THE TREE EVALUATION PROBLEM

CONTEXT & RECENT RESULTS

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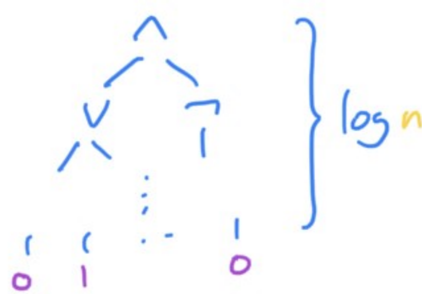
TREE EVALUATION

$$NC^1 \subseteq L \subseteq NL \subseteq NC^2 \subseteq P$$

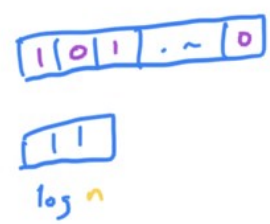
Tree Evaluation

$$NC^1 \subseteq L \subseteq NL \subseteq NC^2 \subseteq P$$

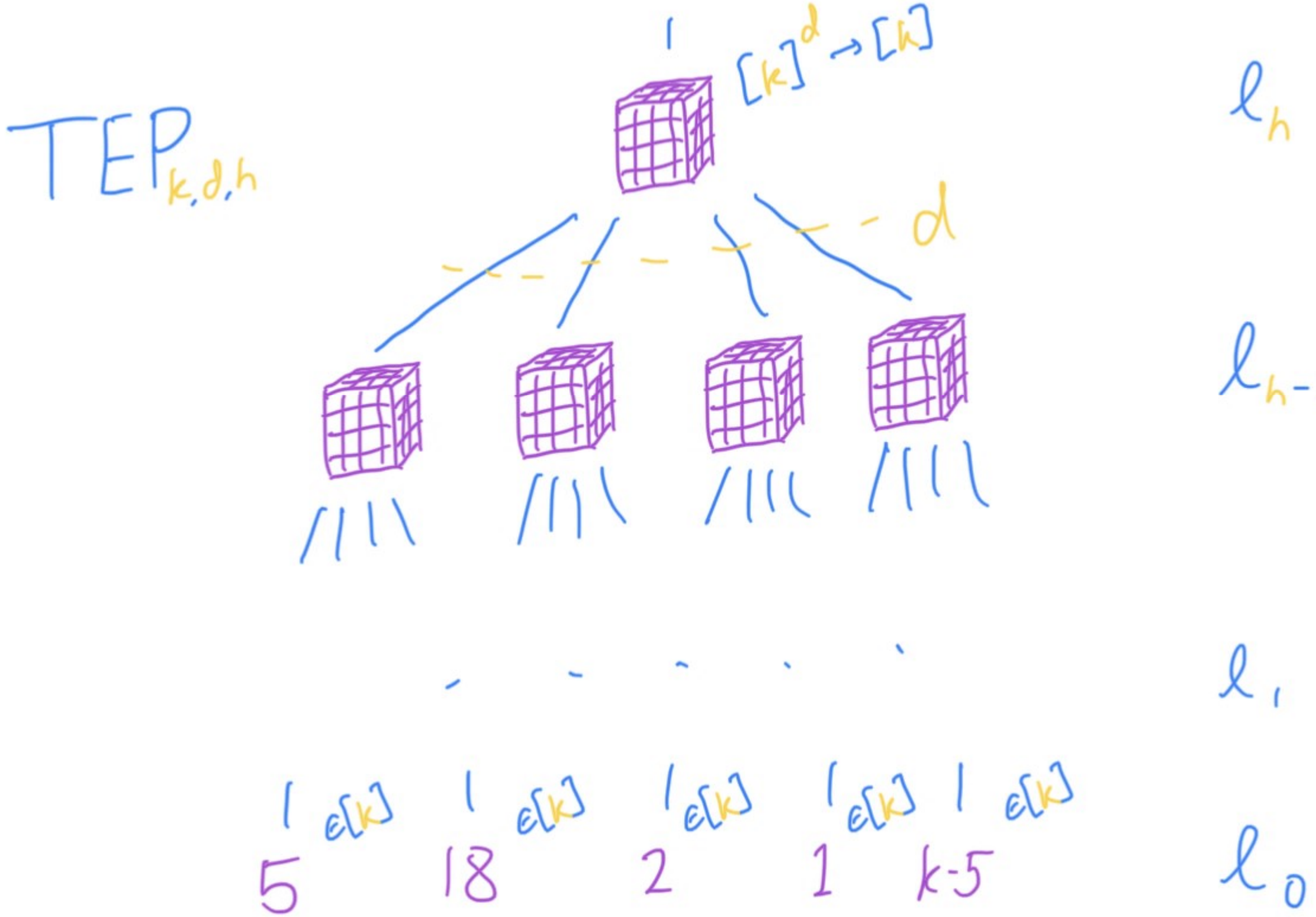
NC^1



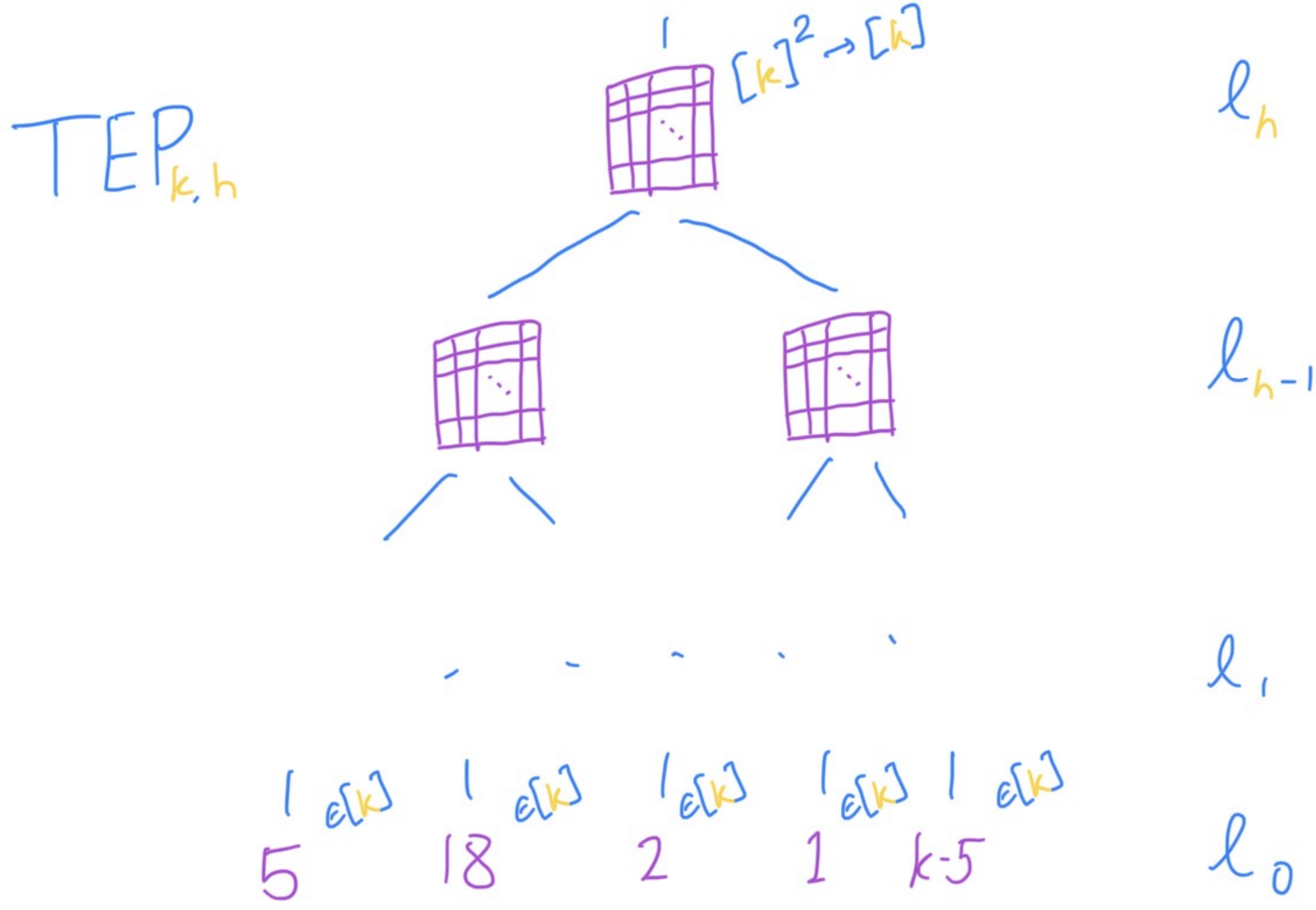
L



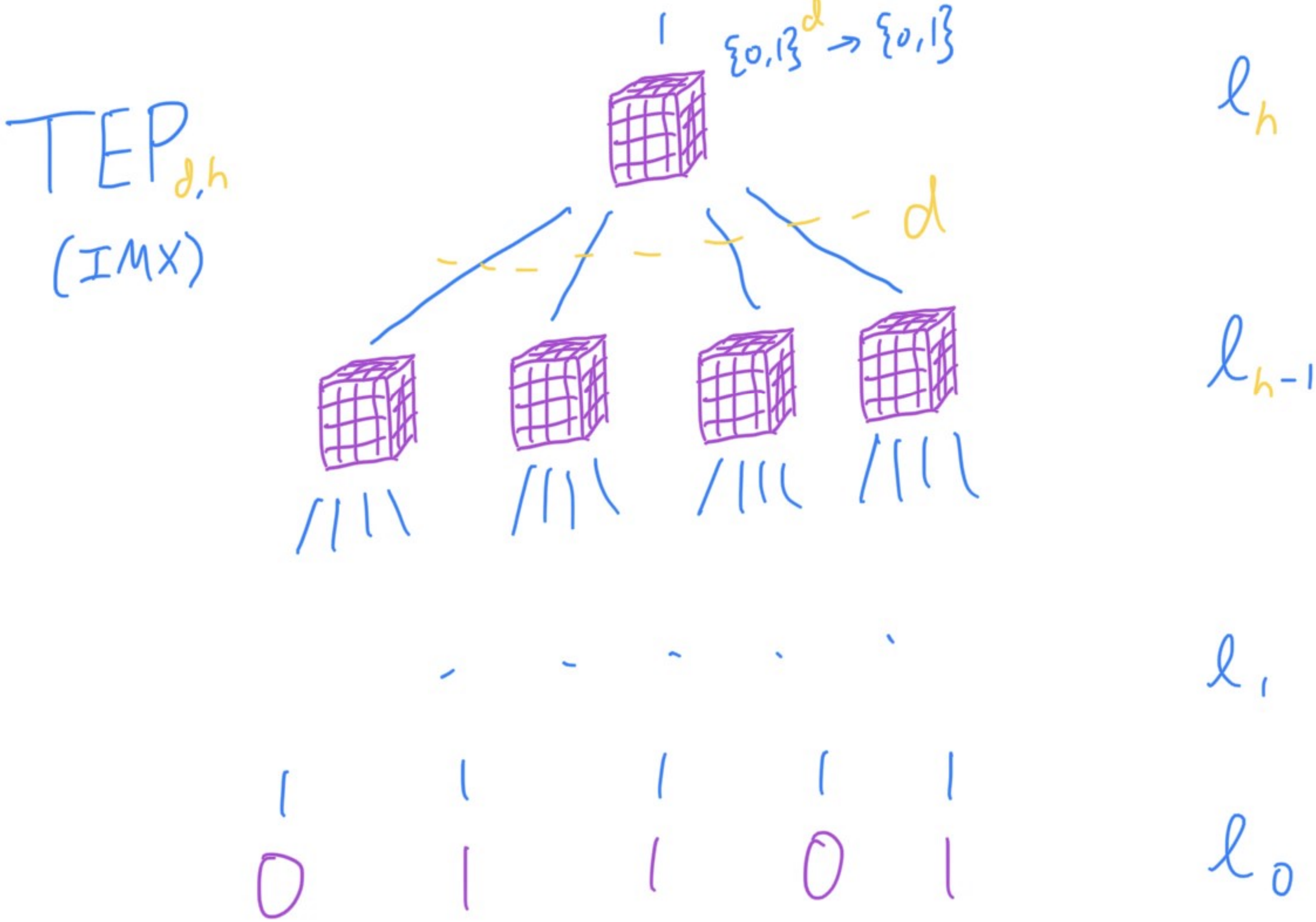
Tree Evaluation



TREE EVALUATION



Tree Evaluation



TREE EVALUATION

