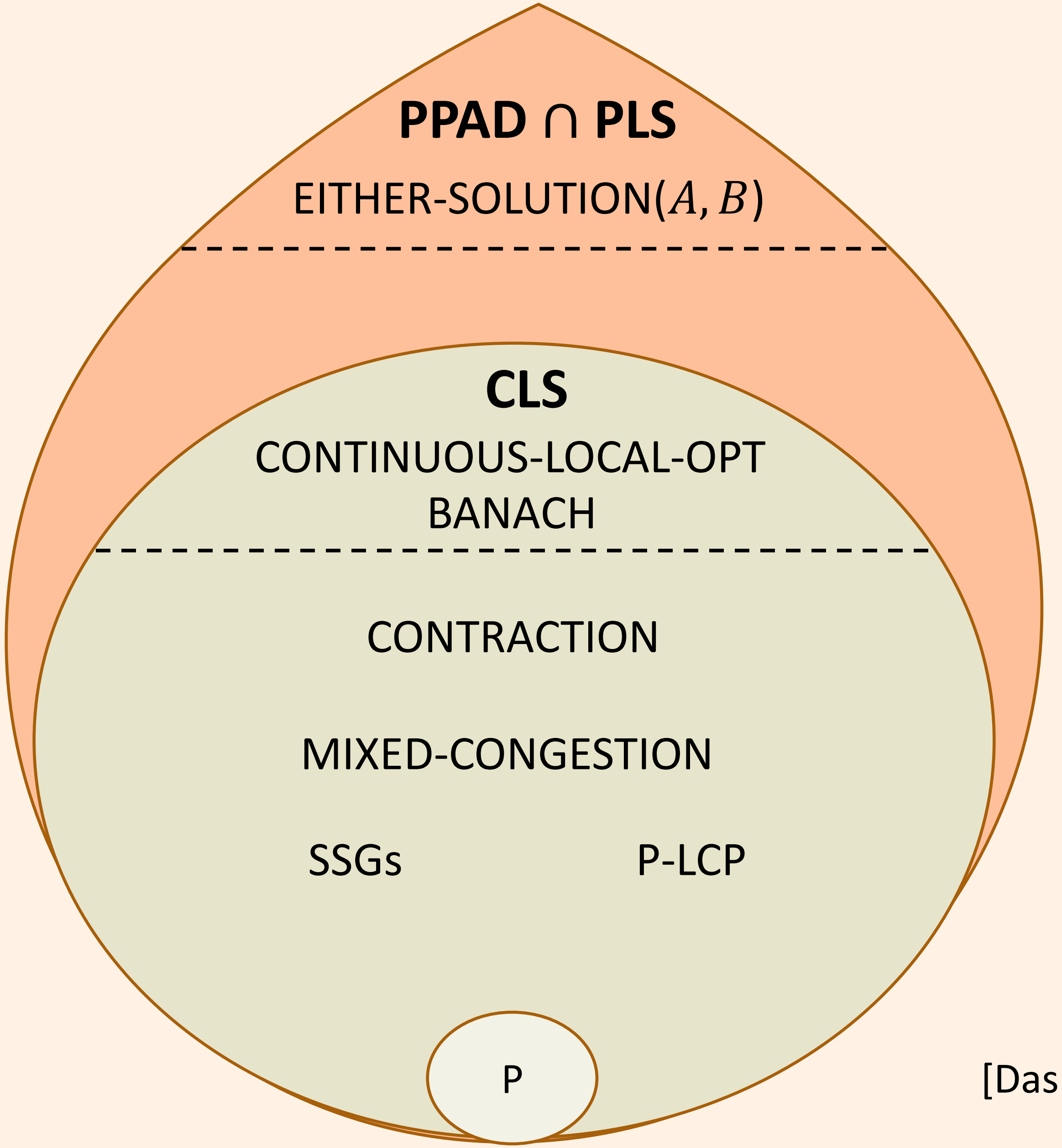

[Daskalakis-Tzamos-Zampetakis, 2018]

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→ in CLS: $p(x) \coloneqq f(x)$ and $g(x) \coloneqq x - \eta \nabla f(x)$

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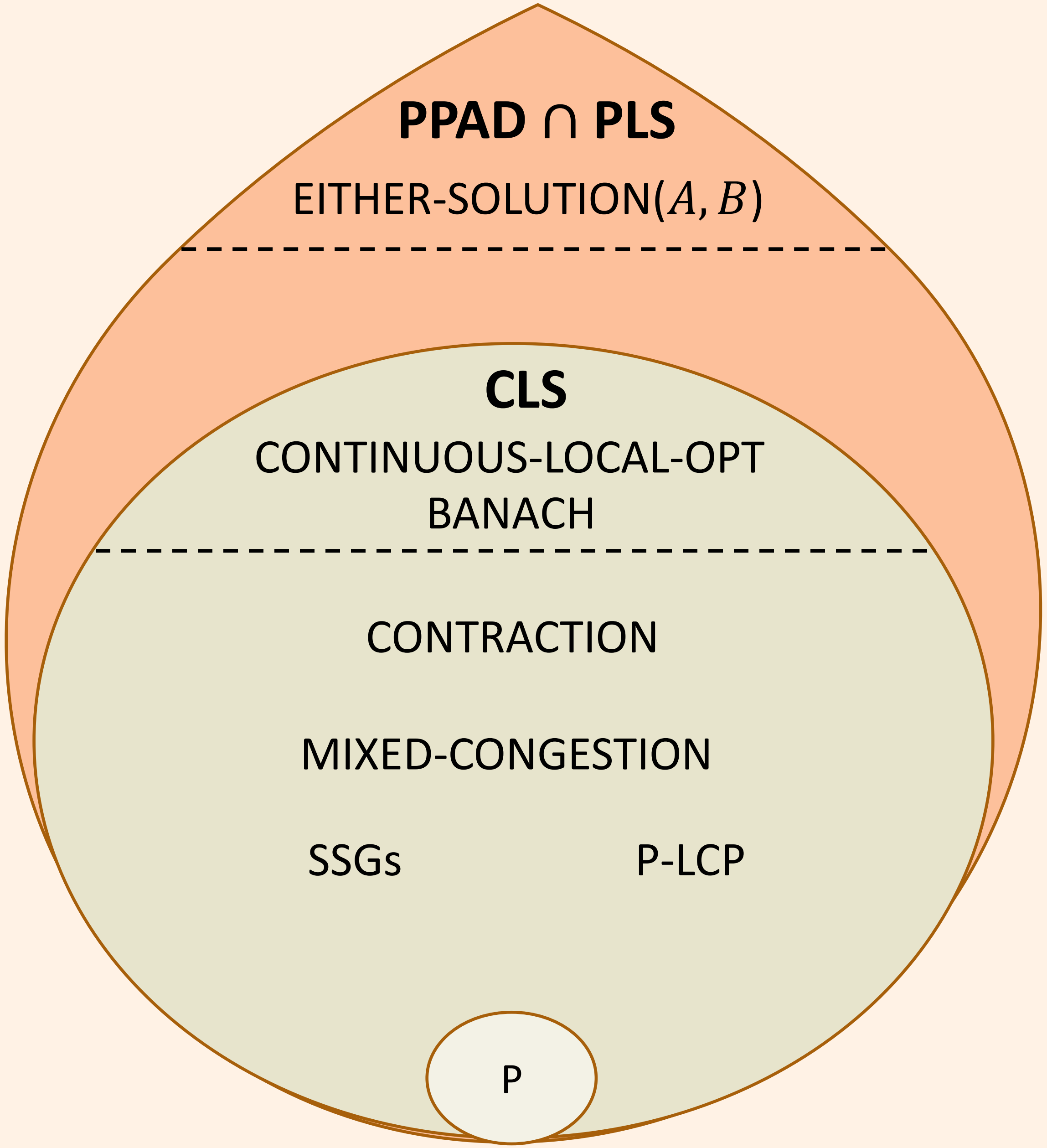
→ polynomial-time equivalent!



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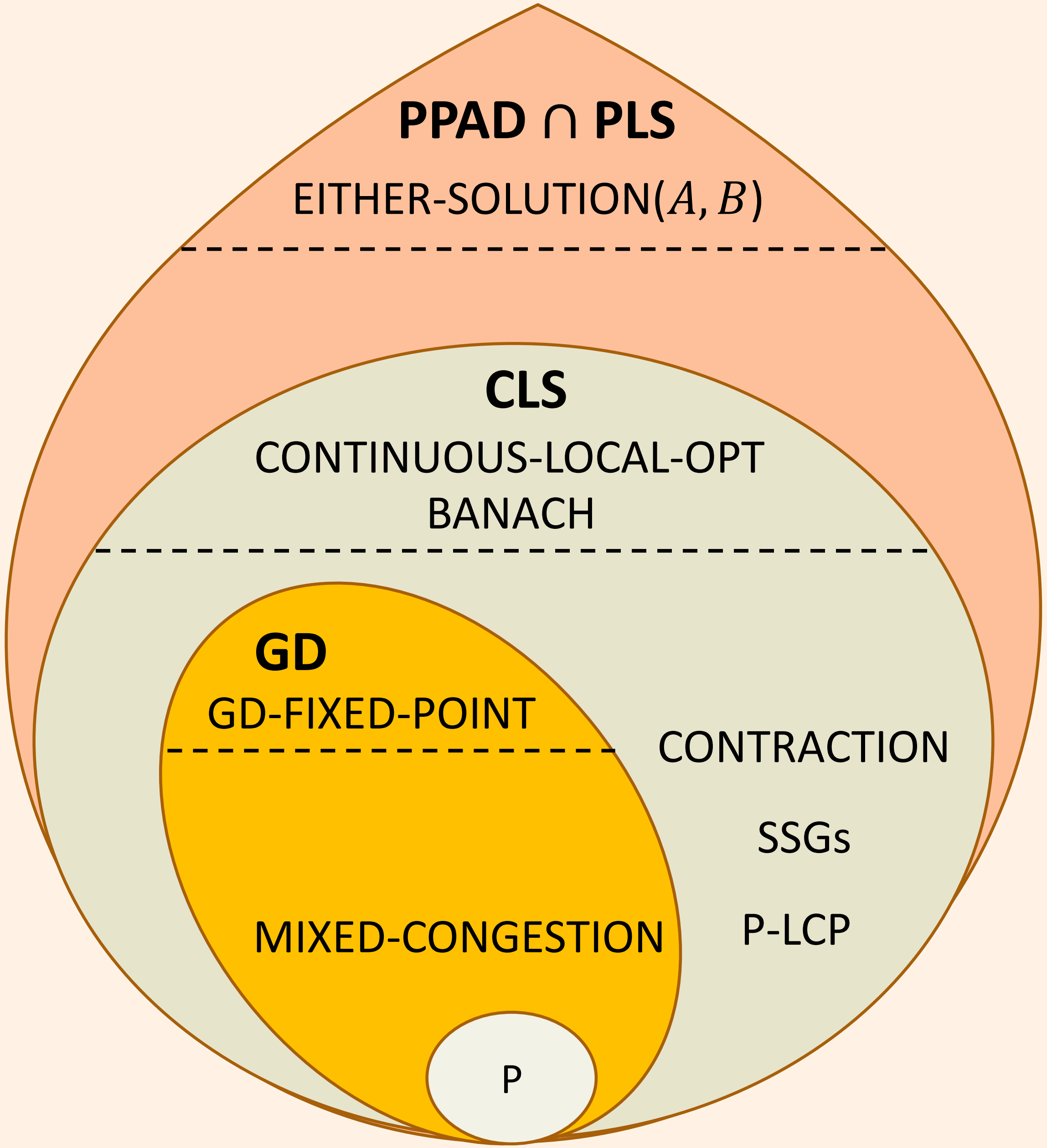
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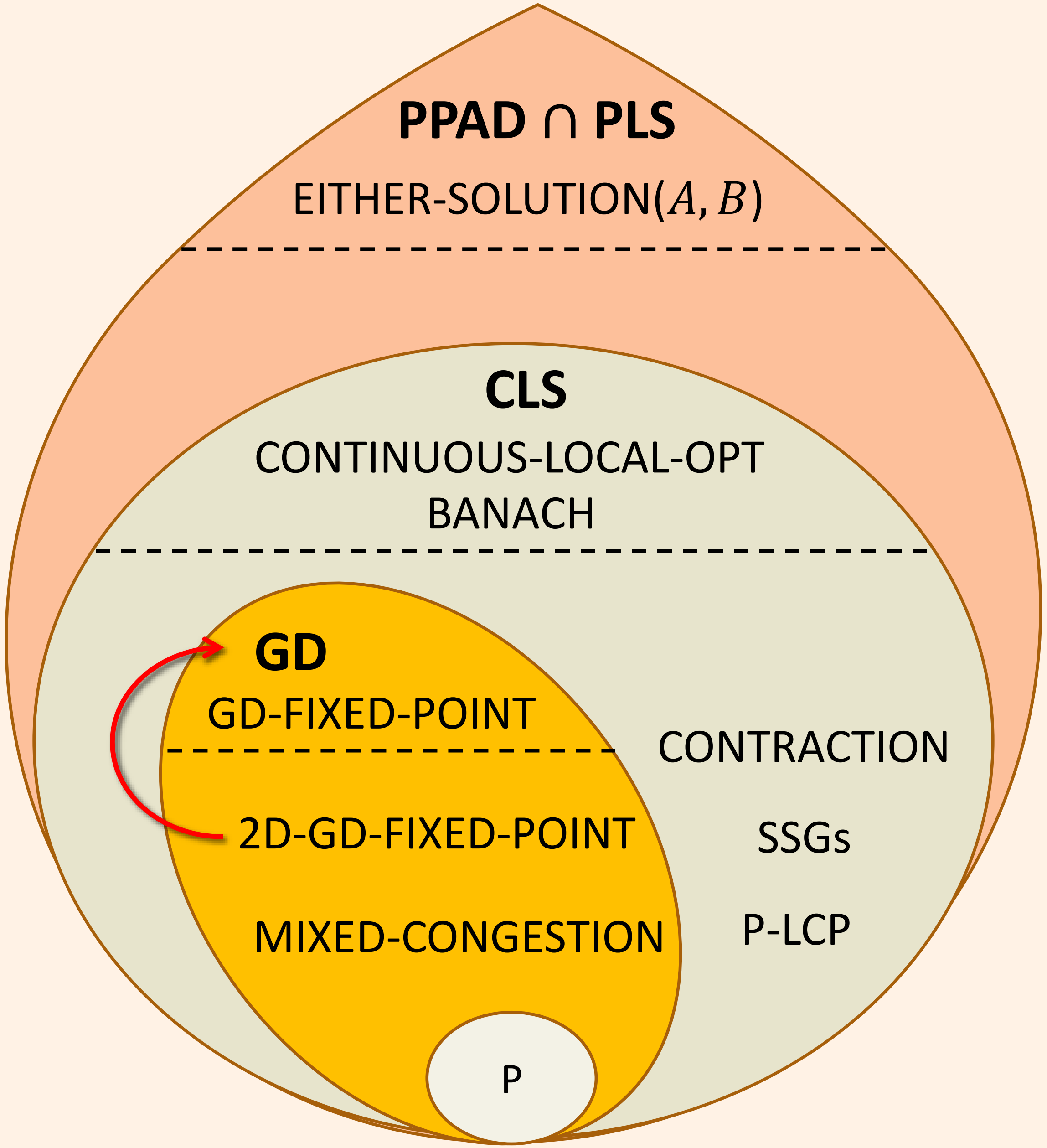
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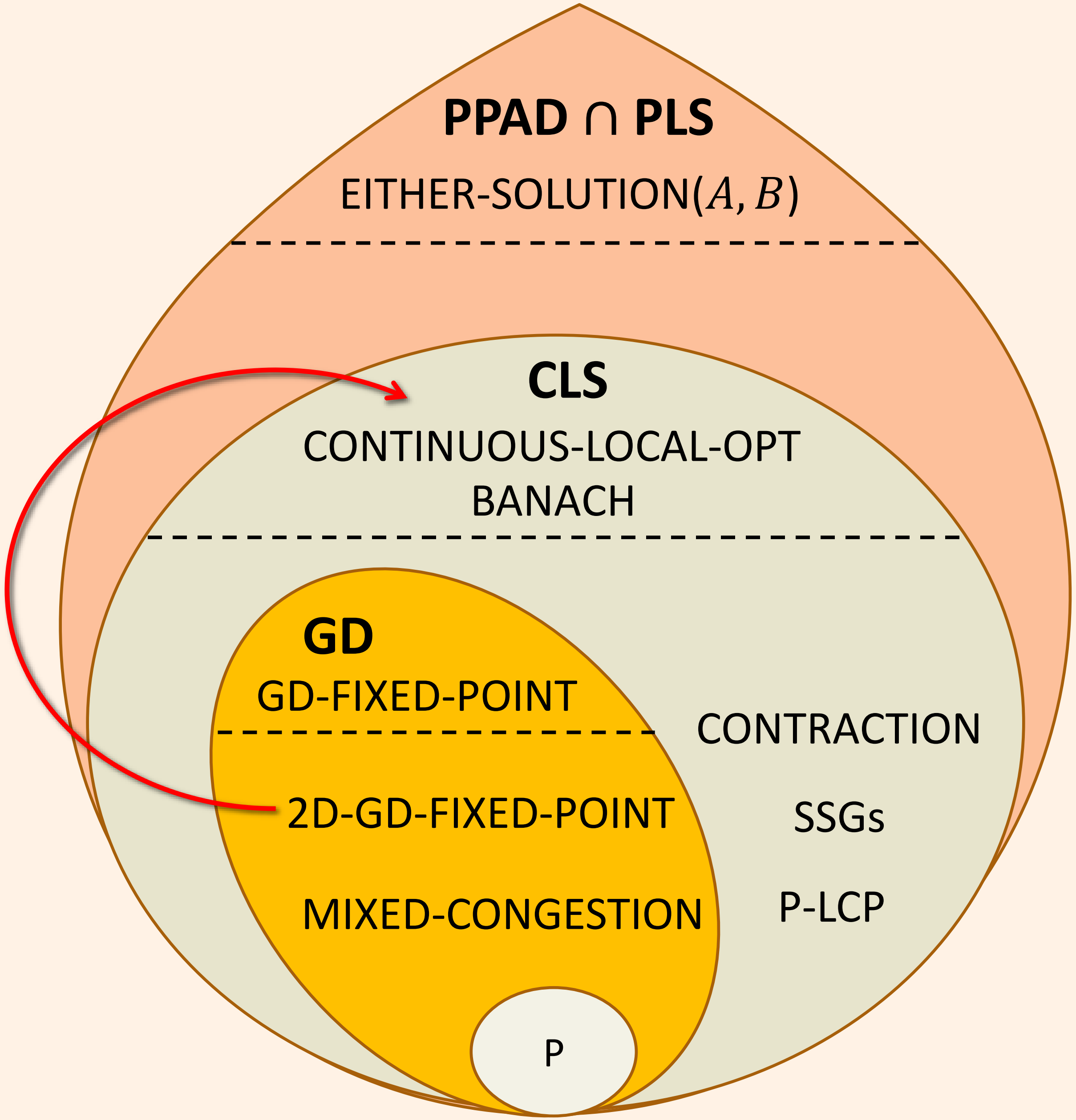