$$\mathrm{TEP}_{k,d,h} \in P$$

(or even NC^2)

Tree Evaluation

$$\mathrm{TEP}_{k,d,h} \in P$$

(or even NC^2)

$$NC^1 \leq \mathrm{TEP}_{2,2,\log n}$$

Tree Evaluation

$$\leq \qquad \mathrm{TEP} \qquad \in$$

$$\mathrm{NC}^1 \subseteq \mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{NC}^2 \subseteq \mathrm{P}$$

Hardness of TEP

<u>Conjecture</u> [KRW'95]: $TEP_{d,h} \notin NC^1$

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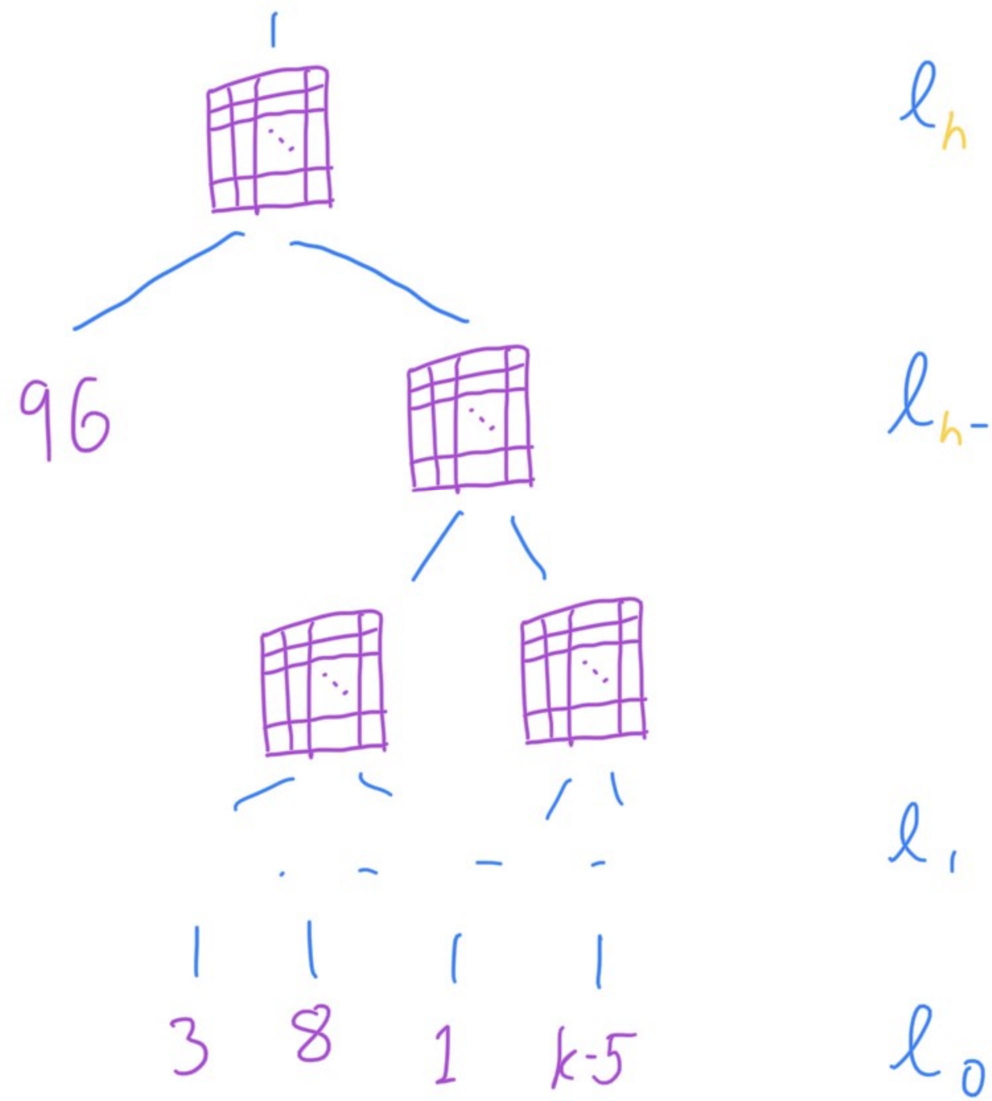
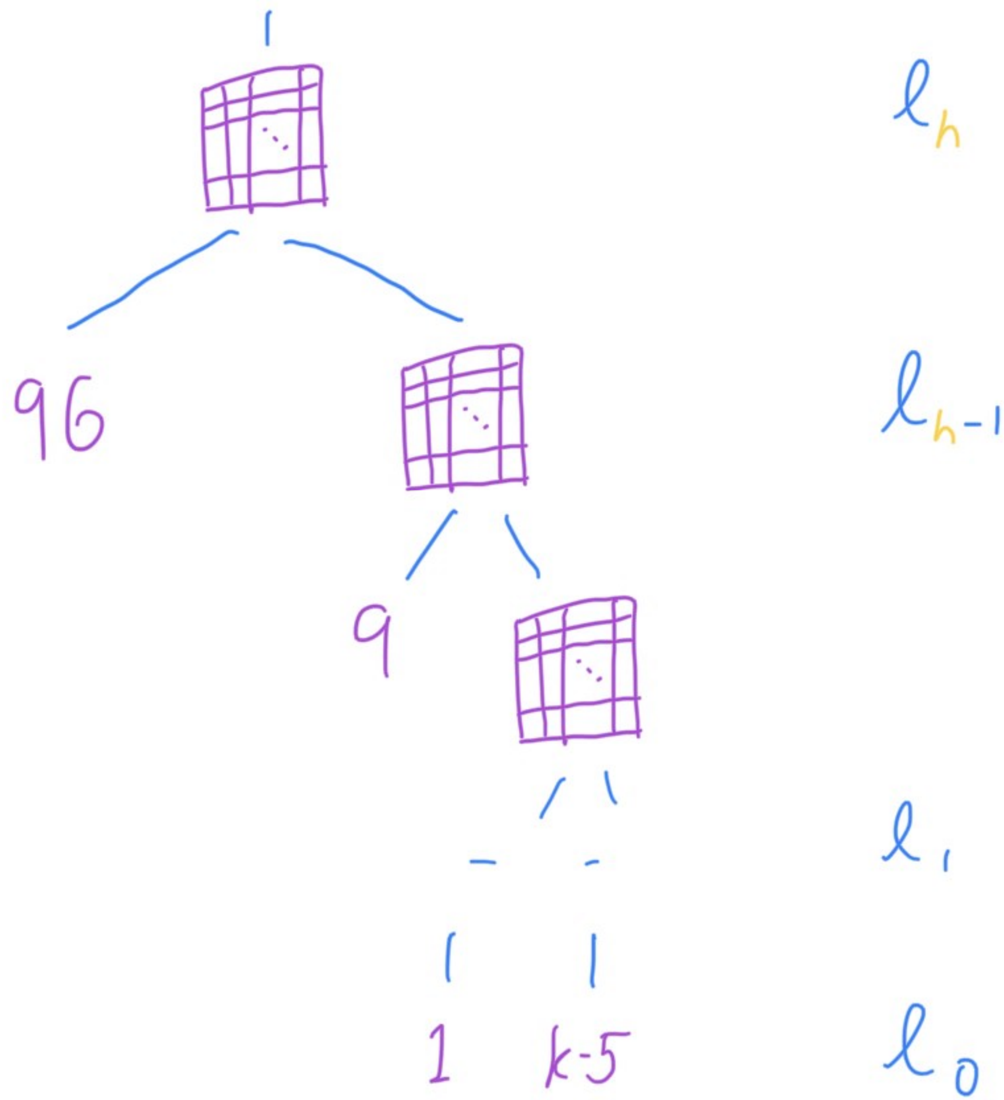
<u>Conjecture</u> [CMWBS'12]: $TEP_{k,h} \notin L$