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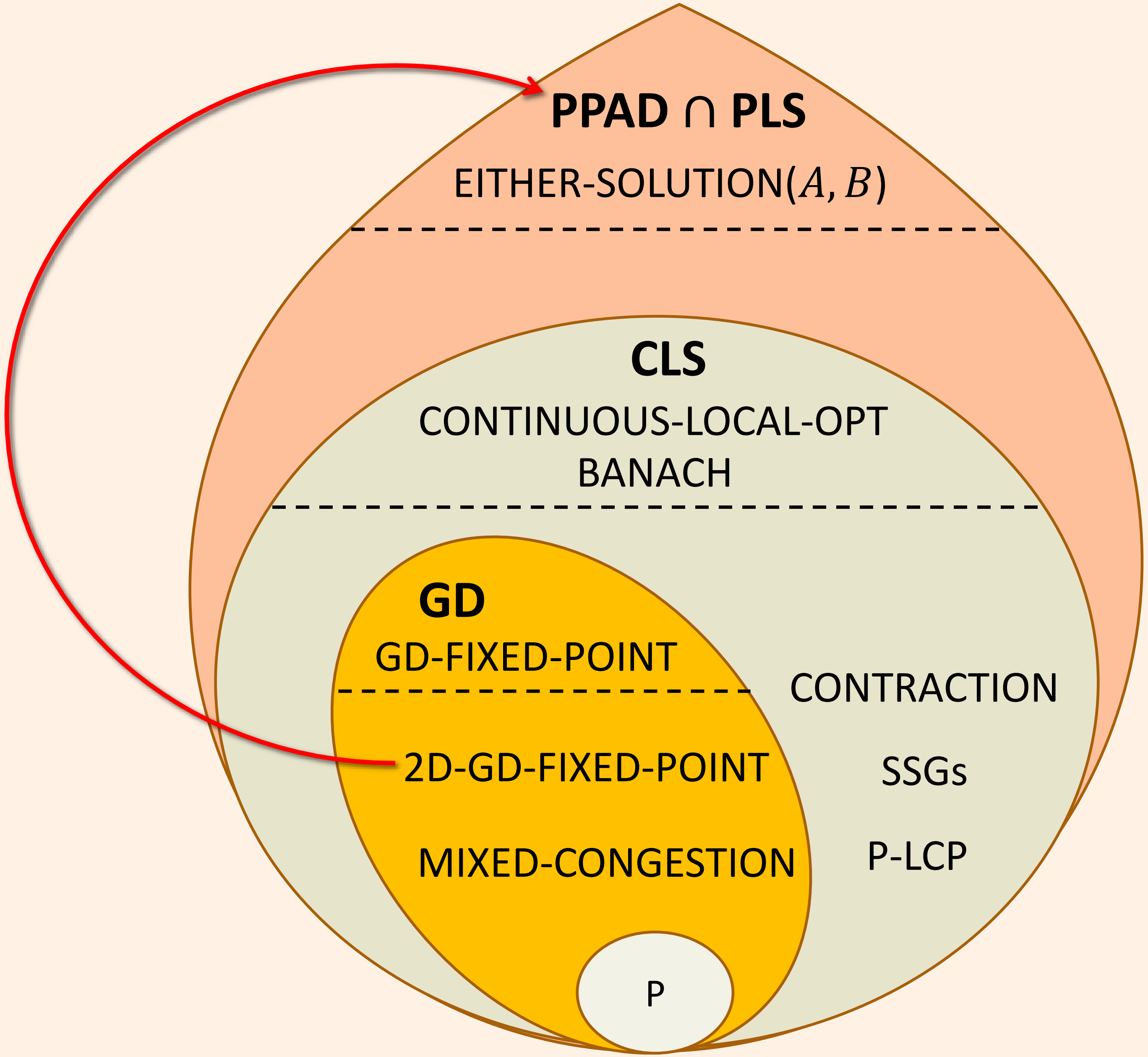
PPAD ∩ PLS
EITHER-SOLUTION(A, B)
CLS
CONTINUOUS-LOCAL-OPT
BANACH
GD
GD-FIXED-POINT
2D-GD-FIXED-POINT
MIXED-CONGESTION
CONTRACTION
SSGs
P-LCP
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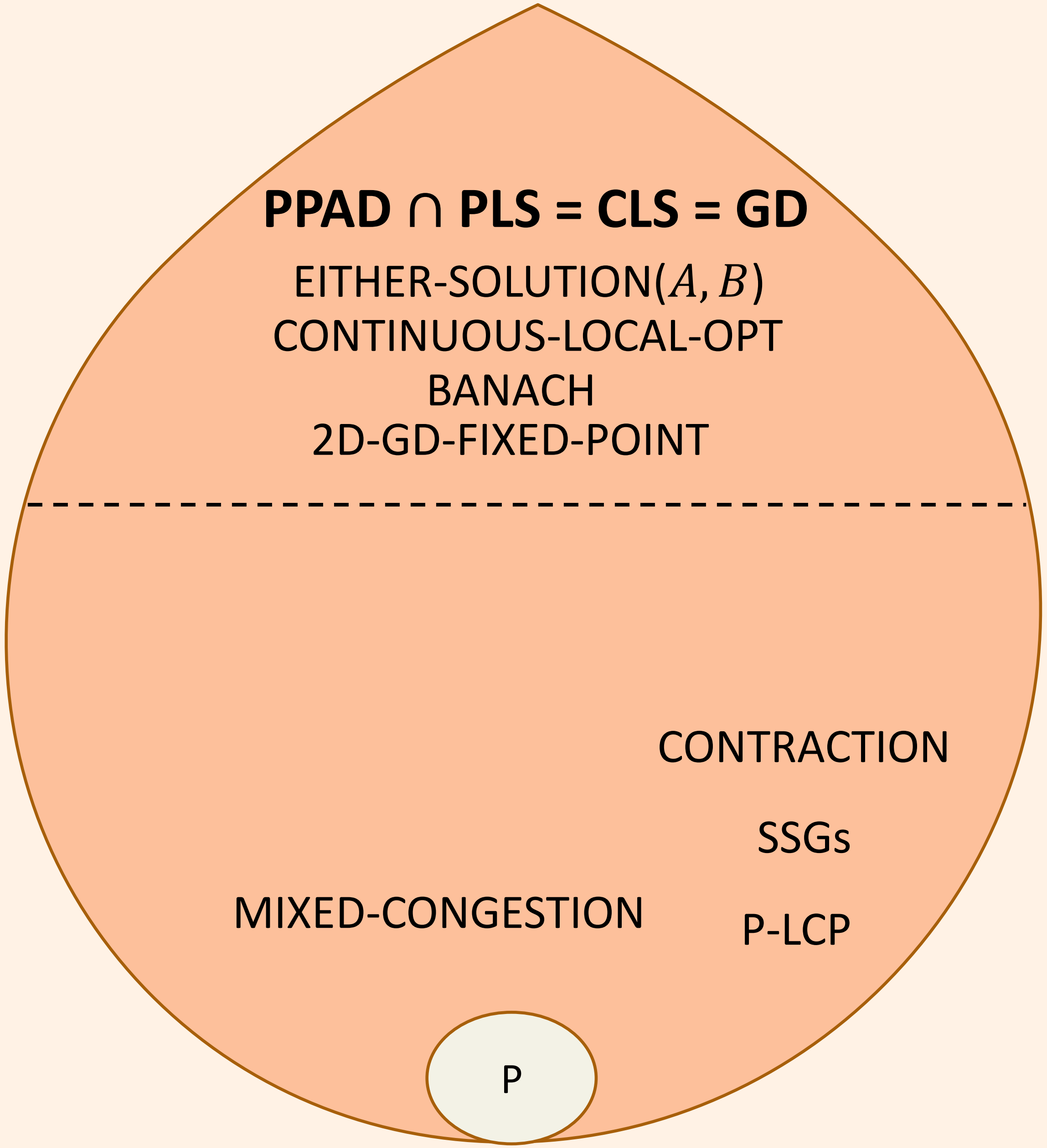
PPAD ∩ PLS = CLS = GD
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Consequences

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- PPAD ∩ PLS is an interesting class!
- It captures continuous local search, and even gradient descent
- CLS and GD are robust with respect to:
  - dimension
  - domain
  - arithmetic circuits
  - …

Proof Sketch

PPAD

Canonical complete problem: **END-OF-LINE**

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Input: directed graph of paths and cycles, and a source

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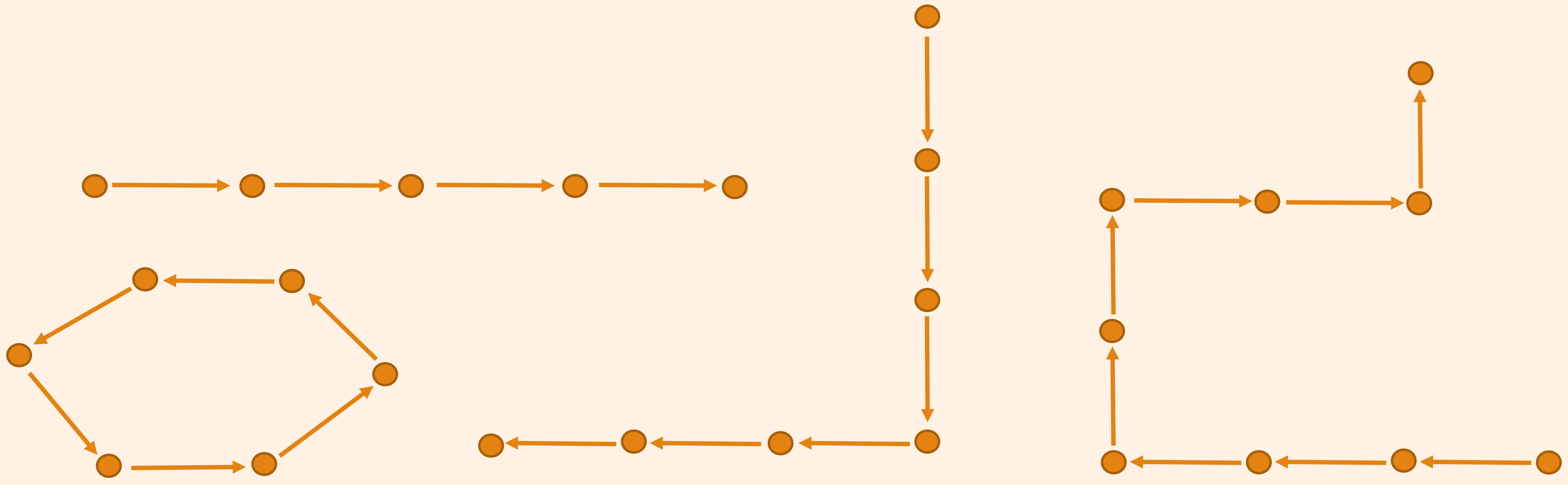

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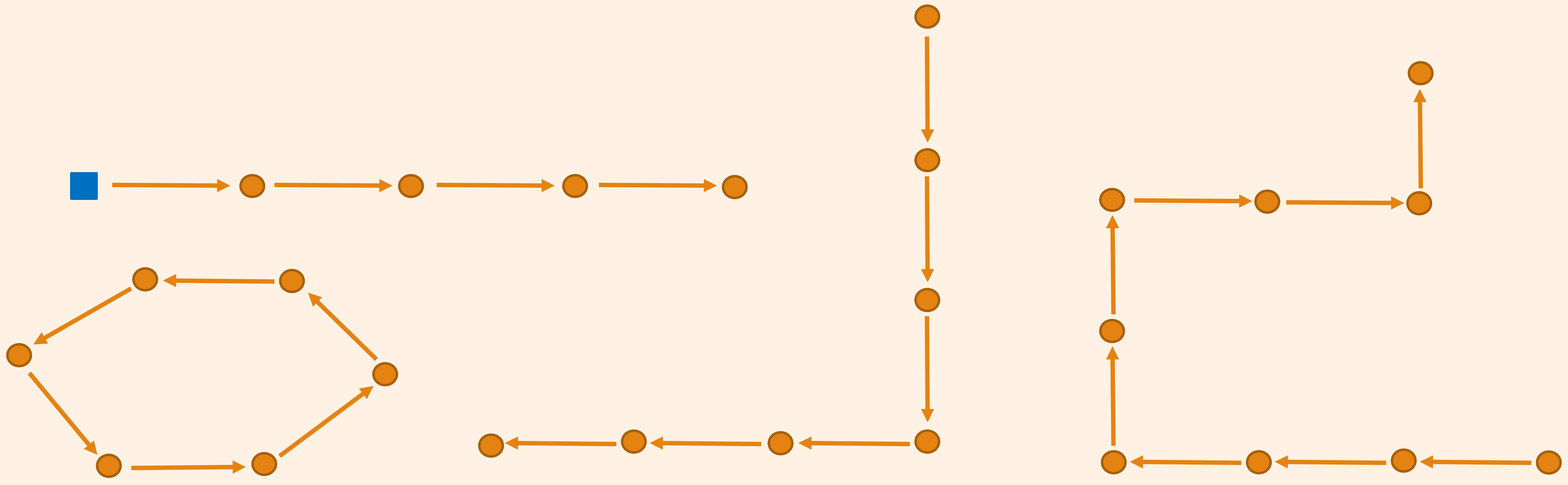
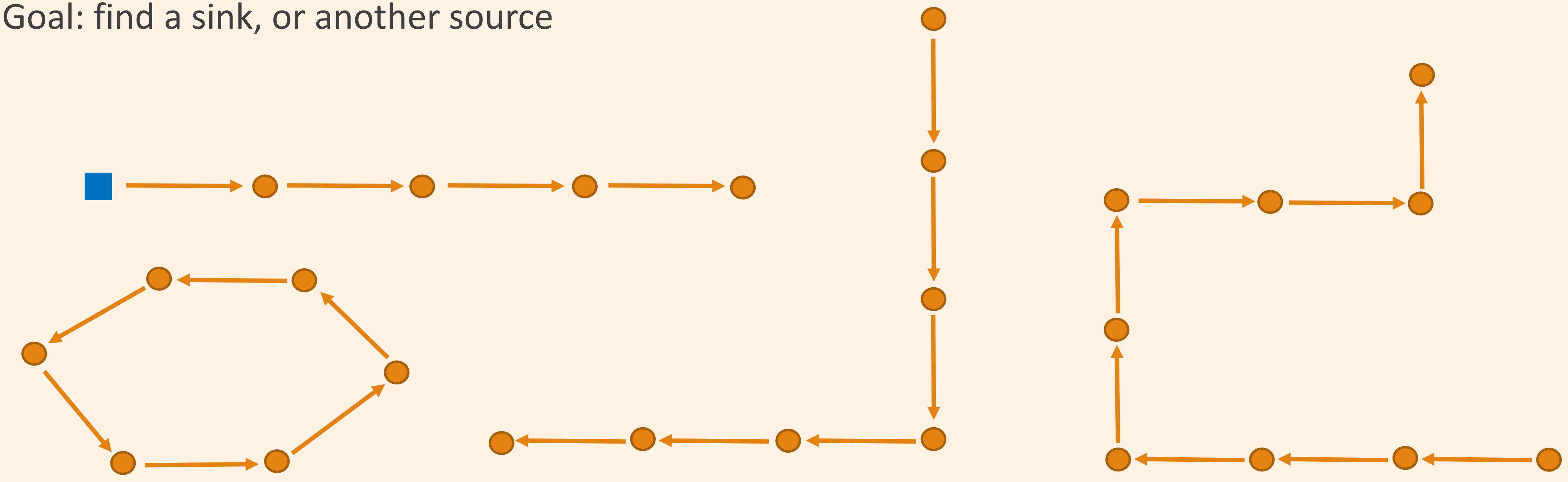