HARDNESS OF TEP



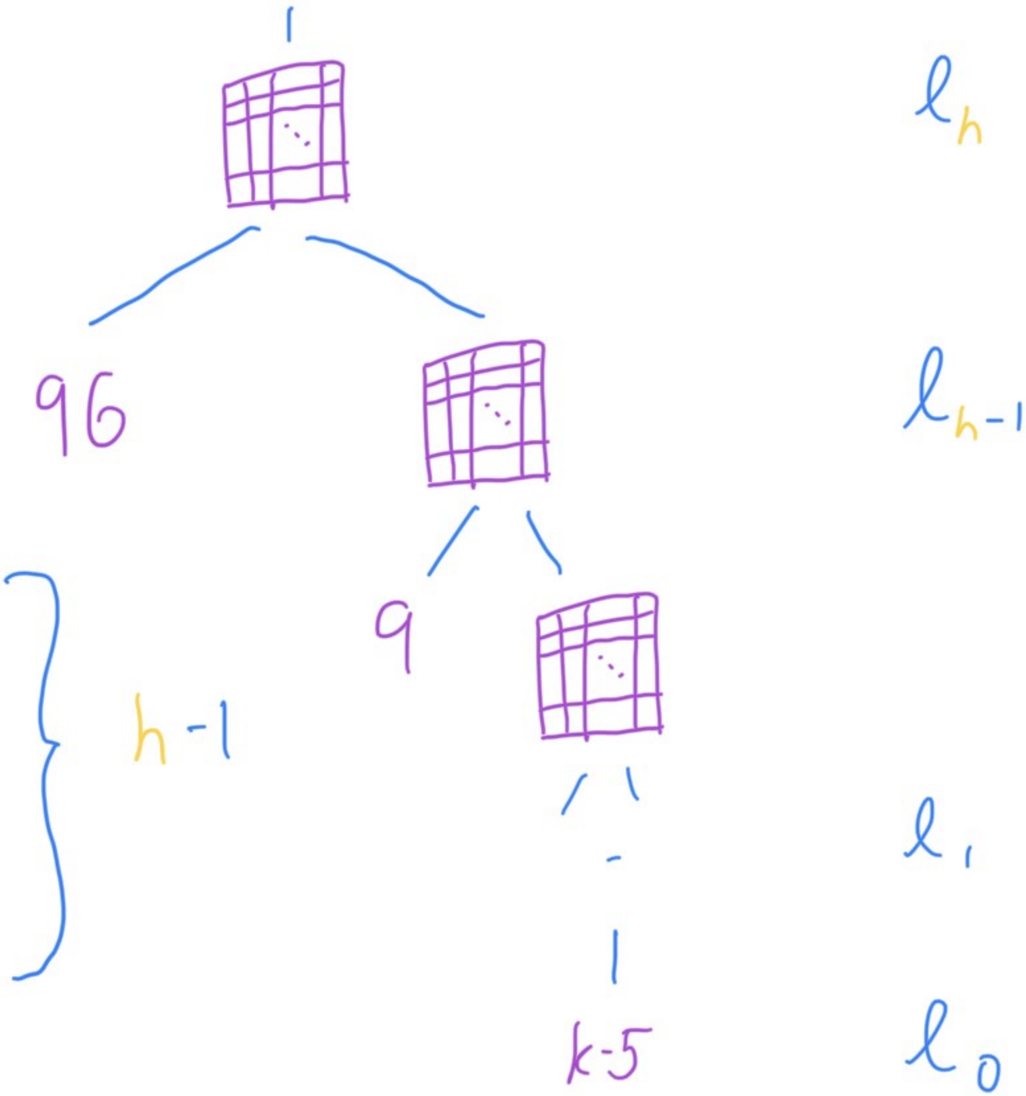
space = log k

Hardness of TEP



$$\text{space} = \log k + \log k$$

HARDNESS OF TEP



$$\text{space} = \underbrace{\log k + \log k + \dots + \log k}_{h-1}$$

HARDNESS OF TEP

<u>CONJECTURE</u> [CMWBS'12]: $TEP_{k,h}$ requires space $\Omega(h \log k) = \Omega(\log^2 n)$

$(|TEP_{k,h}| = 2^h \operatorname{poly} k)$

Hardness of TEP

Conjecture [CMWBS'12]: $\mathrm{TEP}_{k,h}$ requires space $\Omega(h \log k) = \Omega(\log^2 n)$

thrifty ✓

read-once ✓

$(|\mathrm{TEP}_{k,h}| = 2^h \operatorname{poly} k)$

HARDNESS OF TEP

CONJECTURE [CMWBS'12]: $\text{TEP}_{k,h}$ requires

non-deterministic space $\Omega(h \log k) = \Omega(\log^2 n)$

thrifty ✓

read-once ✓

$(|\text{TEP}_{k,h}| = 2^h \text{poly}\, k)$

HARDNESS OF TEP

CONJECTURE [KRW'95]: $TEP_{d,h}$ requires depth $\Omega(dh) = \Omega(\log^2 n / \log\log n)$

$(|TEP_{d,h}| = d^h \, 2^d)$

Easiness of TEP?

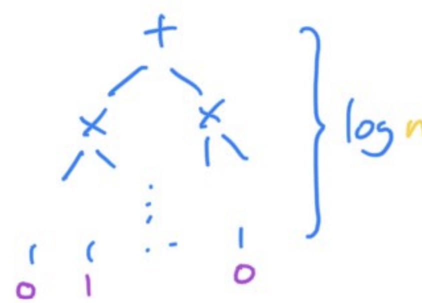
Barrington's Theorem: NC^1 can be computed with permutation BPs of $\mathrm{poly}(n)$ length and width 5.

Easiness of TEP?

Theorem [BC'89]: $\#NC^1 \subseteq L$.

Easiness of TEP?