PPAD

Canonical complete problem: **END-OF-LINE**

Input: directed graph of paths and cycles, and a source

Goal: find a sink, or another source



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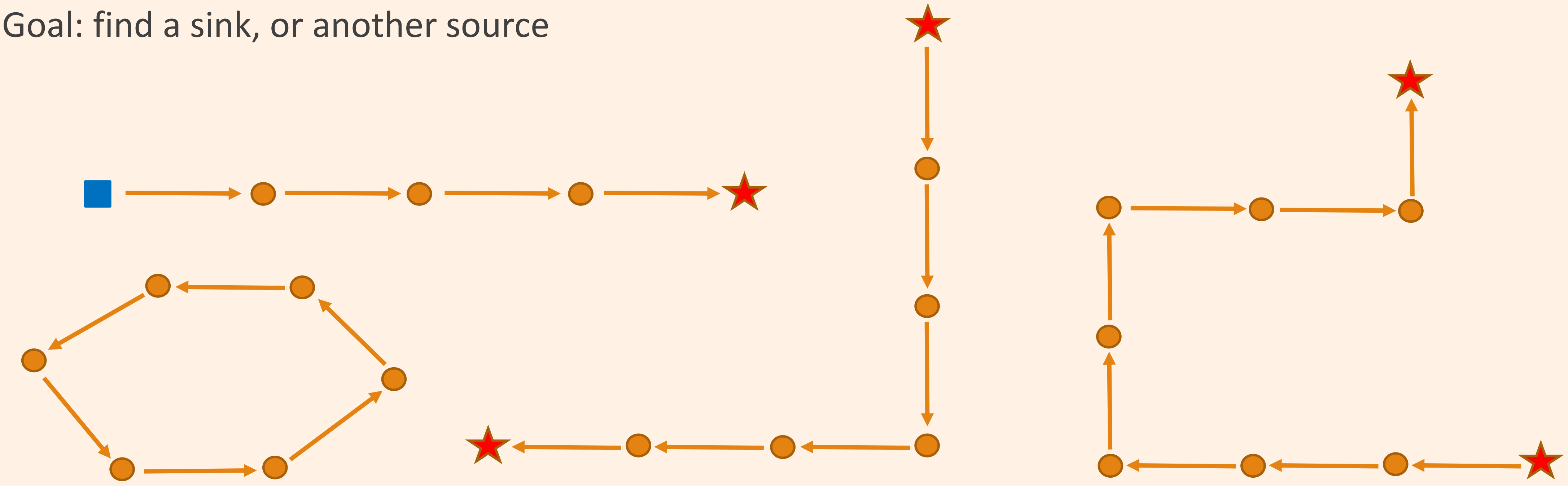

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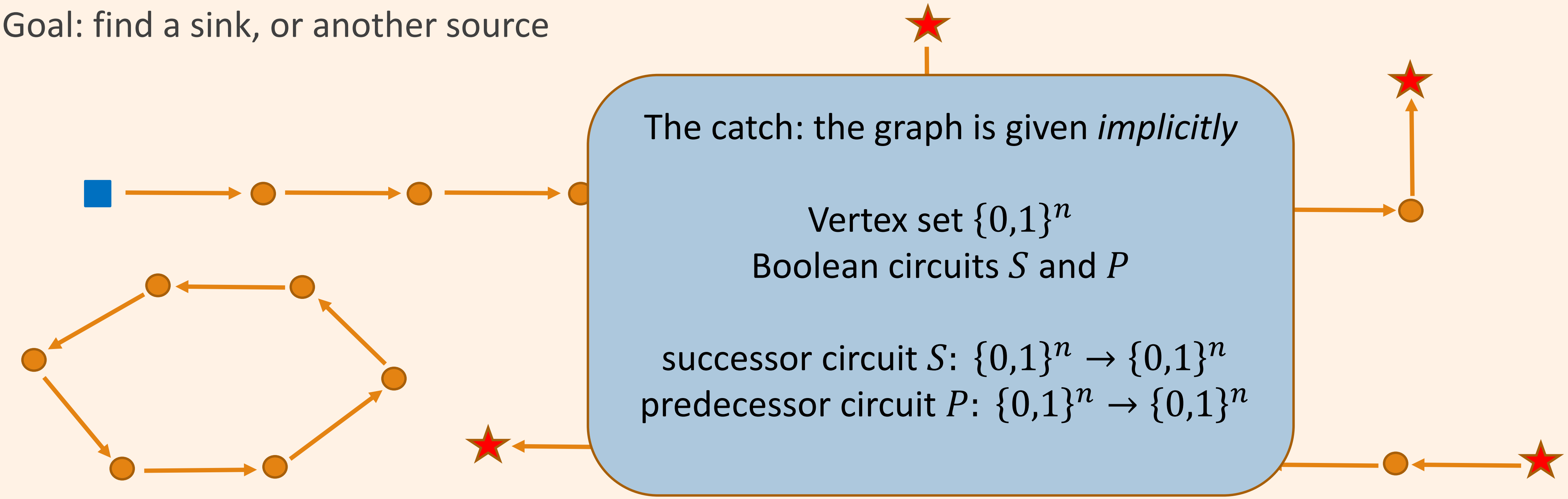

The catch: the graph is given *implicitly*

Vertex set $\{0,1\}^n$
Boolean circuits S and P

successor circuit S: $\{0,1\}^n \rightarrow \{0,1\}^n$
predecessor circuit P: $\{0,1\}^n \rightarrow \{0,1\}^n$

Reduction: high level

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Goal: reduction from EITHER-SOLUTION(END-OF-LINE, LOCAL-OPT) to 2D-GD-FIXED-POINT

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→ Construct a continuously differentiable function $f: [0,1]^2 \rightarrow \mathbb{R}$ such that any gradient descent fixed point yields a solution to the EITHER-SOLUTION instance

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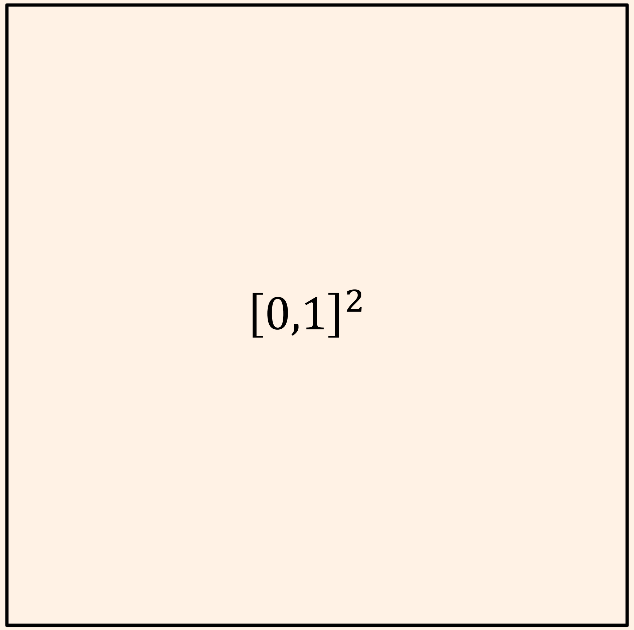

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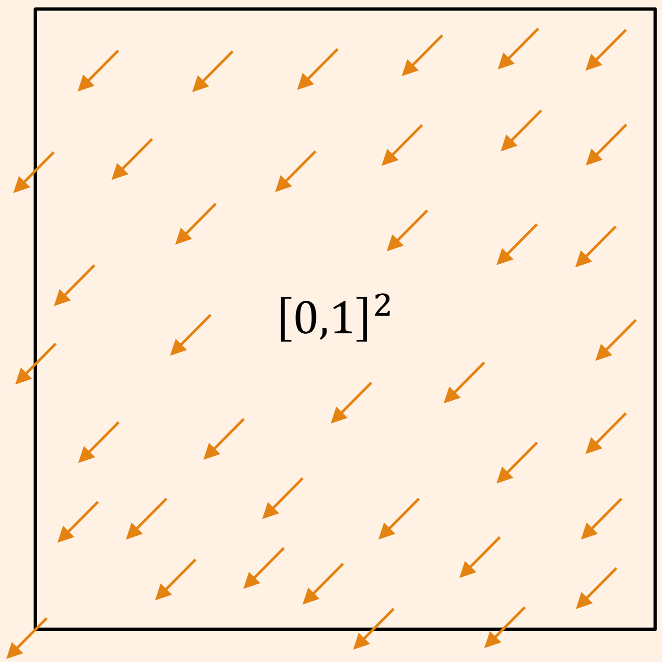

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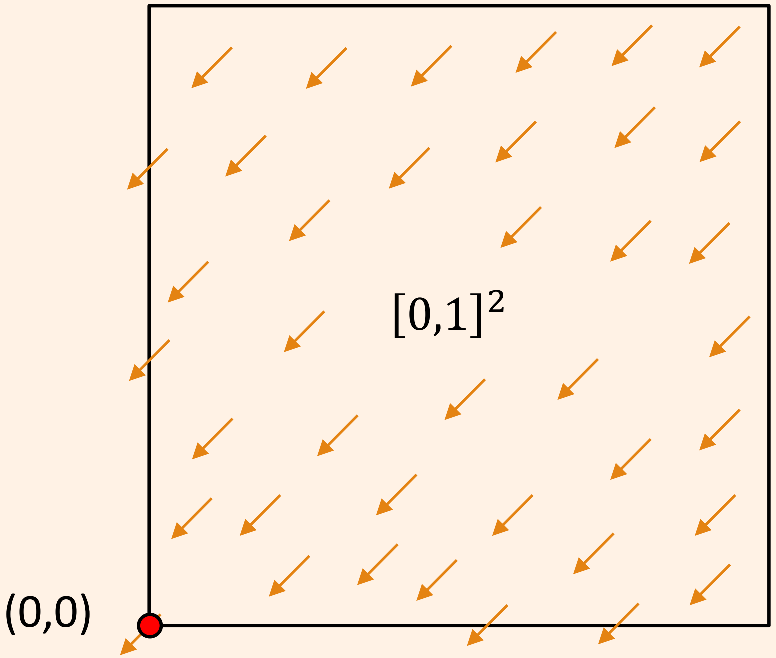

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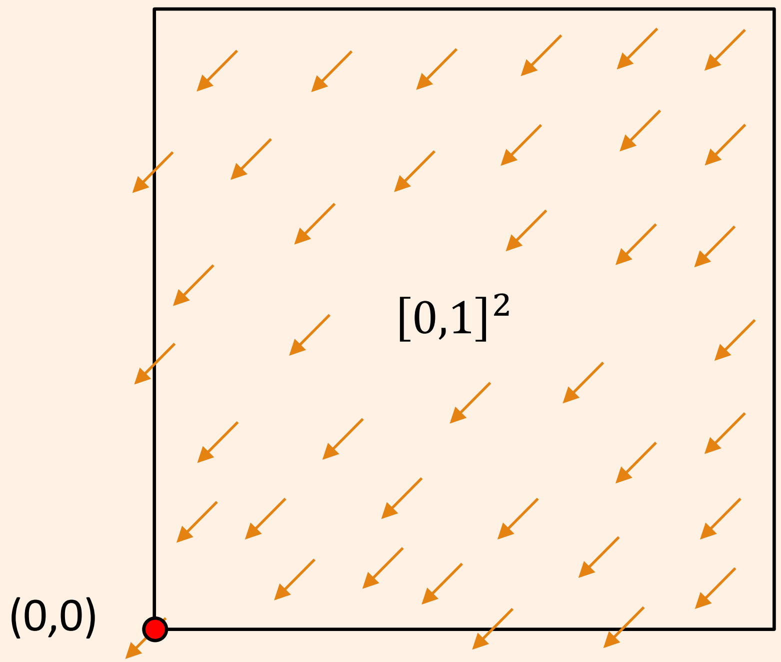

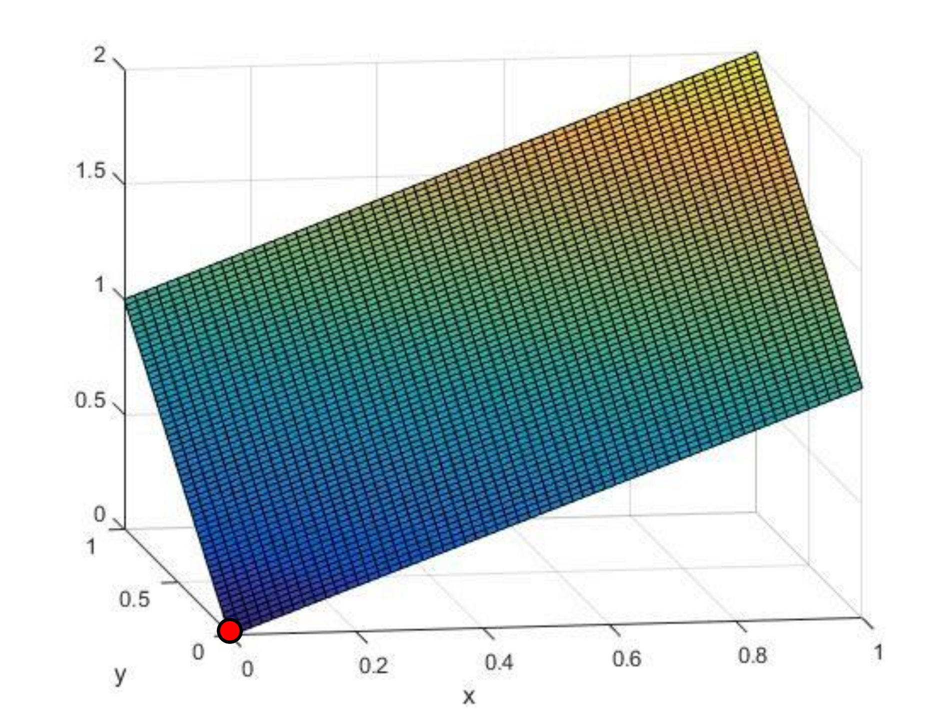

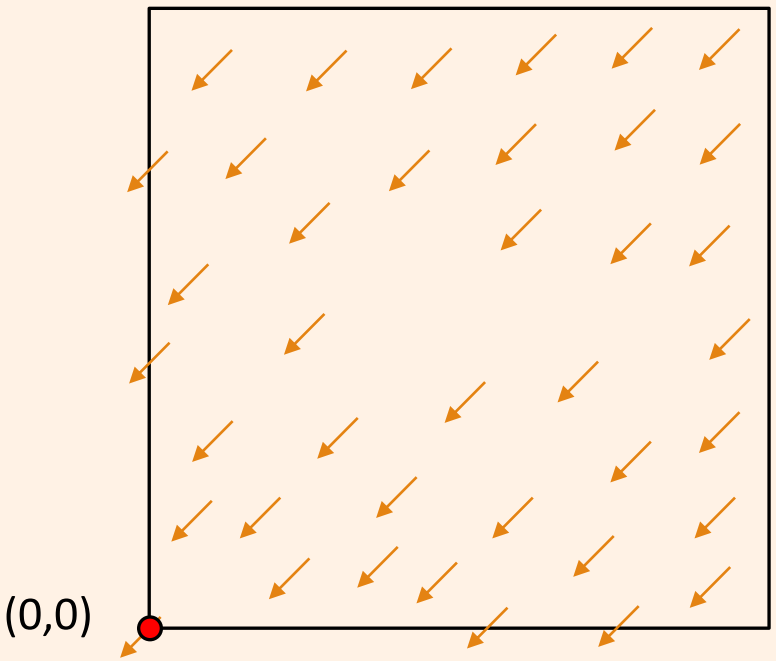
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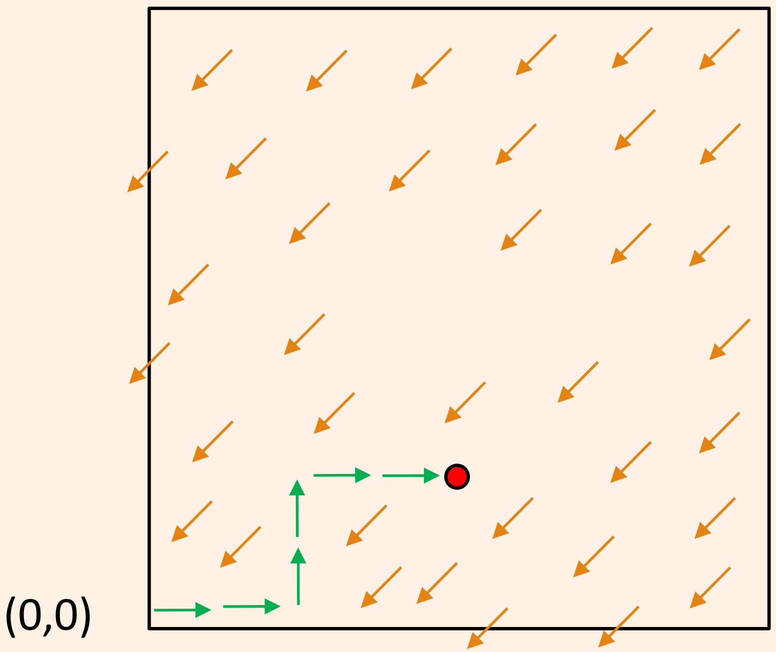
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Warm up: Monotone-End-of-Line

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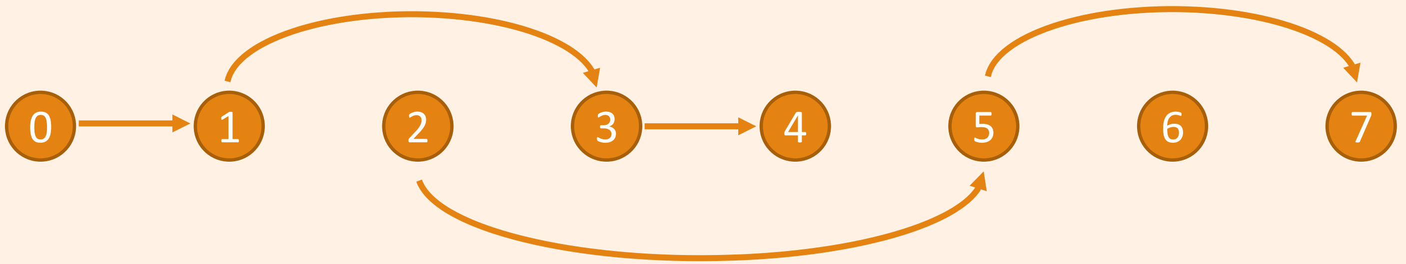
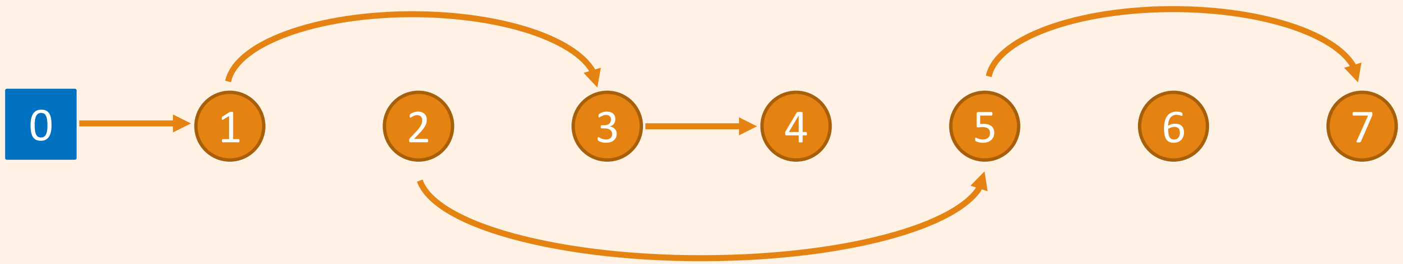


Special case of END-OF-LINE: No backward edges allowed!

Warm up: Monotone-End-of-Line



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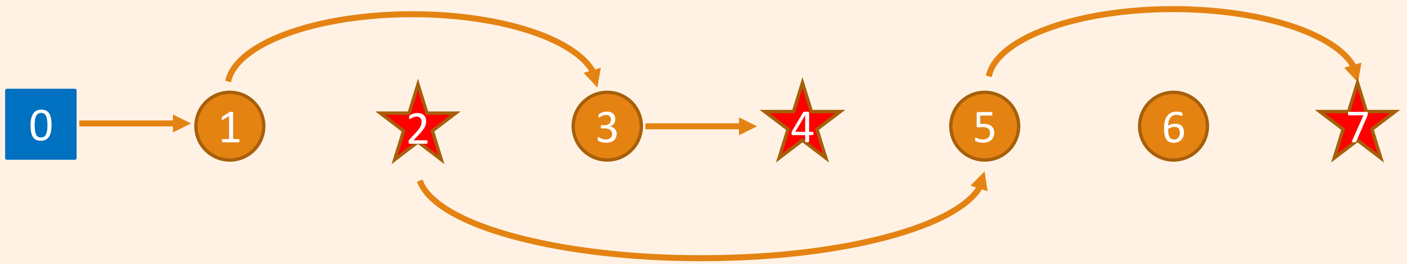