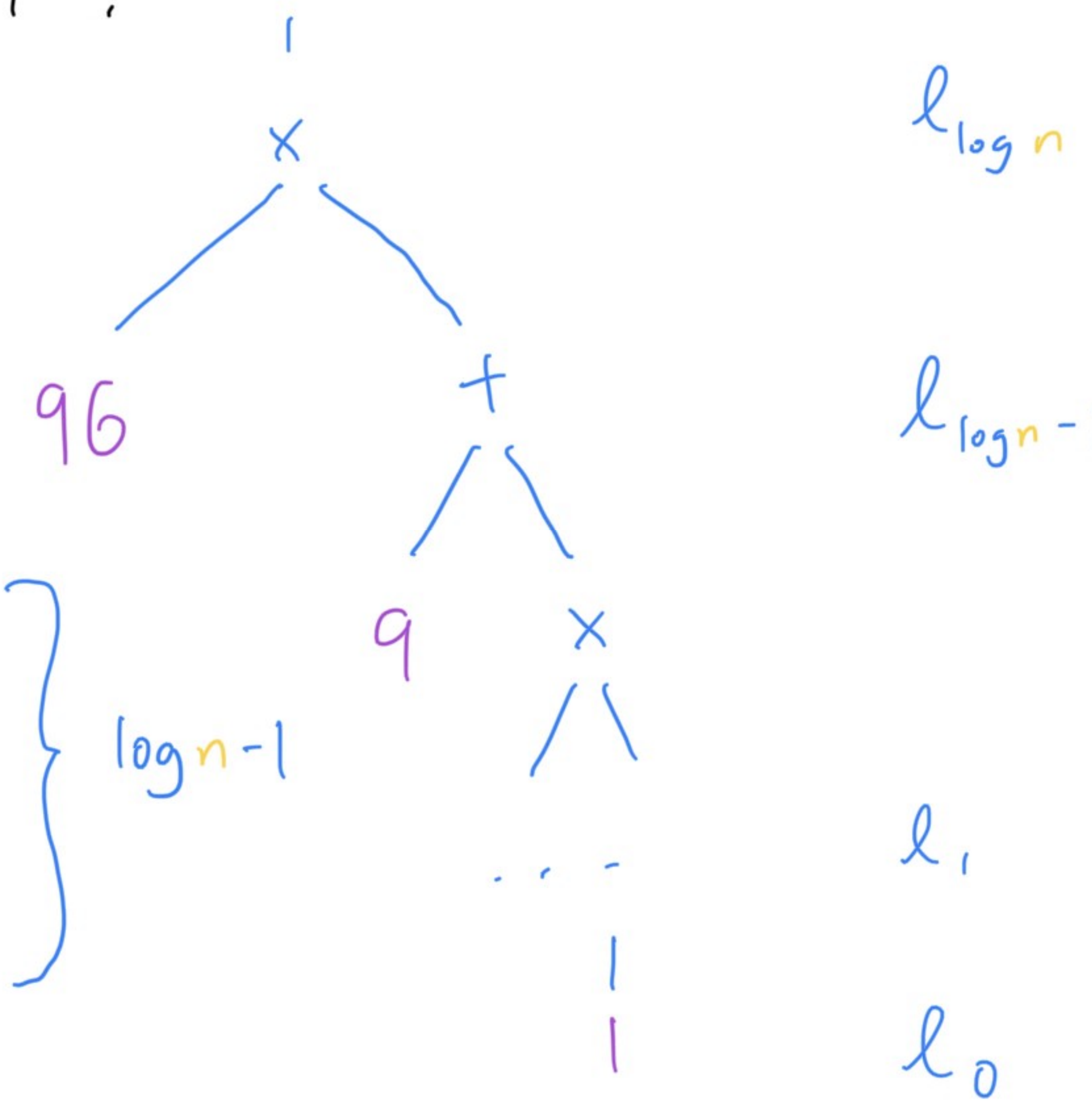
$\#NC^1$



$\#NC^1$

$NC^1 \subseteq L \subseteq NL \subseteq NC^2 \subseteq P$

Easiness of TEP?



$$\text{space} = \underbrace{\log n + \log n + \cdots + \log n}_{\log n - 1}$$

Ben-Or & Cleve

Proof [BC'89]:

two uses of space:

1) storage

2) computation

BEN-OR & CLEVE

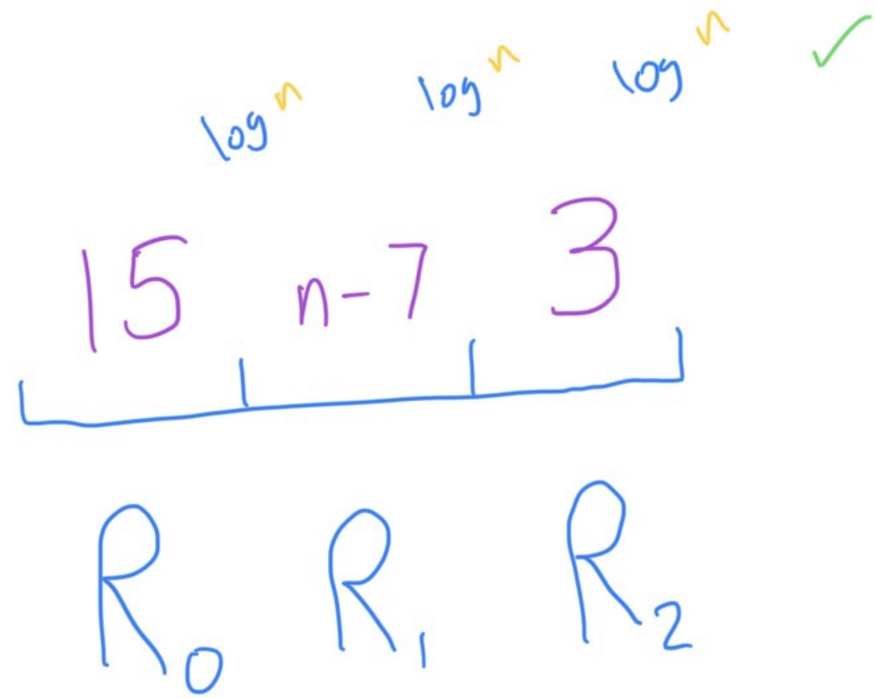
PROOF [BC'89]:

two uses of space:

1) storage
2) computation

BOTH AT ONCE?

Ben-Or & Cleve



Ben-Or & Cleve

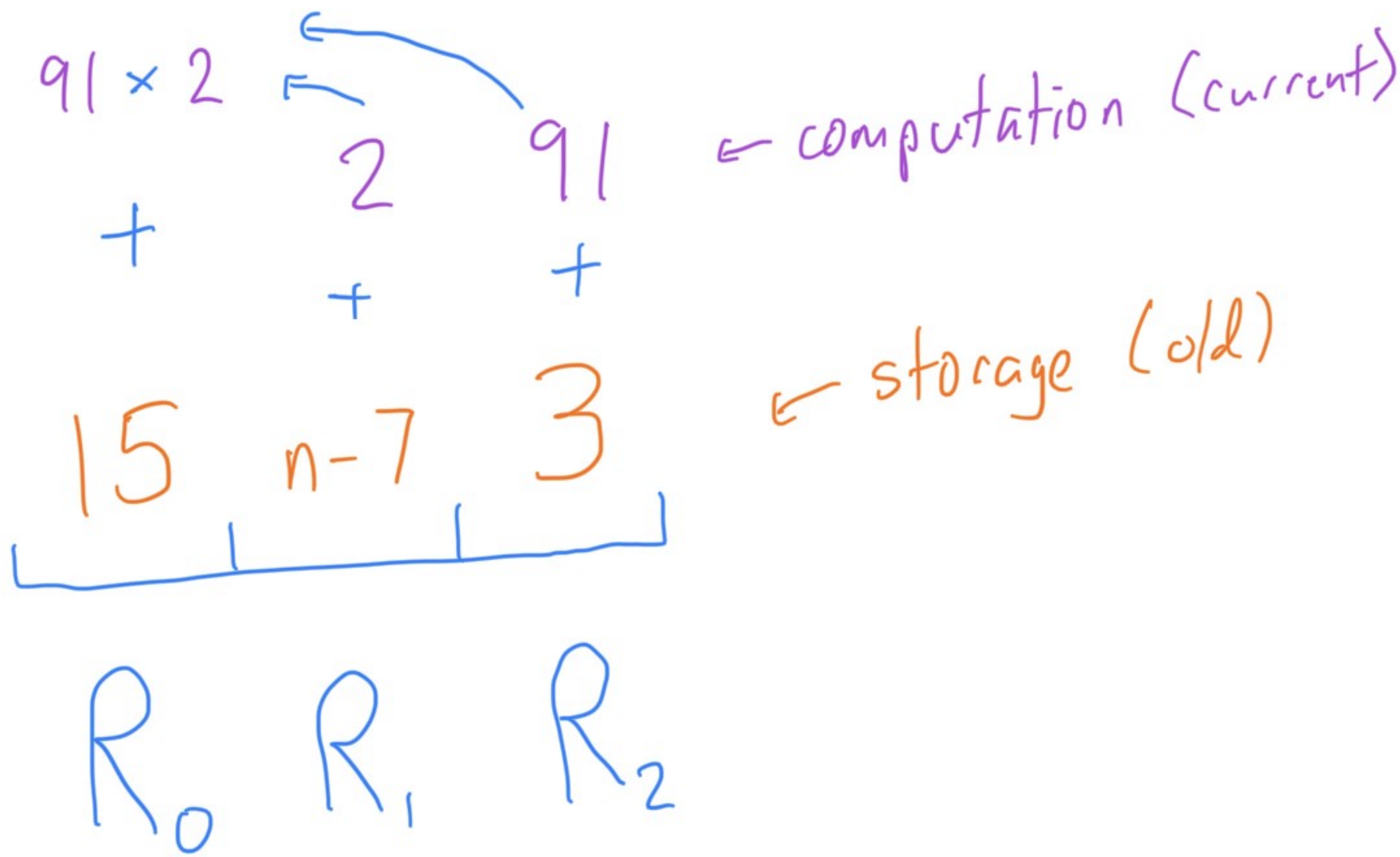
91 ← computation (current)