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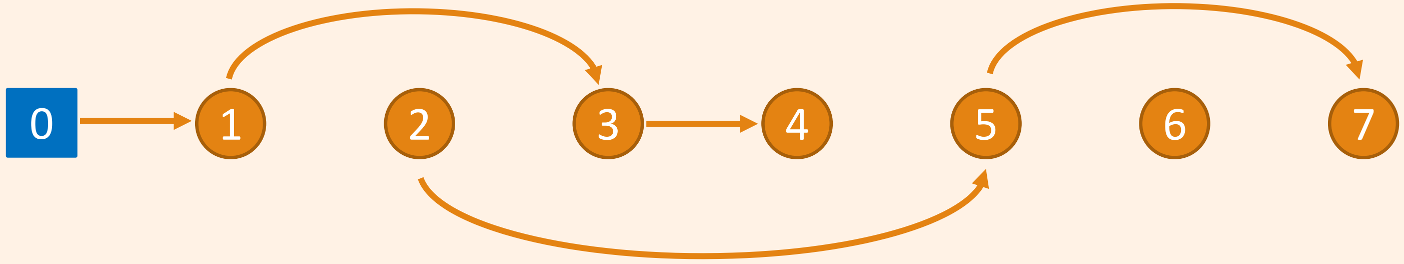
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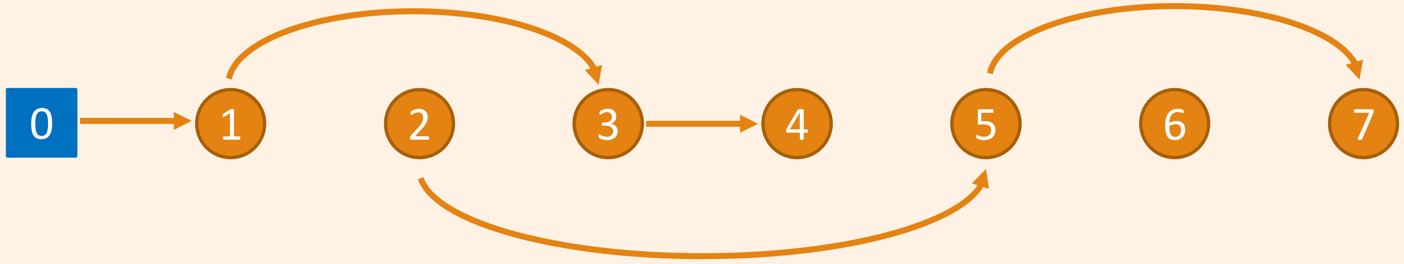
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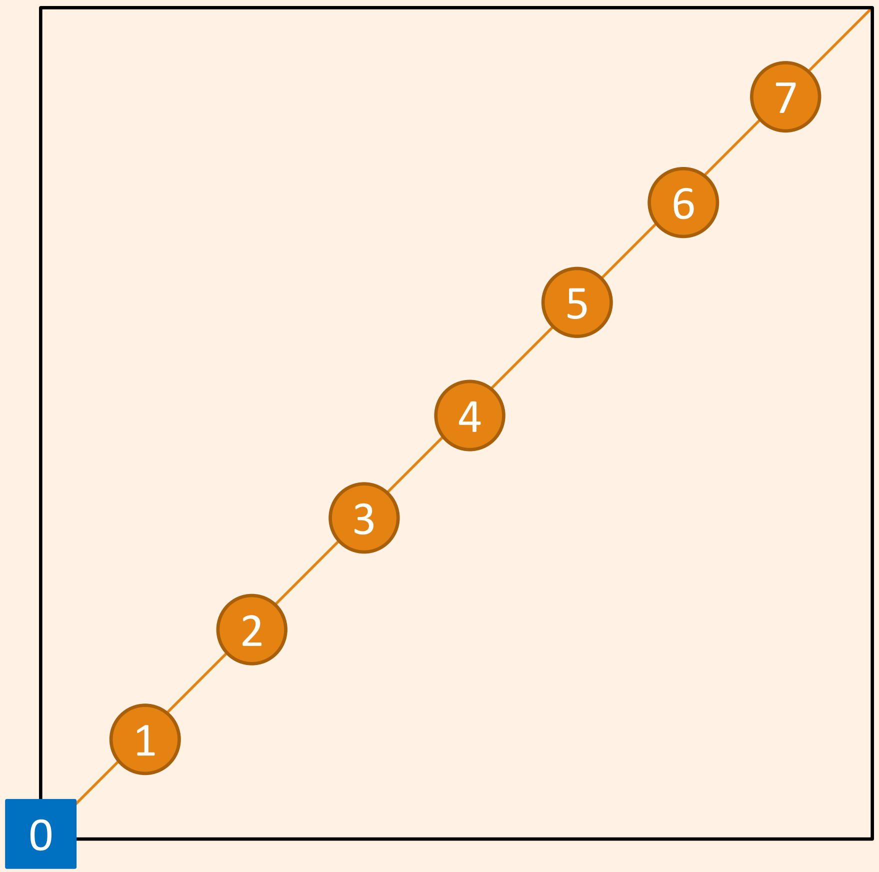
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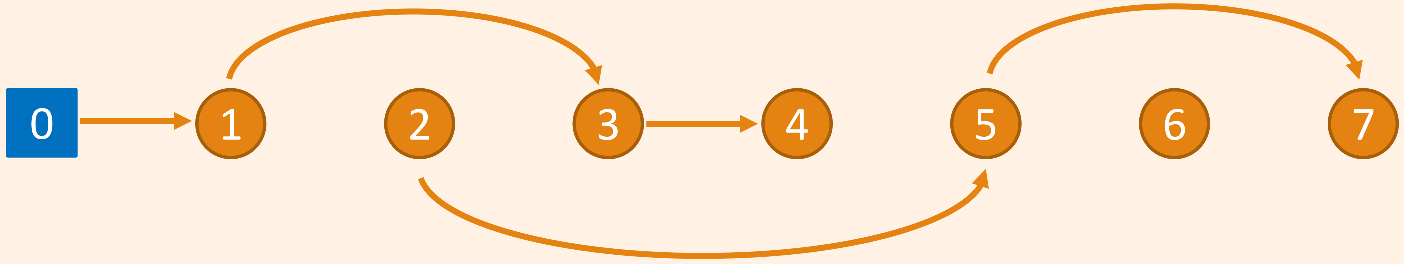
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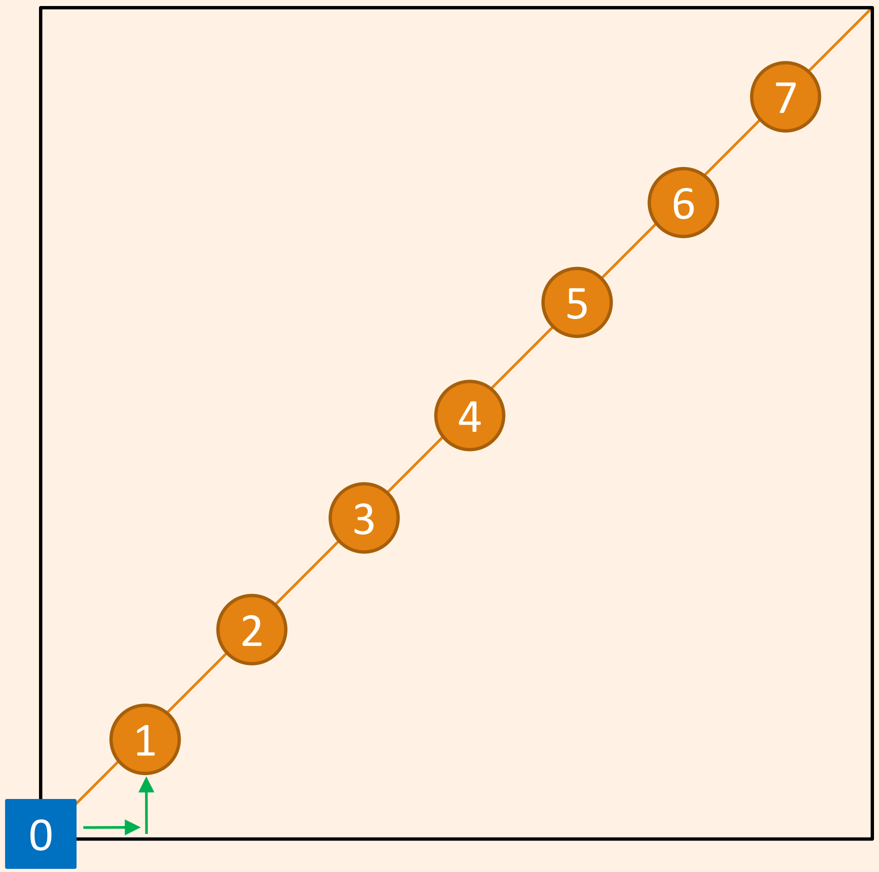
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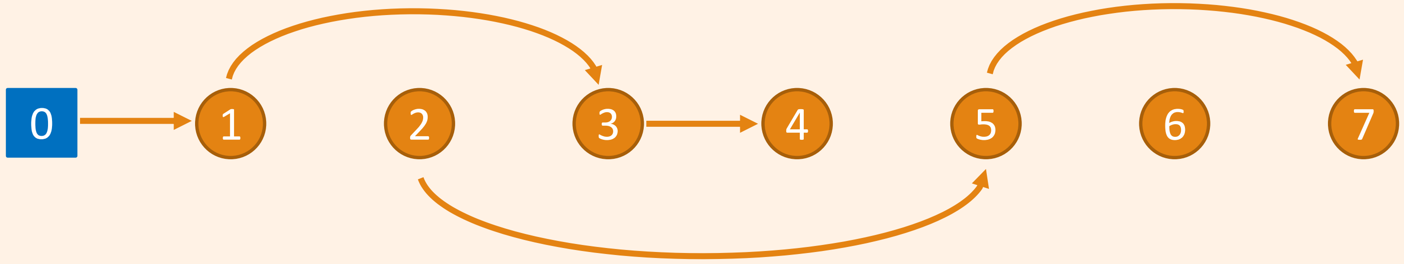
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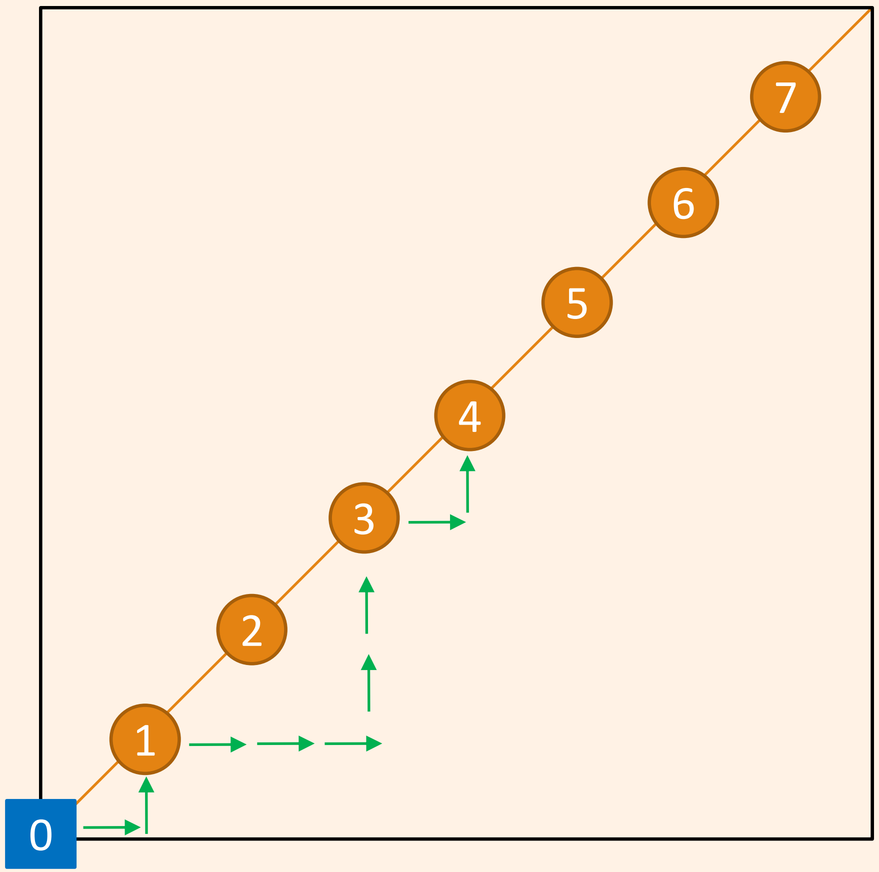
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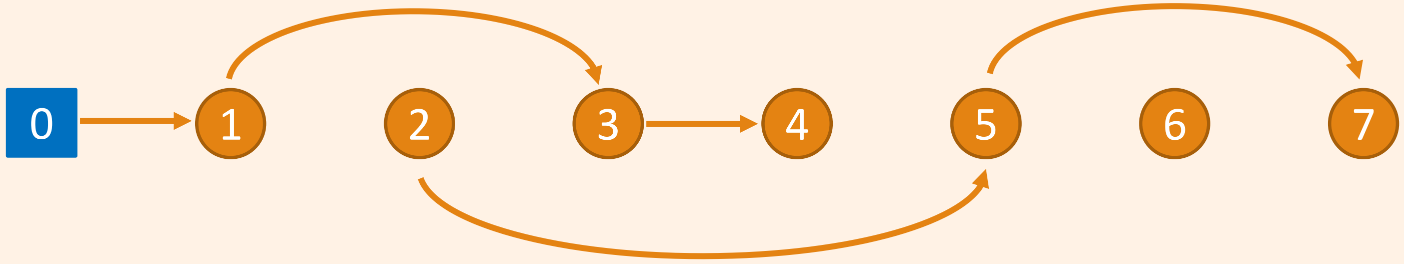
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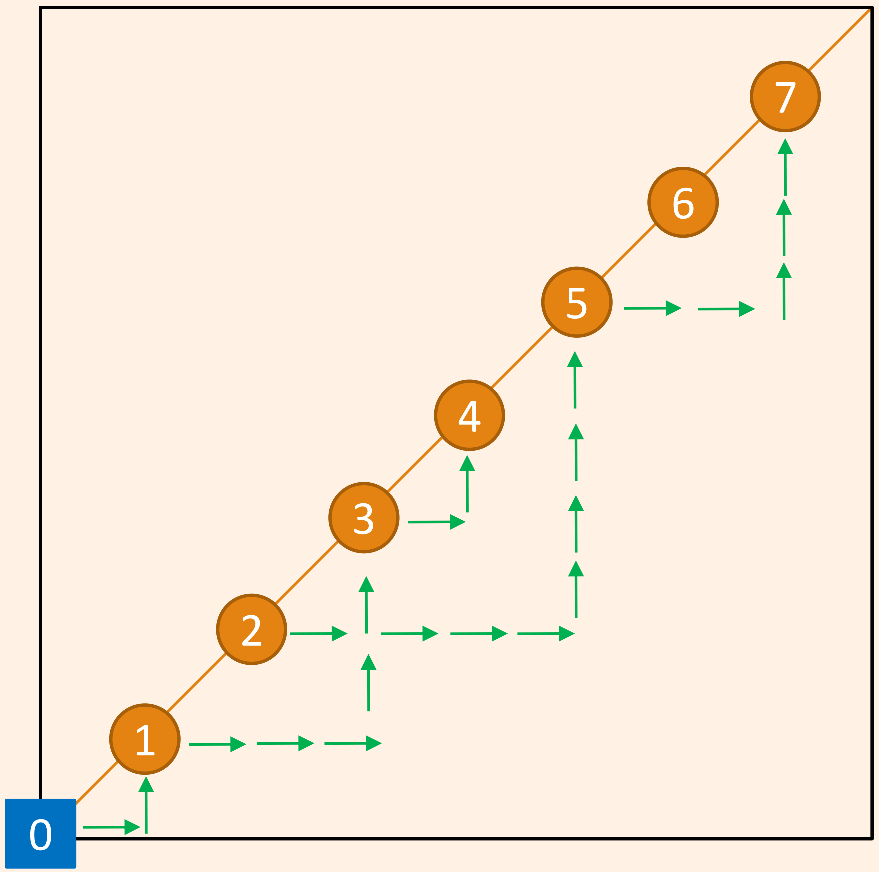
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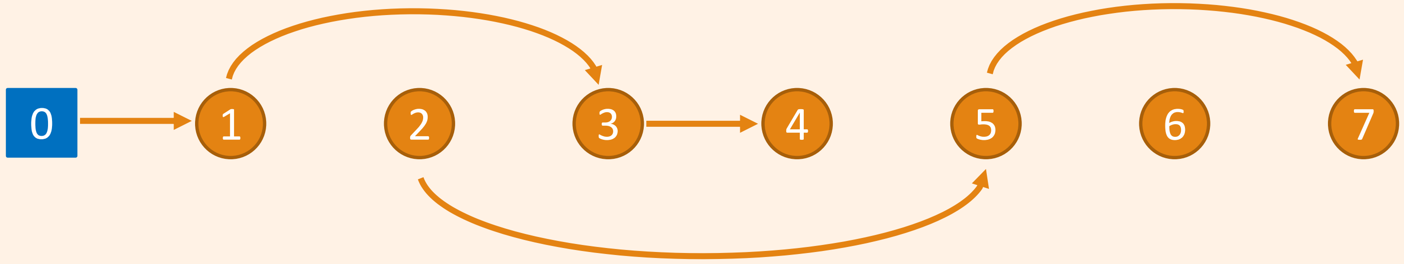
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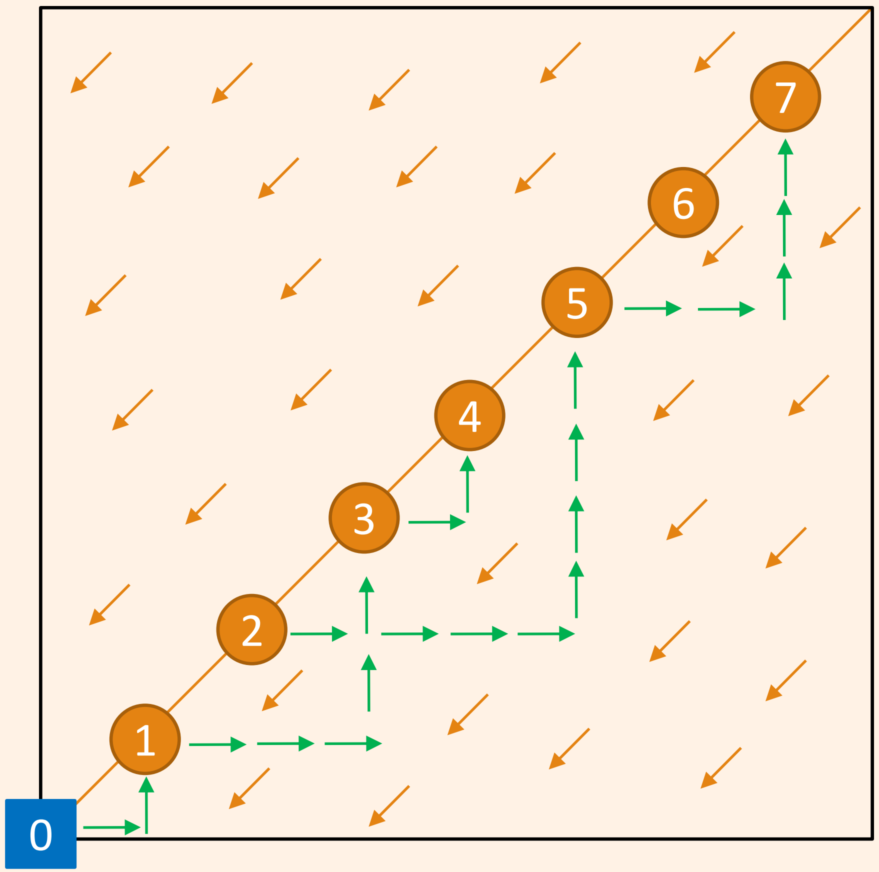
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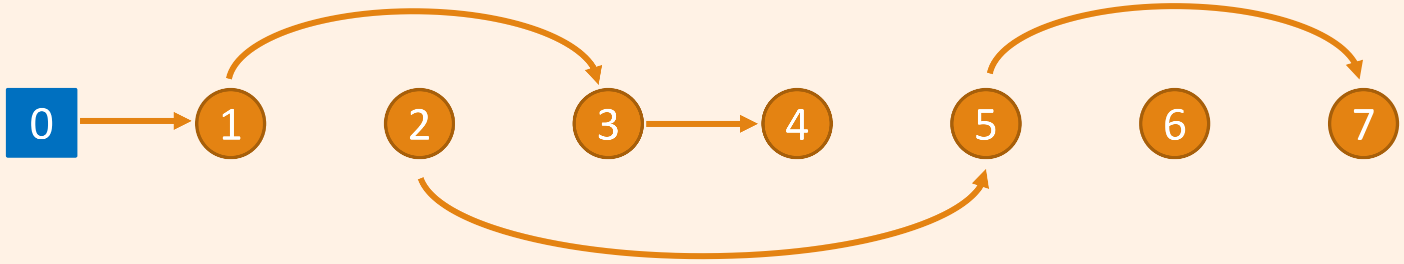
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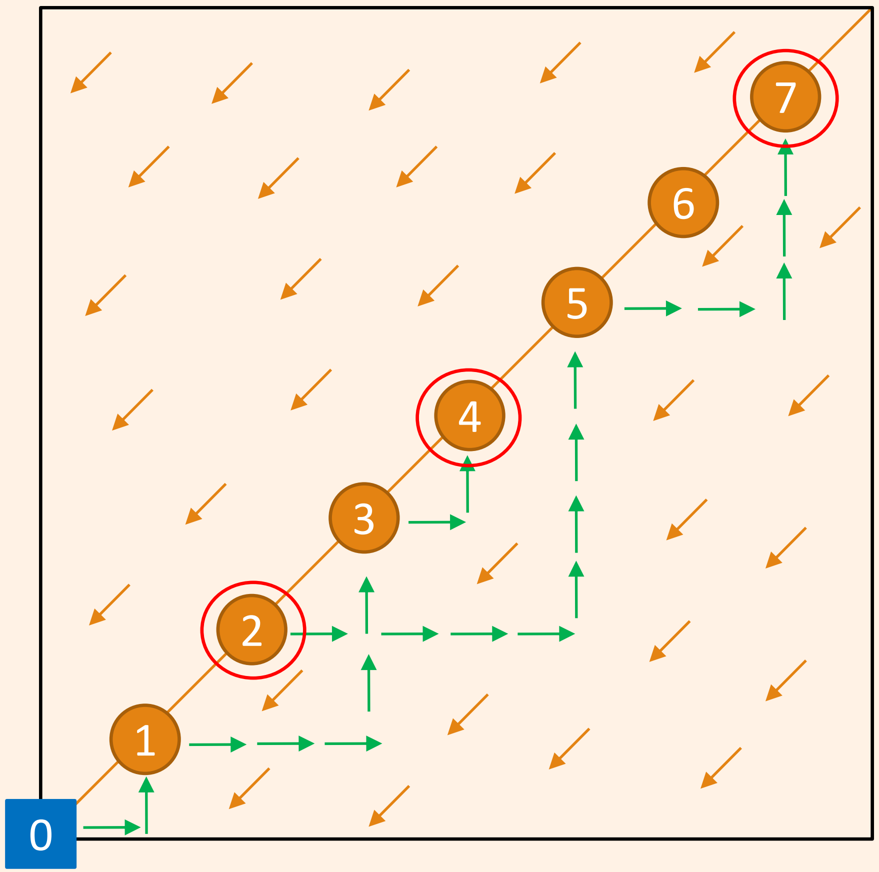
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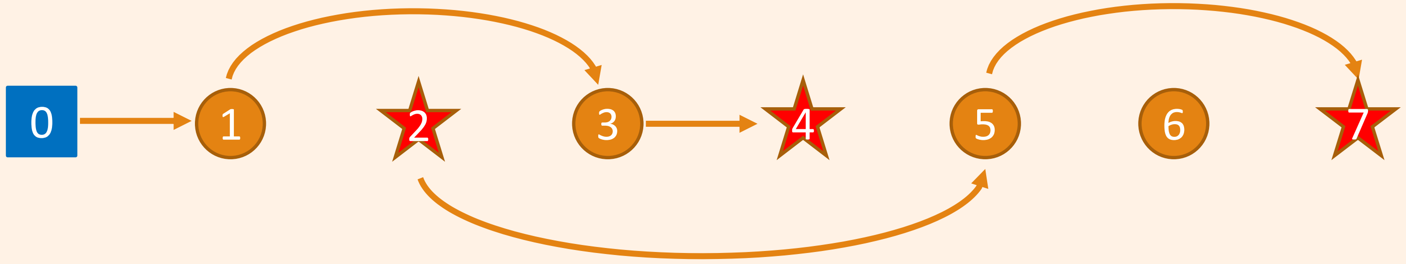
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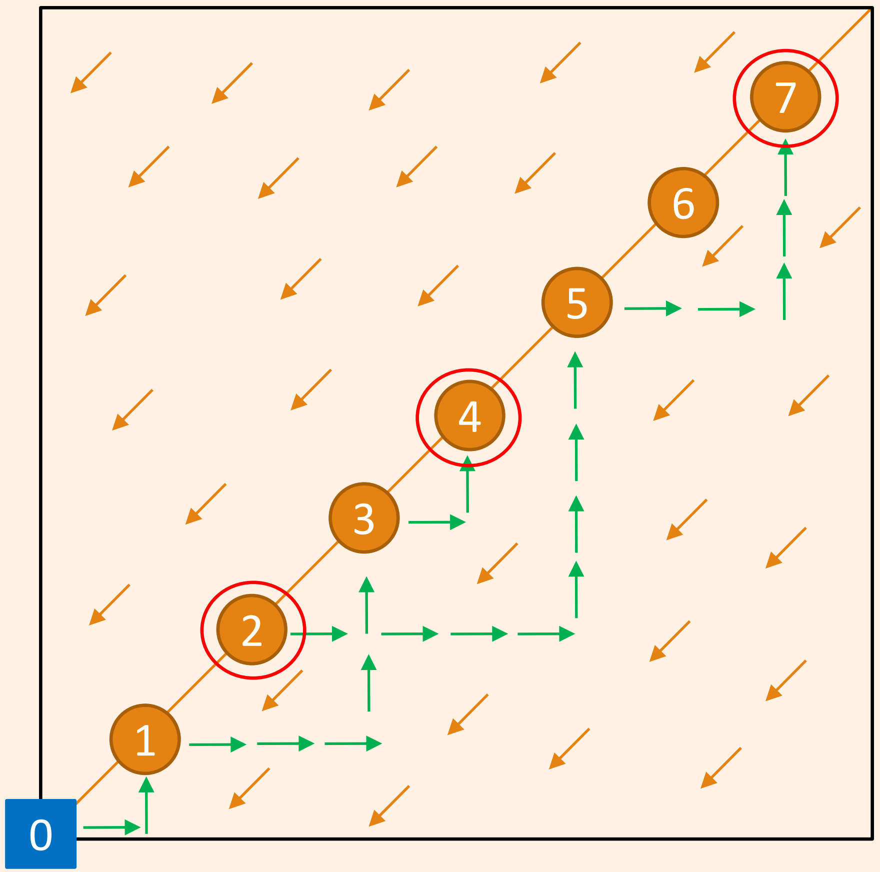
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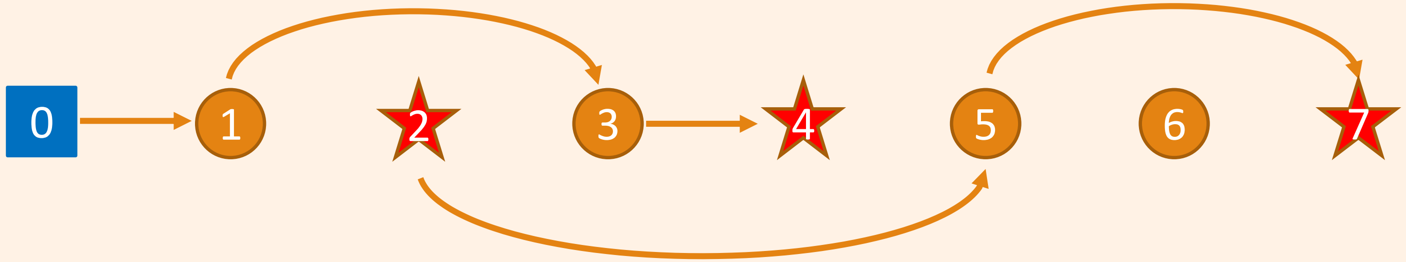

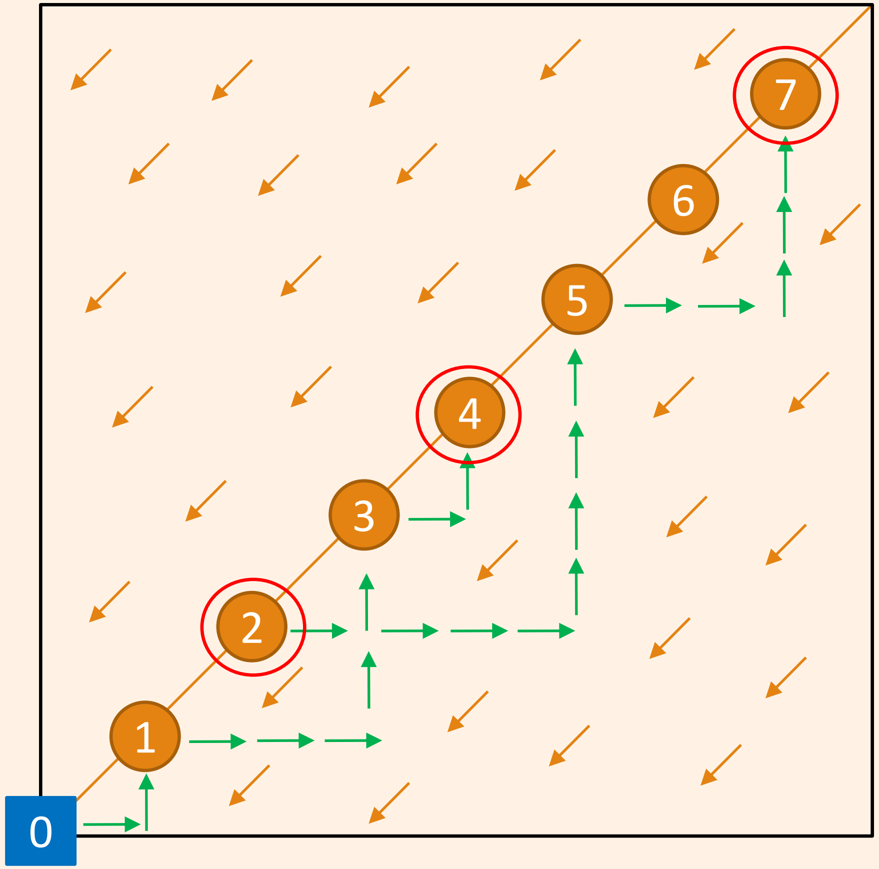

Locally computable!

[Hubáček-Yogev, 2017] for CLS

Back to standard End-of-Line

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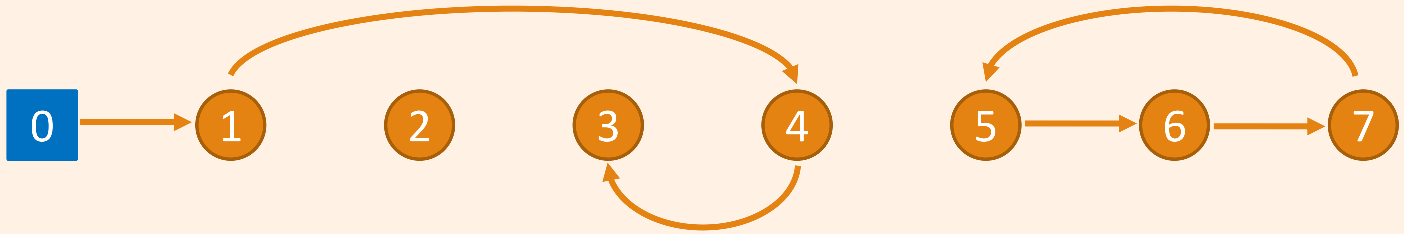
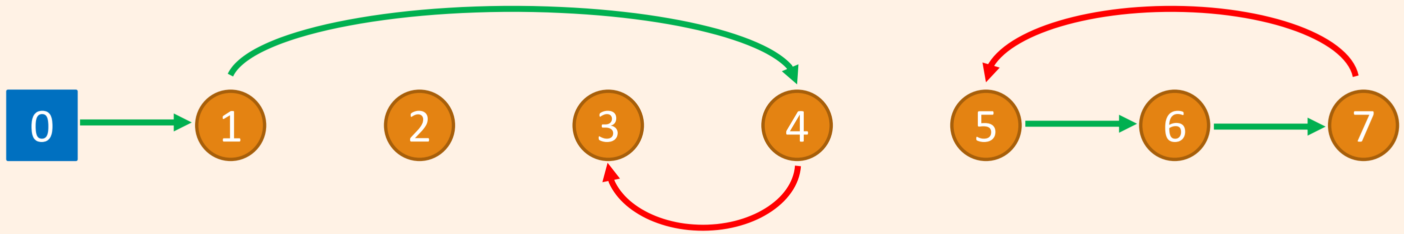
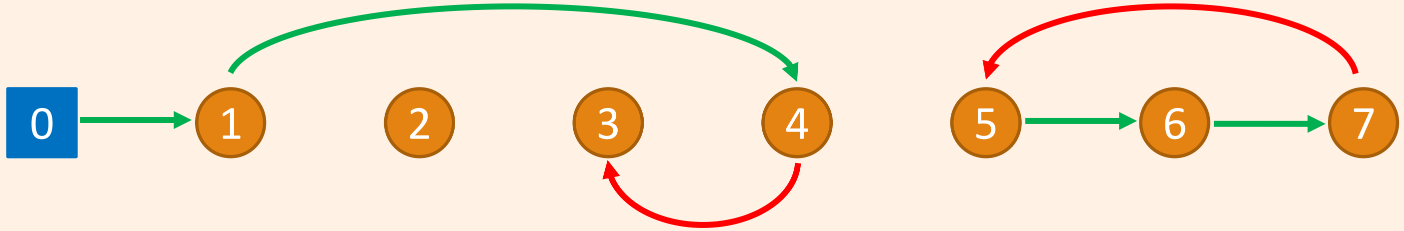