$+$

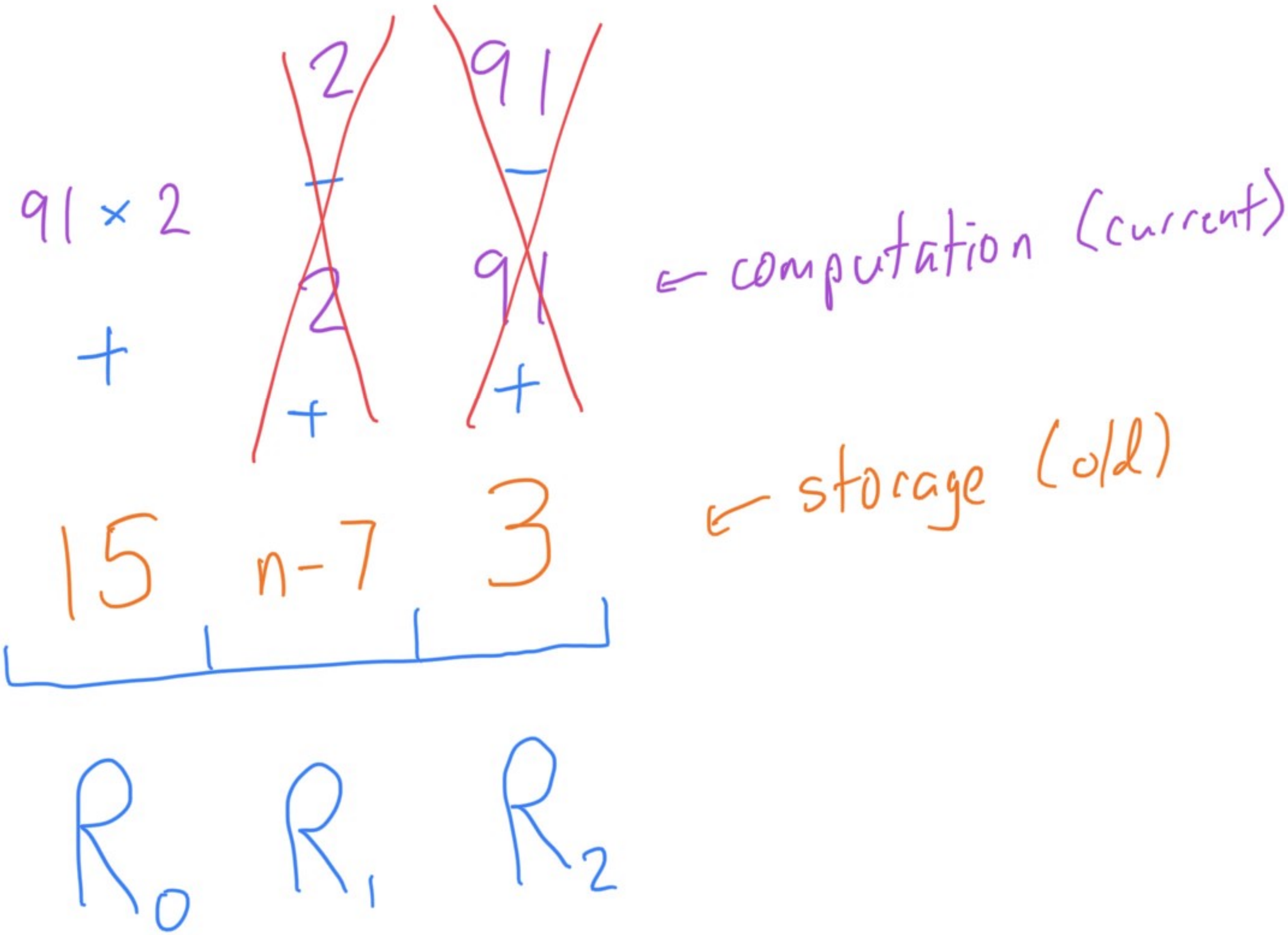
3 ← storage (old)

15	$n-7$	3
R_0	R_1	R_2

Ben-Or & Cleve



Ben-Or & Cleve



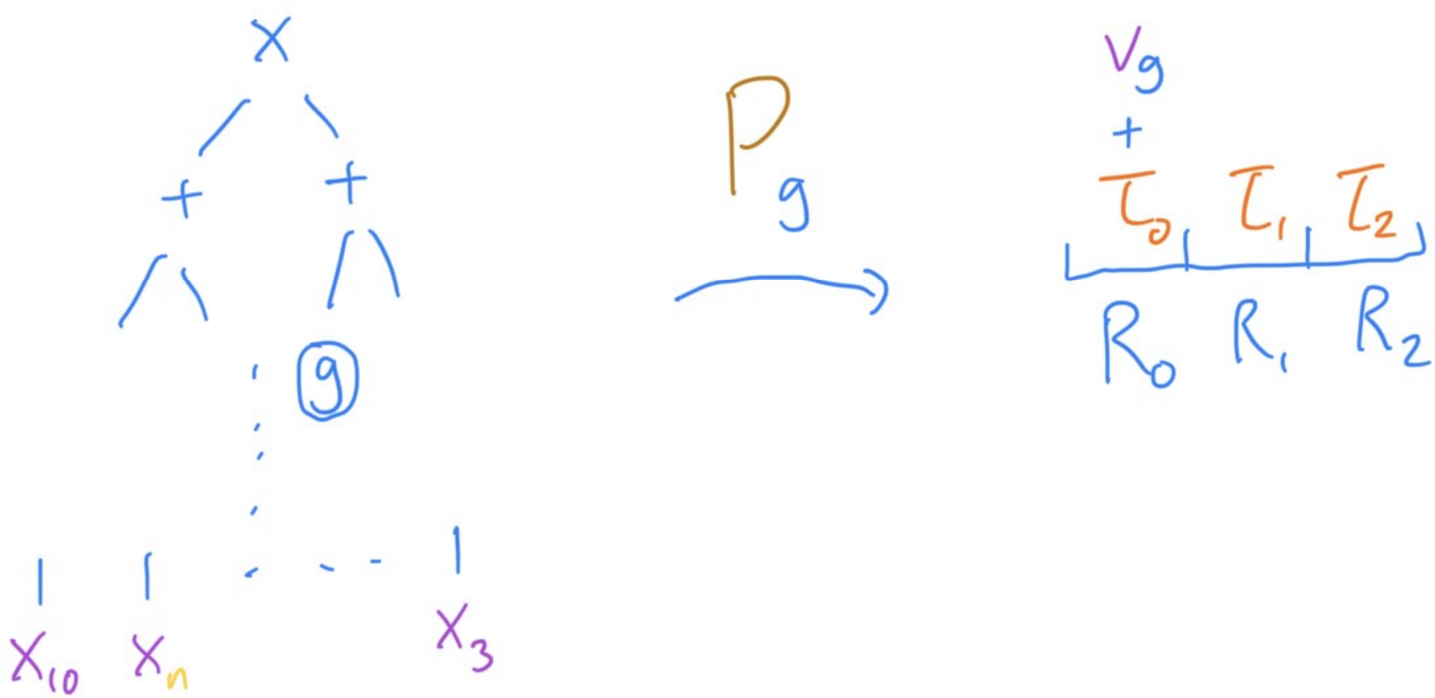
2
91
91×2
−
−
2
91
+
+
+
← computation (current)
← storage (old)
15
n-7
3
R_0
R_1
R_2