Back to standard End-of-Line



green edges: forward
red edges: backwards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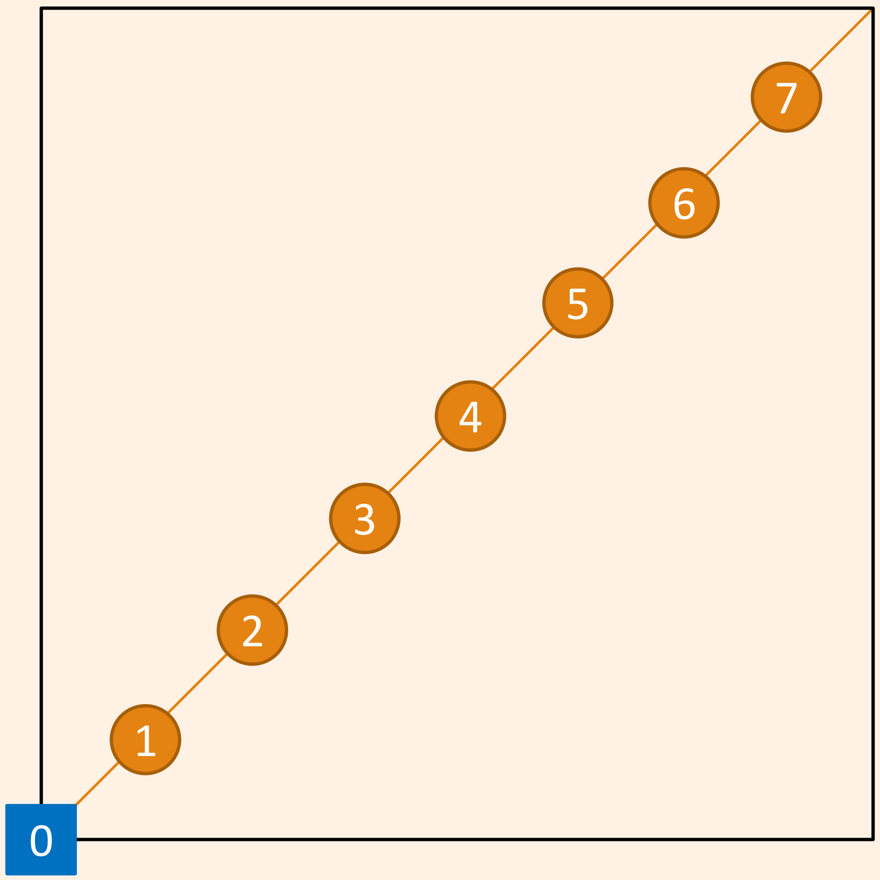
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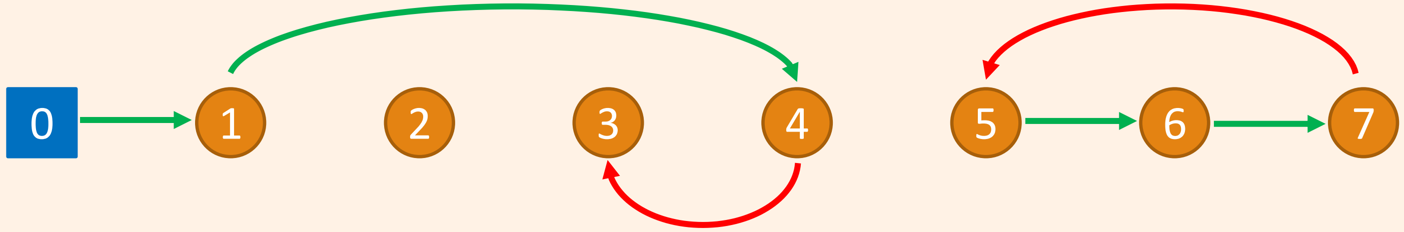
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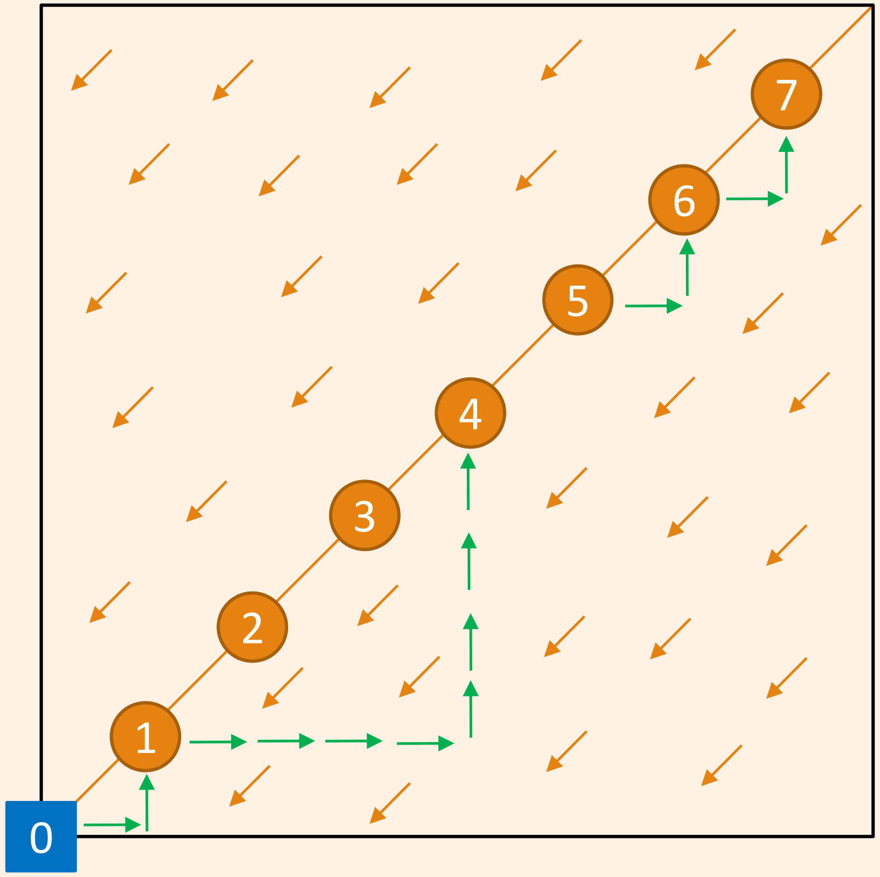
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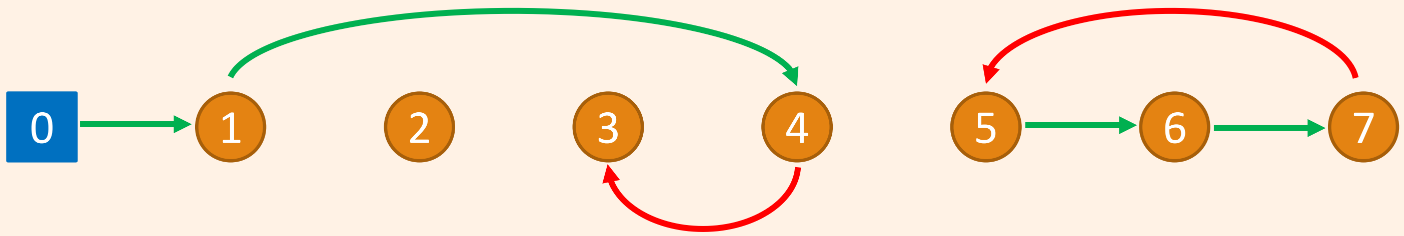
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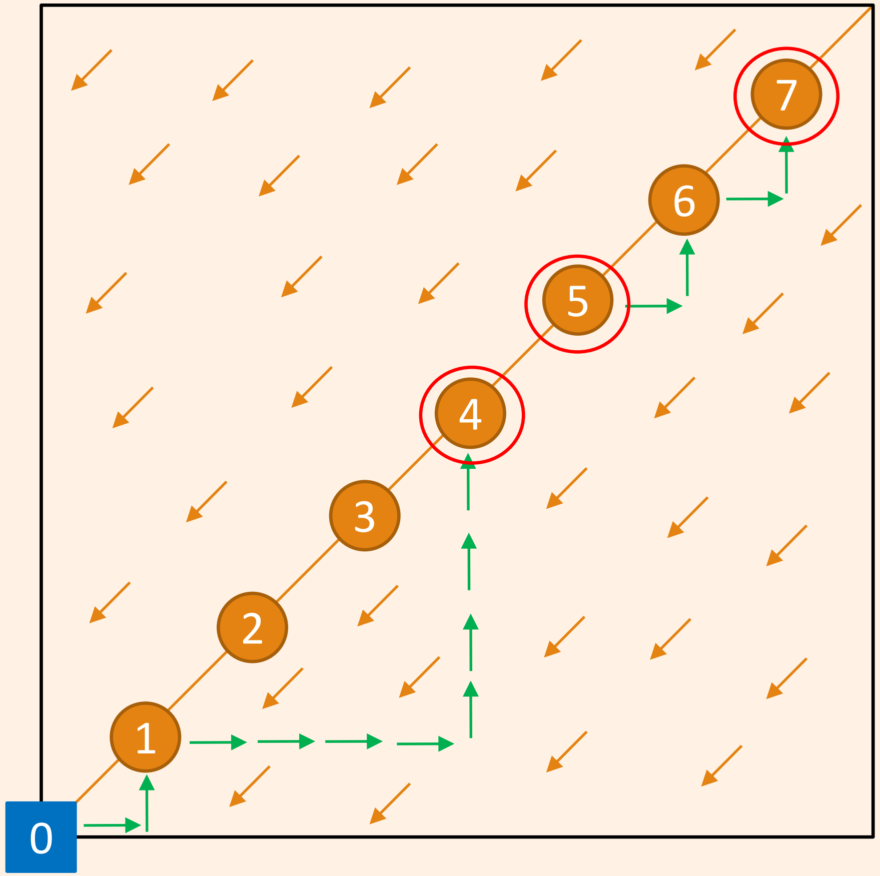
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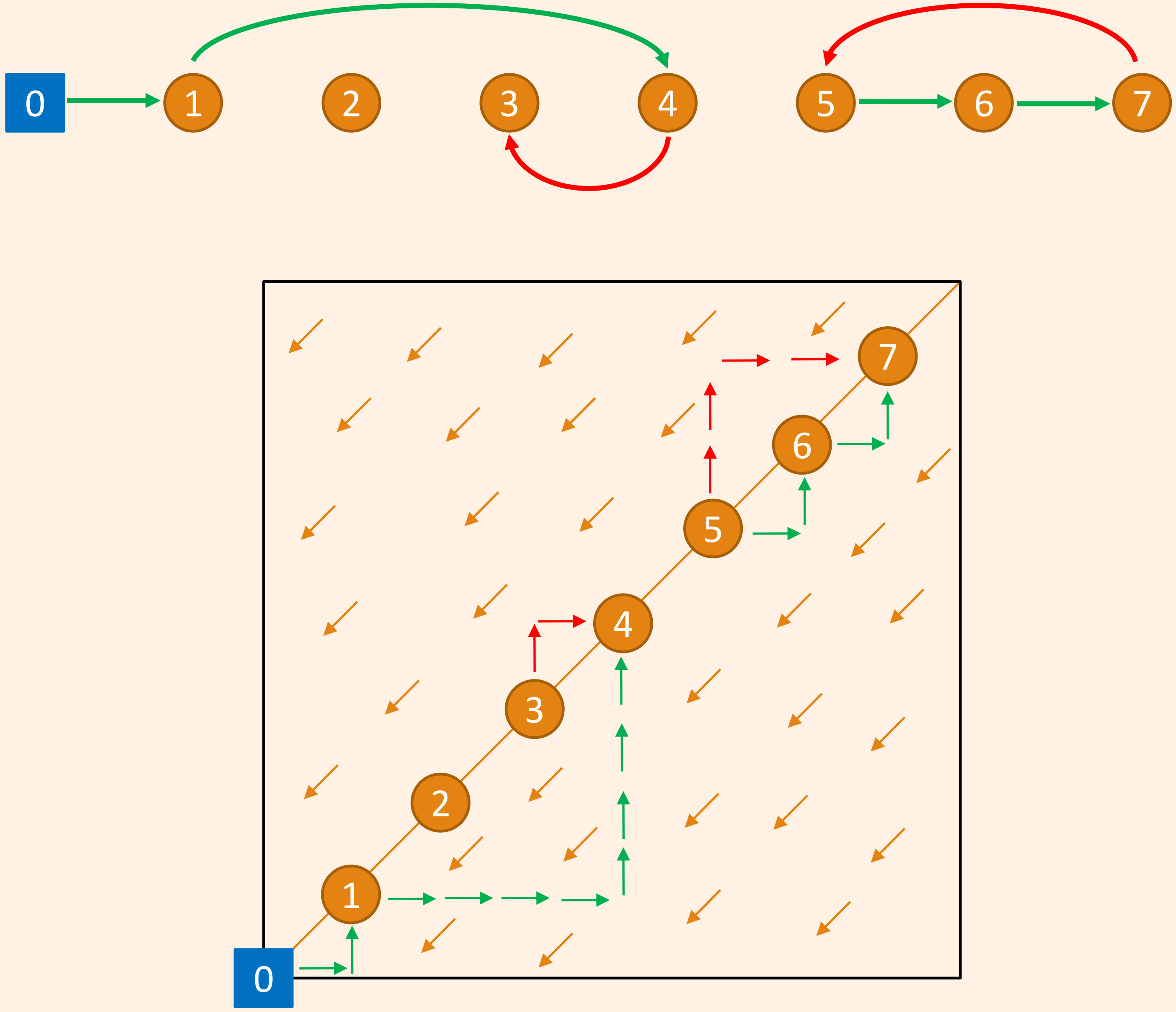
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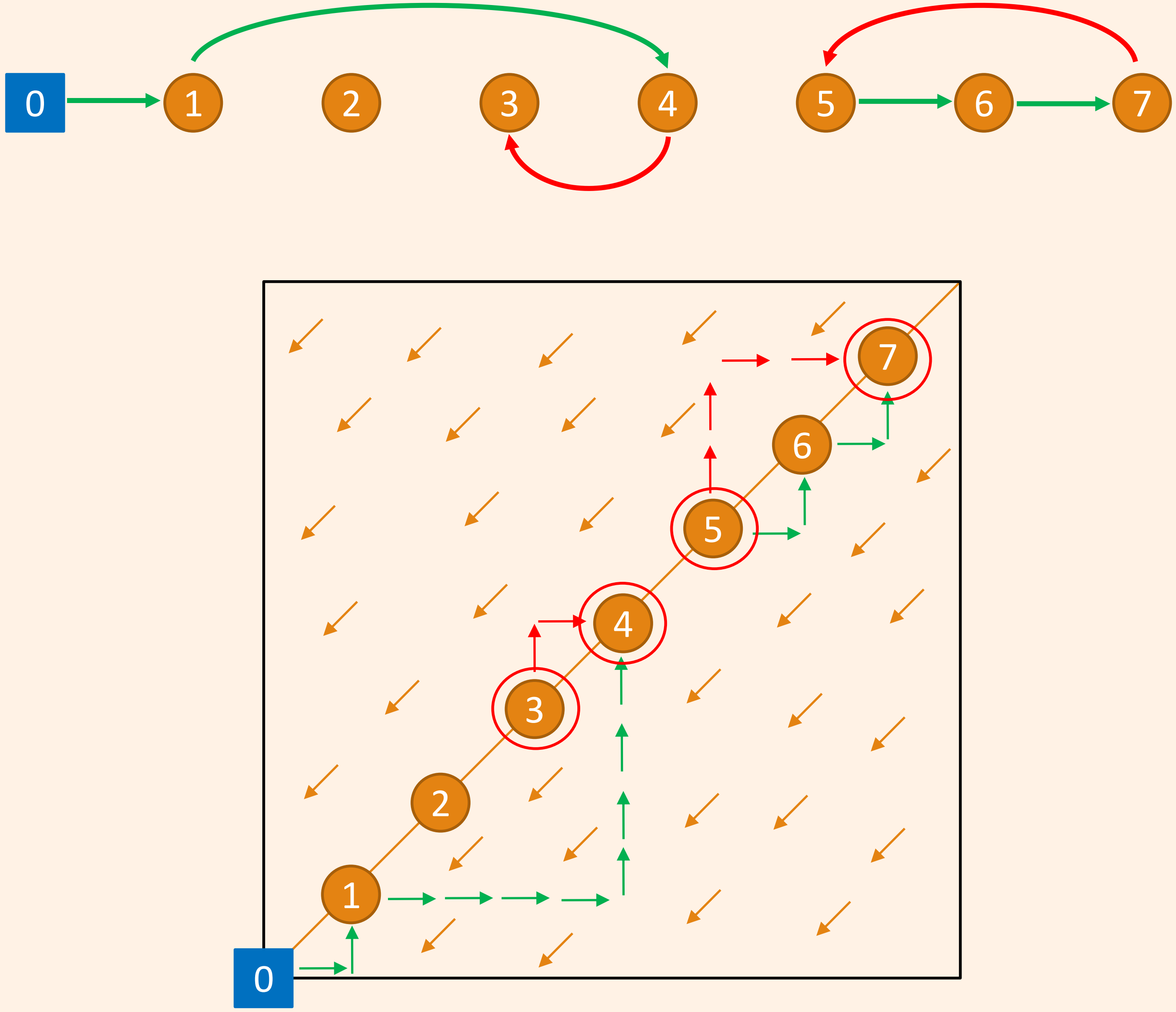
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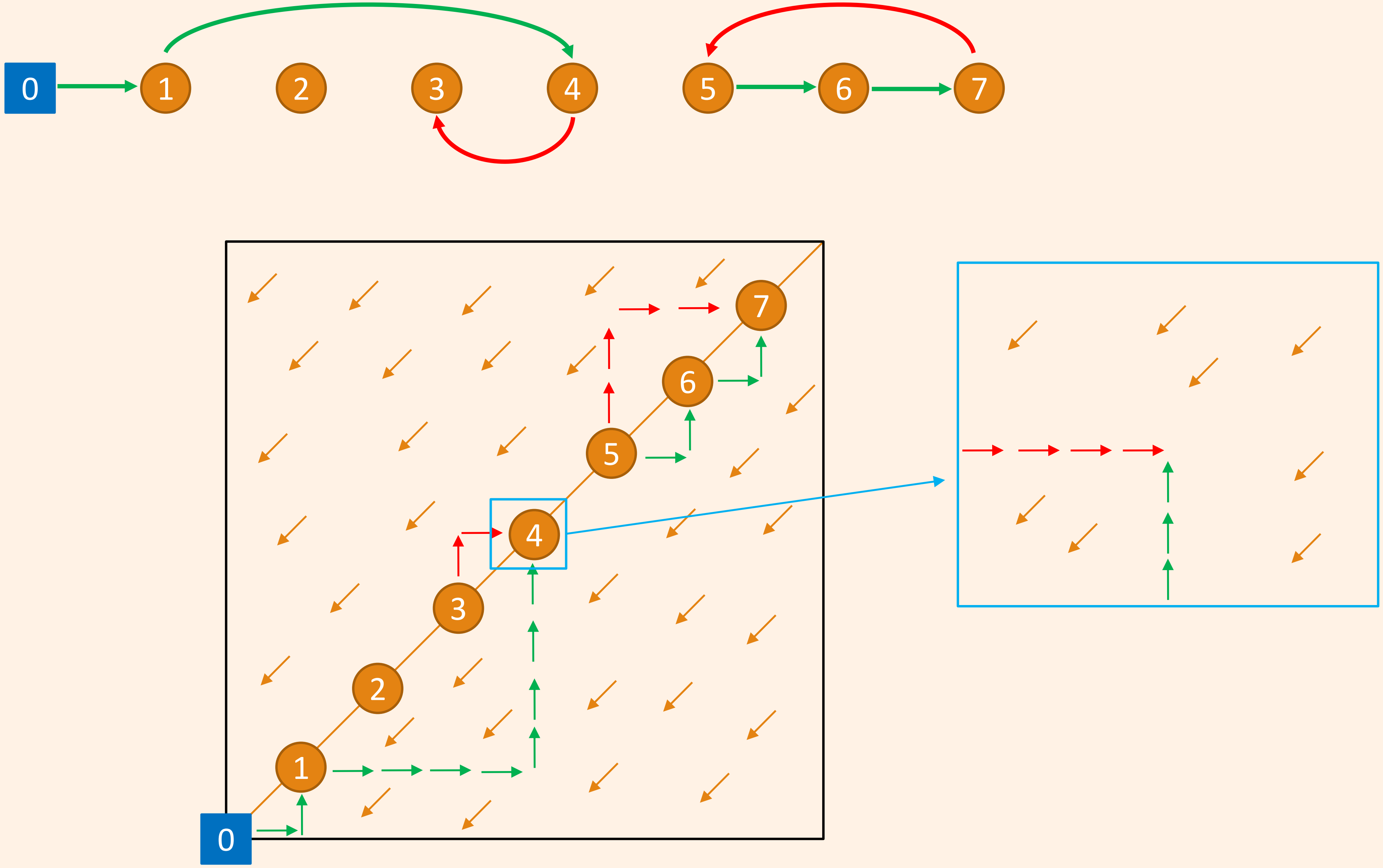

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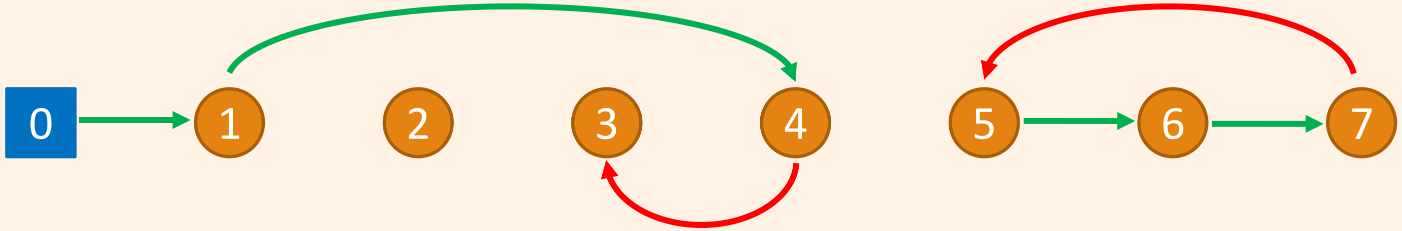
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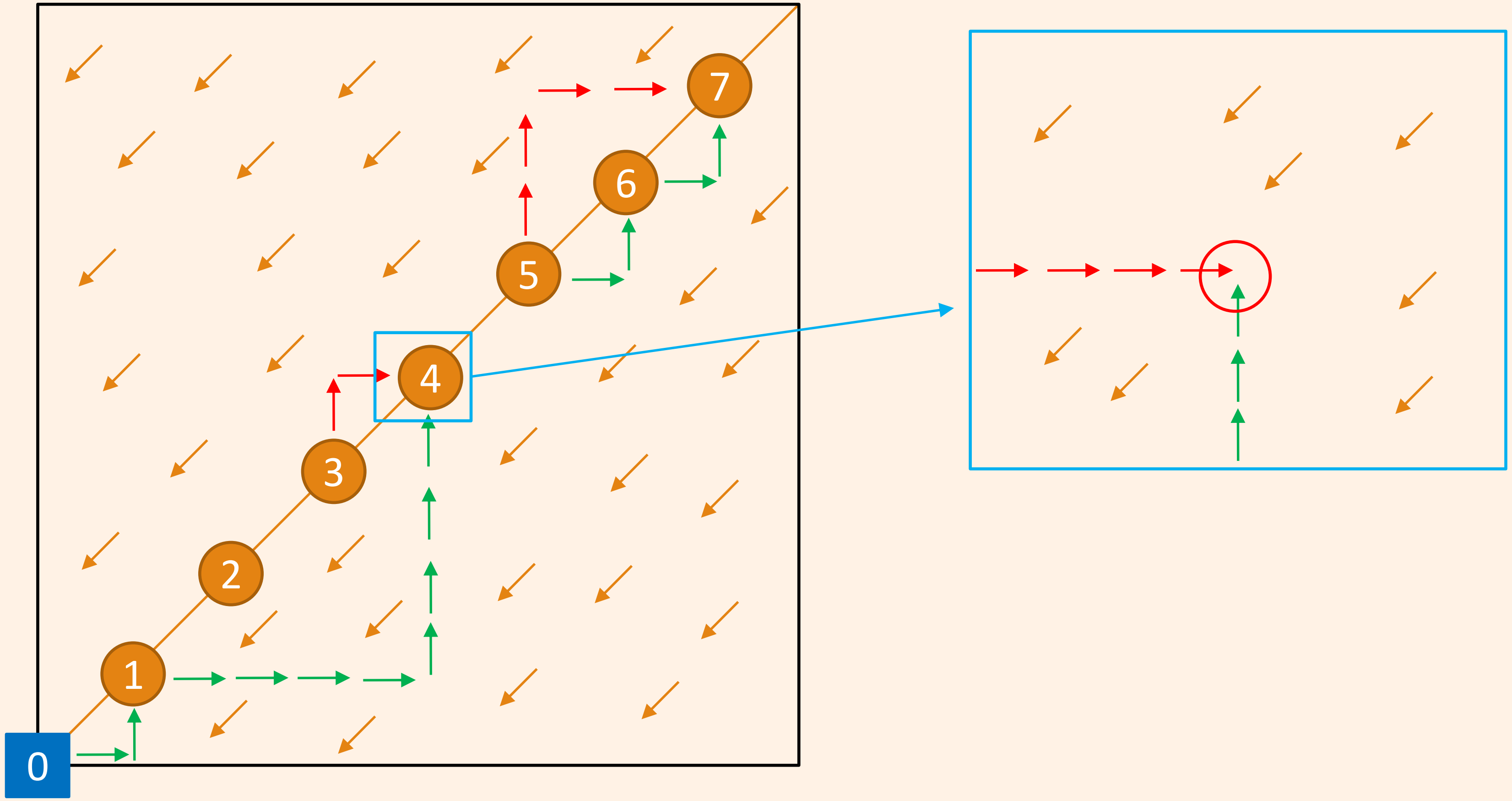
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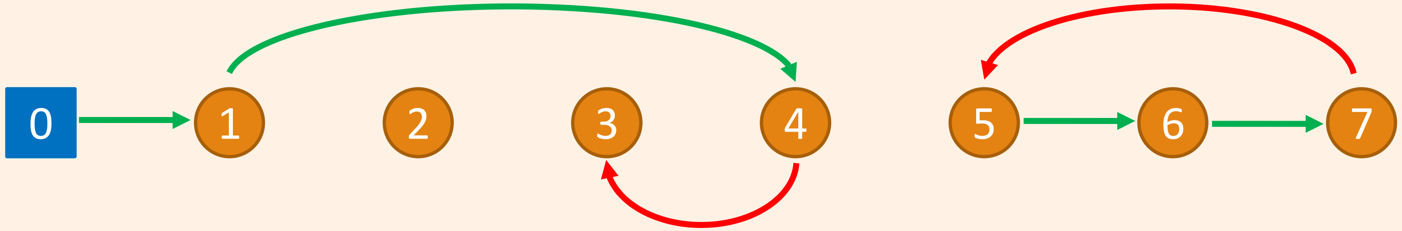
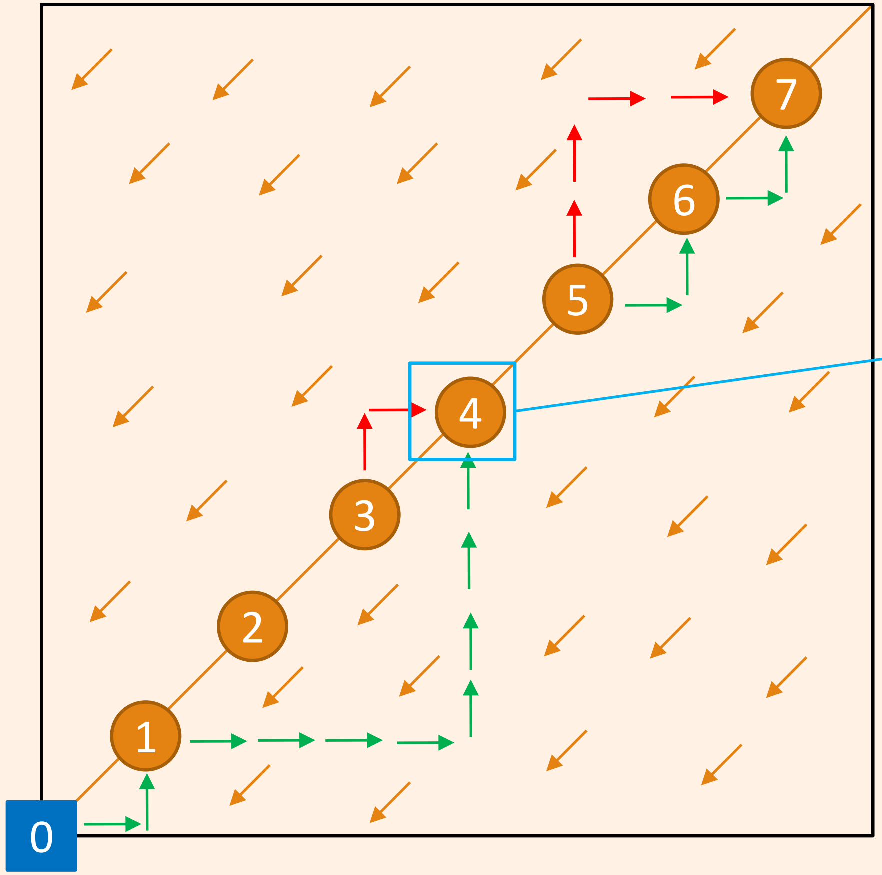
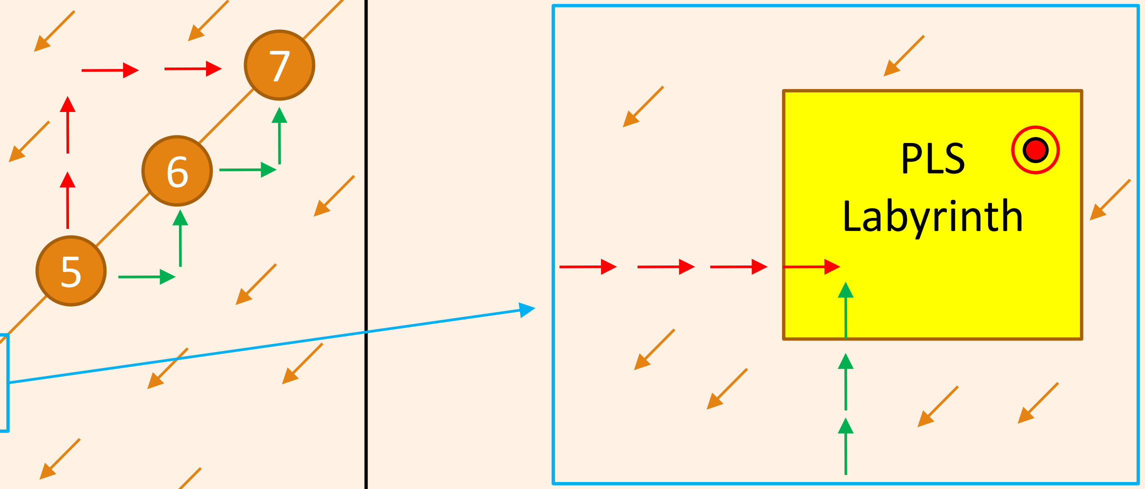
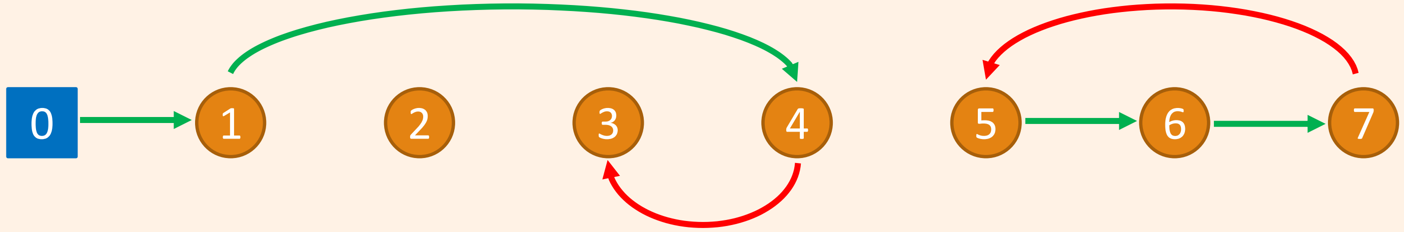






Requires solving the PLS instance!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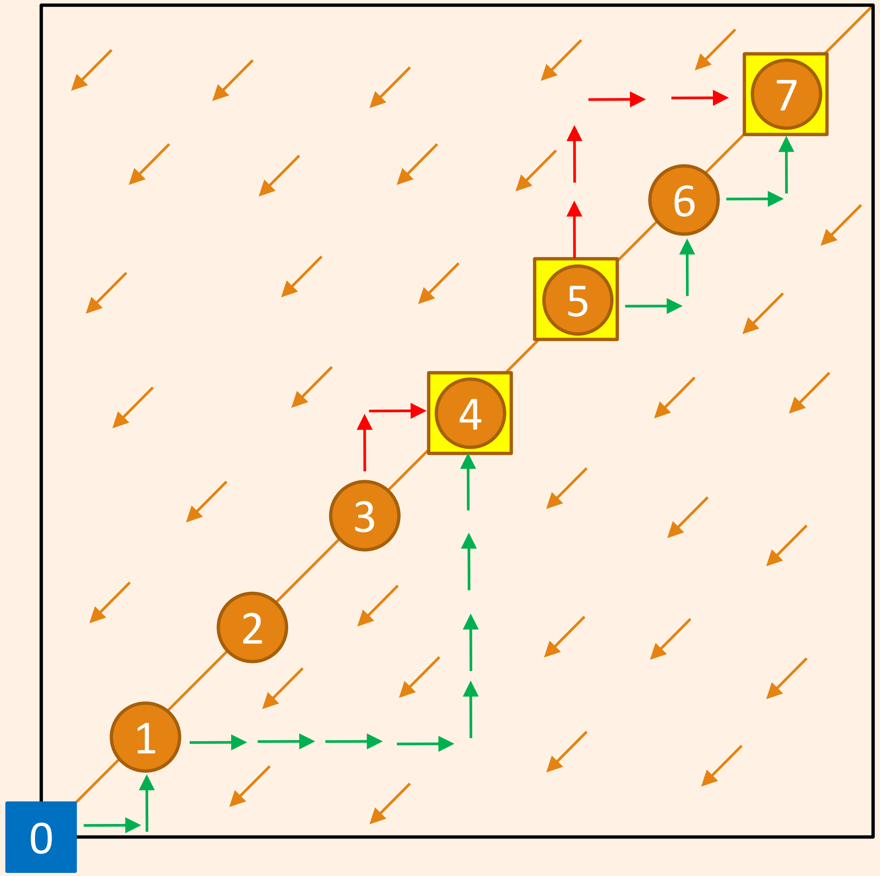
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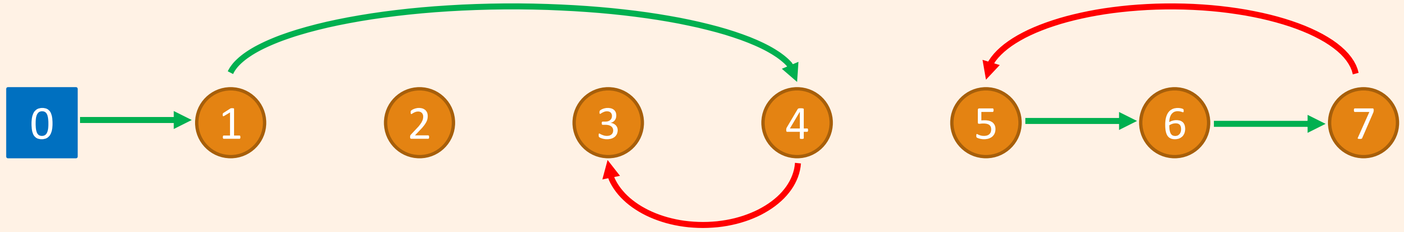
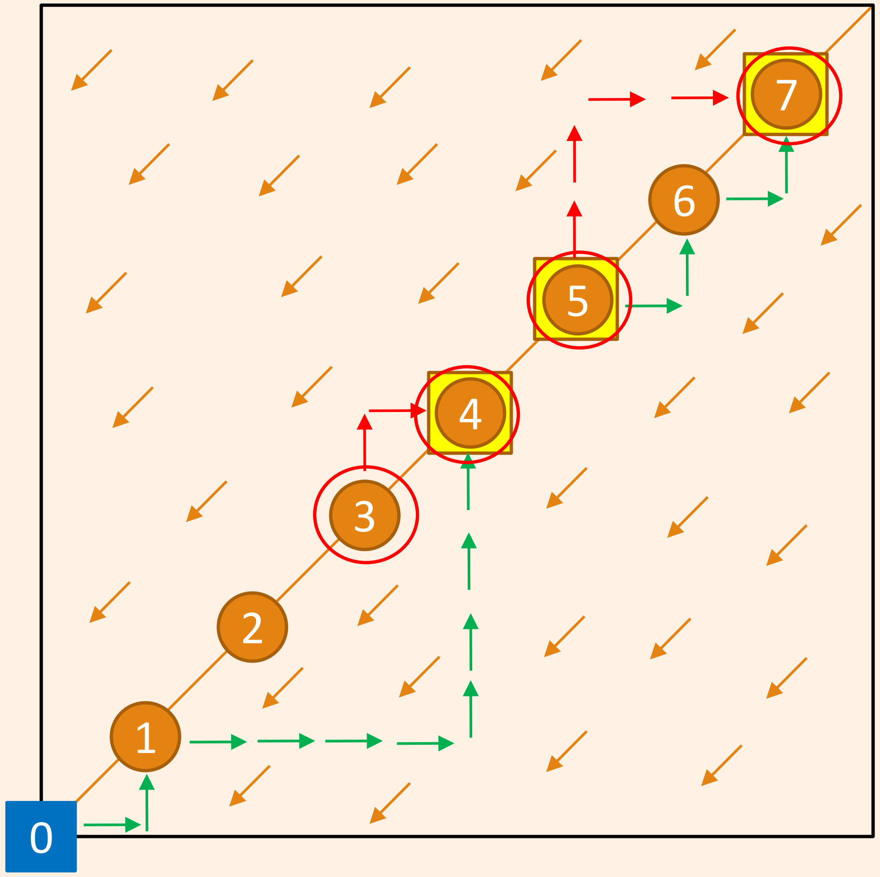
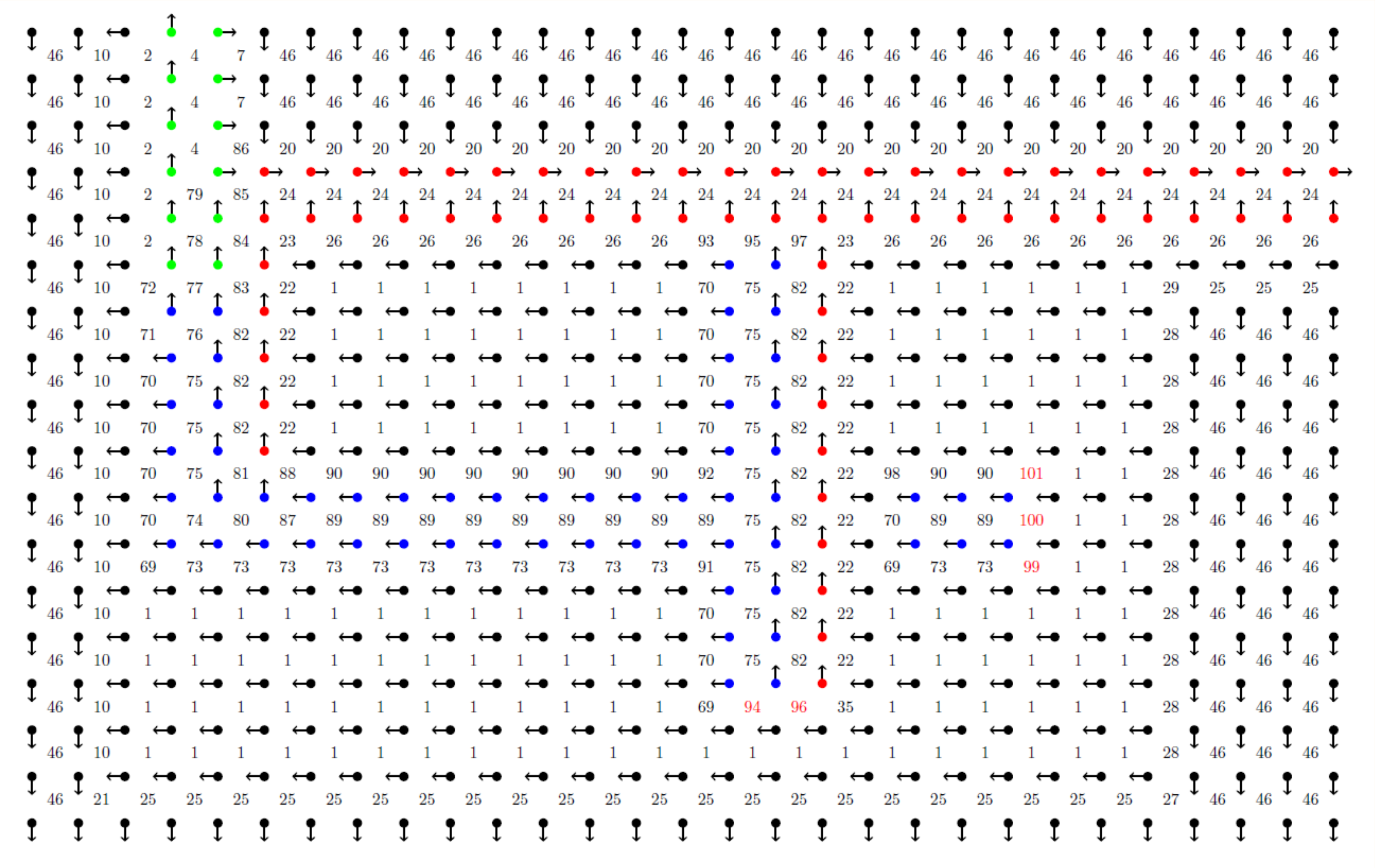




→ to find a gradient descent fixed point, we have to solve the PPAD problem or the PLS problem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[Babichenko-Rubinstein, 2020]

2D-GD-FIXED-POINT $\leq$ MIXED-CONGESTION $\leq$ POLYNOMIAL-KKT

Thank You!