Ben-Or & Cleve

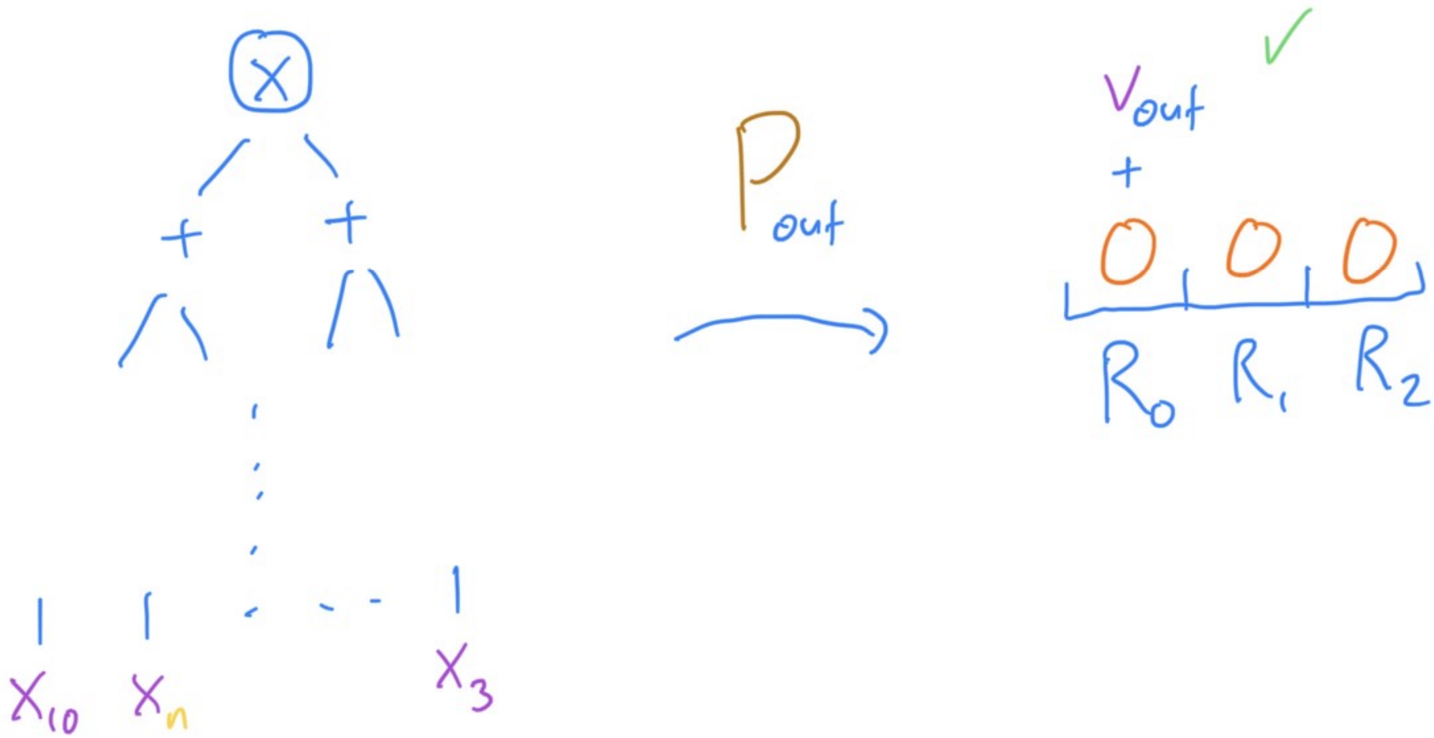
<u>Lemma</u>: $\forall g \in \mathcal{C}$, $\exists P_g$ s.t.

$$P_g : \quad \begin{aligned} R_0 &\leftarrow R_0 + v_g \\ R_1 &\leftarrow R_1 \\ R_2 &\leftarrow R_2 \end{aligned}$$

Ben-Or & Cleve

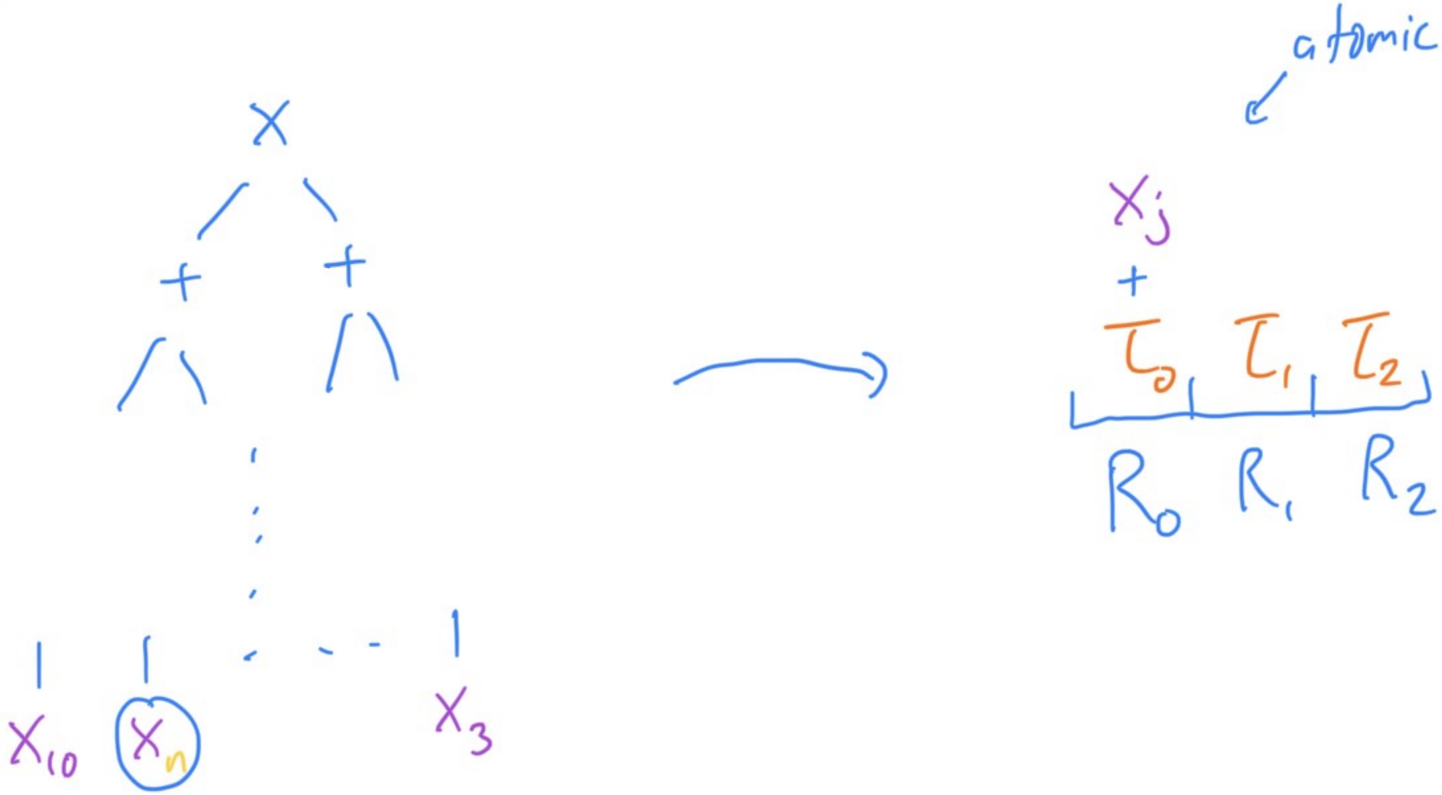


Ben-Or & Cleve



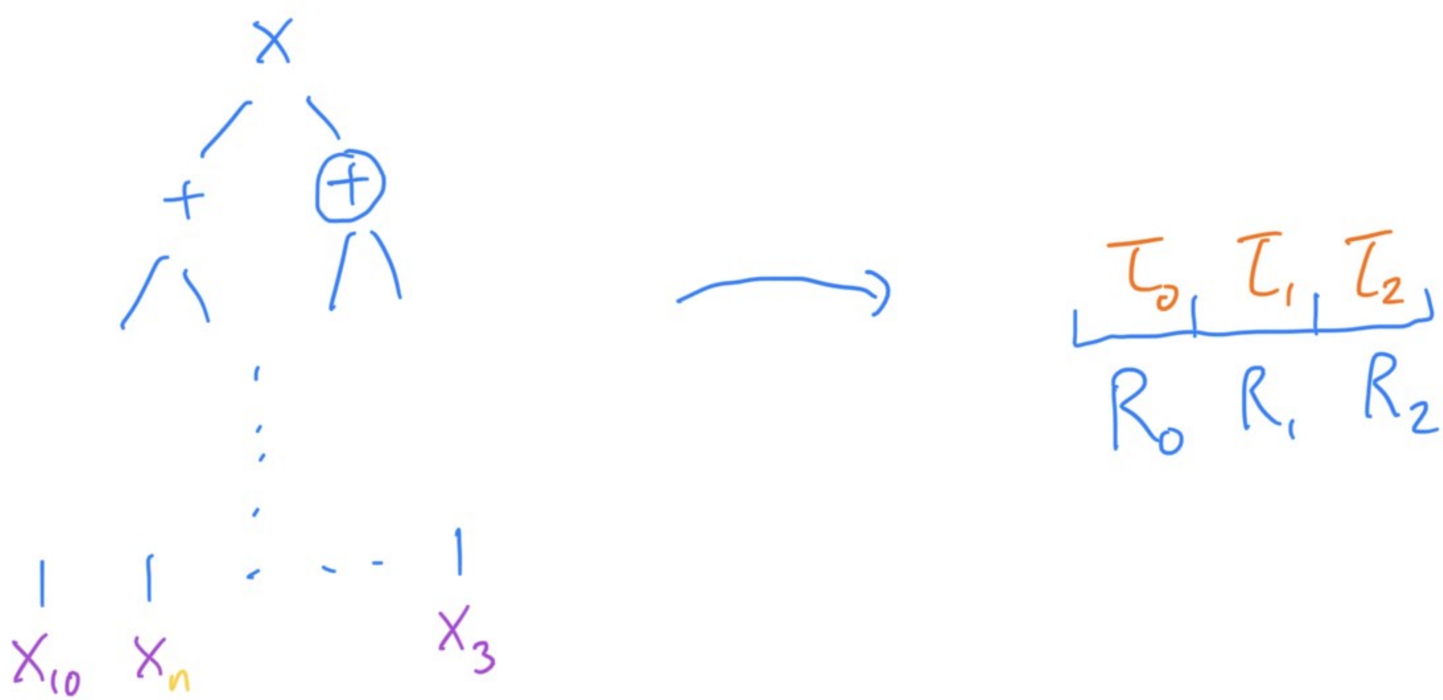
Ben-Or & Cleve

base case: $g = x_j$



Ben-Or & Cleve

case 1 : $g = g_1 + g_2$



Ben-Or & Cleve

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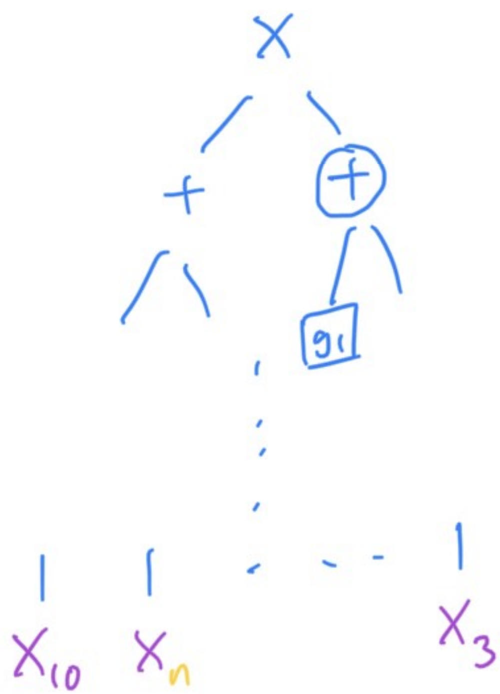
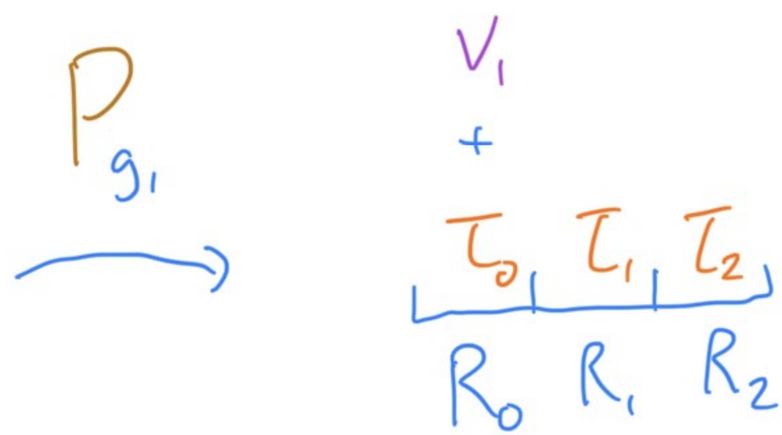
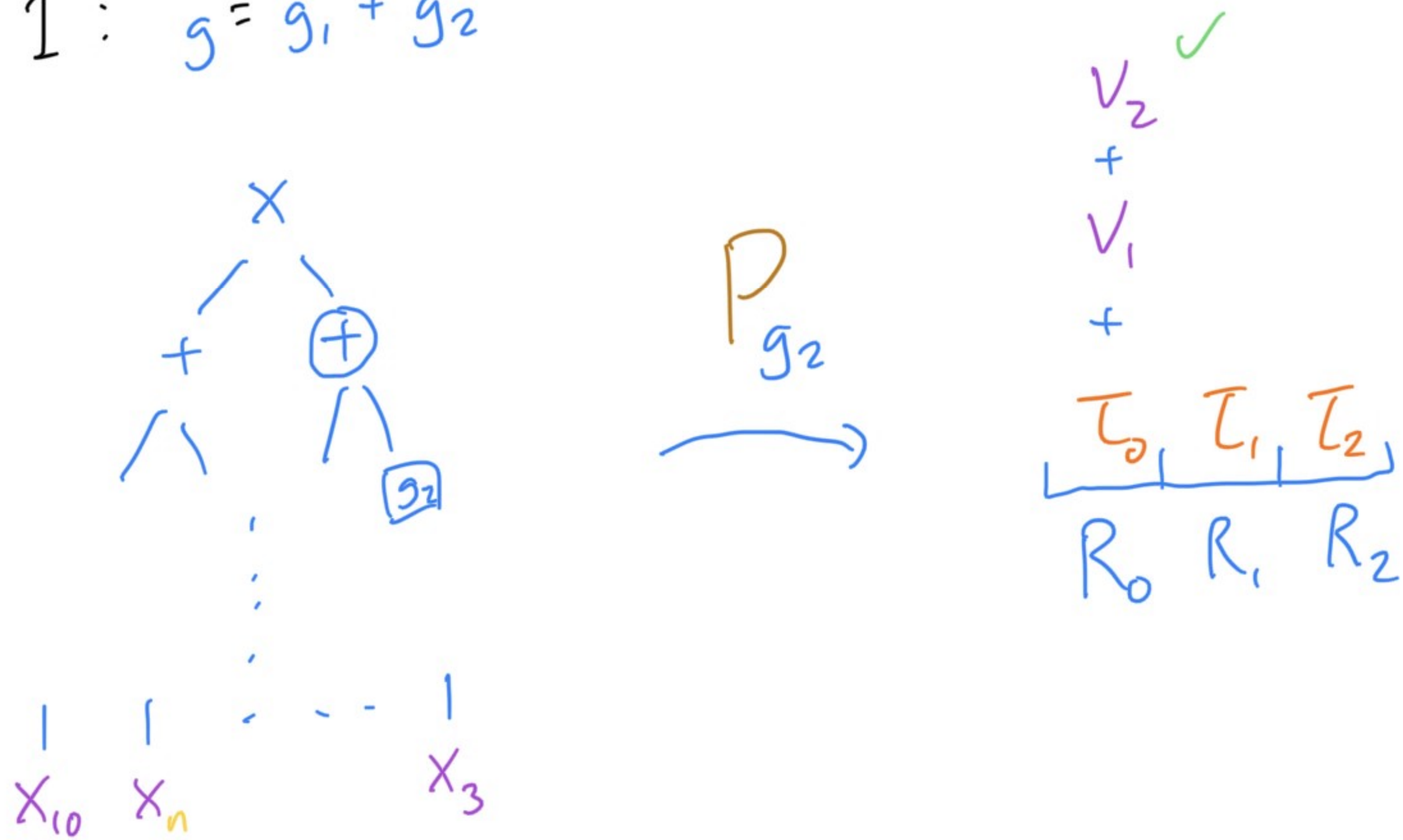




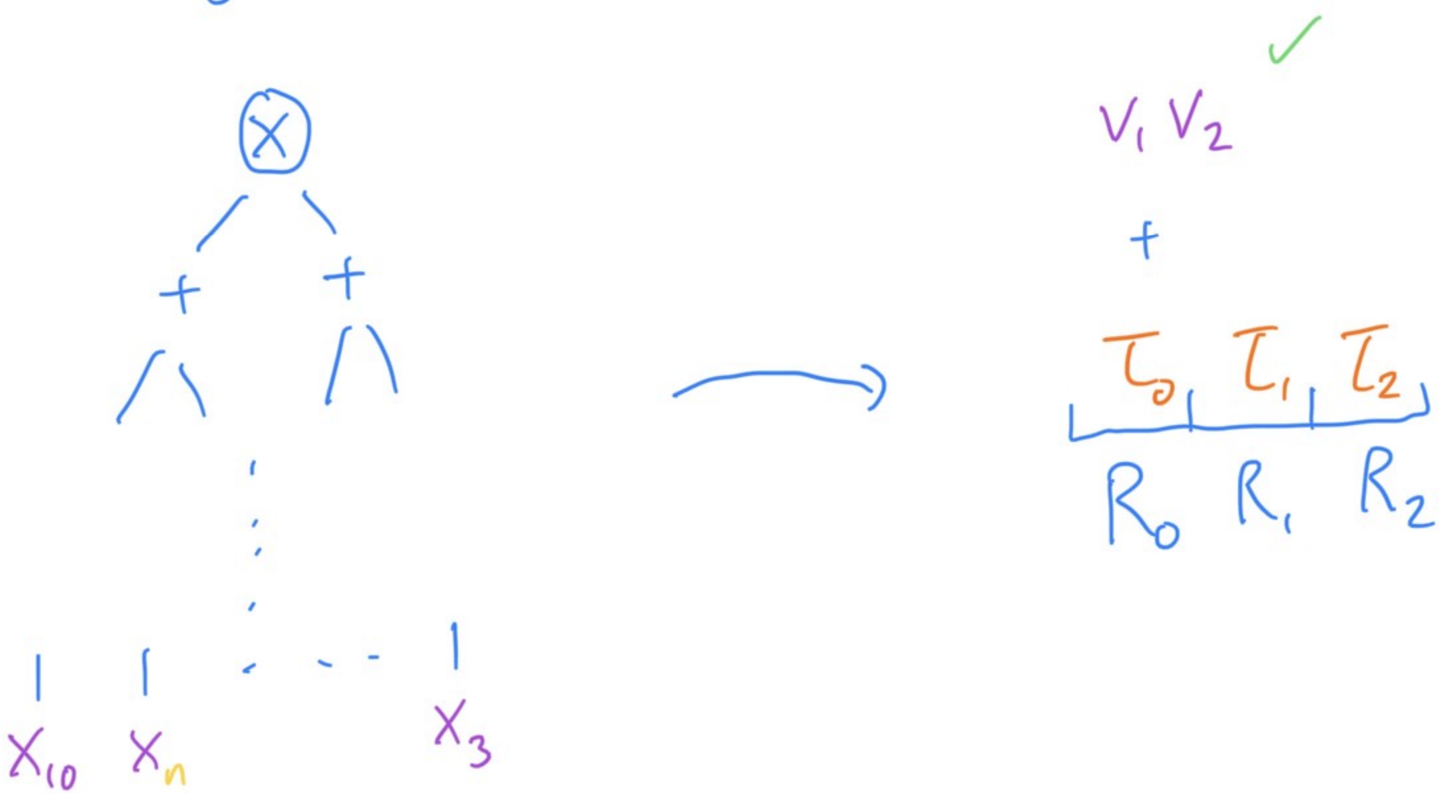
Ben-Or & Cleve

case 1 : $g = g_1 + g_2$



Ben-Or & Cleve

case 2 : $g = g_1 \times g_2$



BEN-OR & CLEVE

$$\text{space} = \underbrace{3 \log n}_{R_0\ R_1\ R_2} + \underbrace{O(\log n)}_{\text{misc}}$$

$$+ \log \text{runtime}$$

Ben-Or & Cleve

$$\text{space} = \underset{R_0 \; R_1 \; R_2}{3 \log n} + \underset{\text{misc}}{O(\log n)}$$

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$$P_g \leftarrow 4 \text{ calls to } P_{g'}\text{s} + 4 \text{ other instructions}$$

Ben-Or & Cleve

$$\text{space} = \underset{R_0\ R_1\ R_2}{3 \log n} + \underset{\text{misc}}{O(\log n)}$$

$$+ \log \text{runtime}$$

$P_g \leftarrow 4$ calls to $P_{g'}$s $+ 4$ other instructions

$R(h) \le 4 \cdot R(h-1) + 4 \rightarrow 4^{\log n}$ $\text{poly}\, n$ instructions ✓ $\square$

$$\mathrm{SPACE}(f, z) = \mathrm{SPACE}(f) + |z| \text{ ?}$$

$$\mathrm{SPACE}(f, z) = \mathrm{SPACE}(f) + |z| \,?$$

[CMWBS'12]: YES for most f (conjecture)

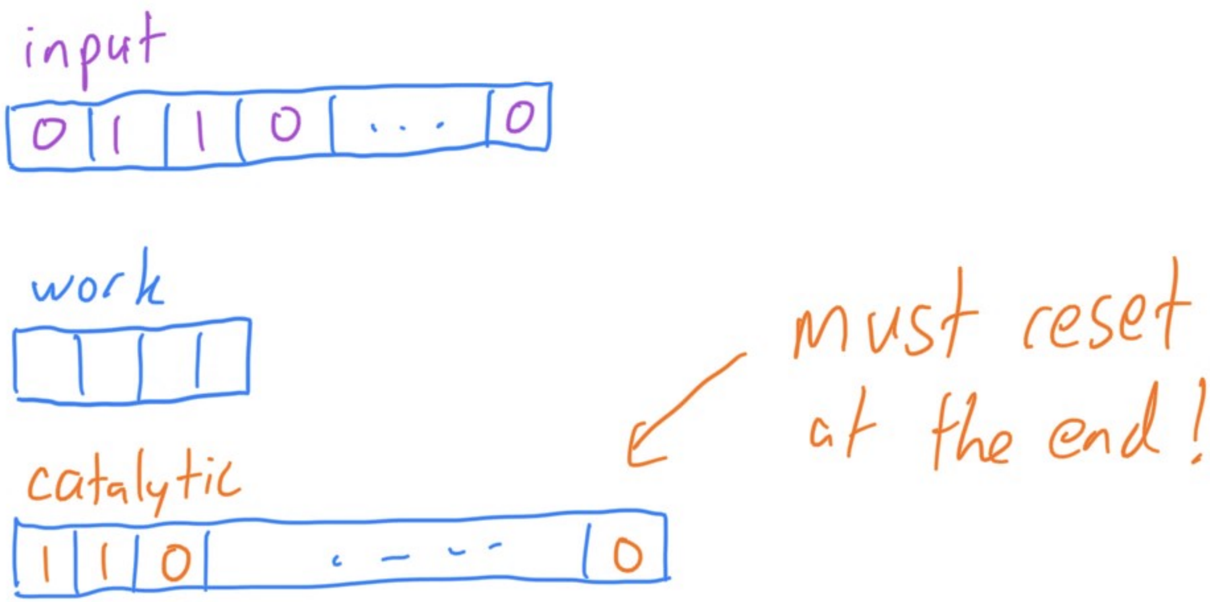
[BC'89]: NO for $f = +, \times$

CATALYTIC COMPUTING

[BCKLS'14]: let's model it

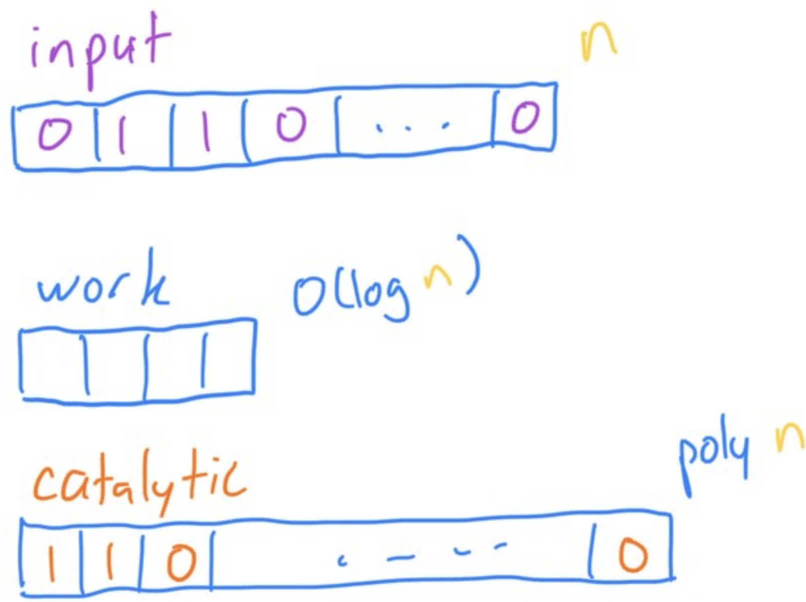
CATALYTIC COMPUTING

[BCKLS'14]: let's model it



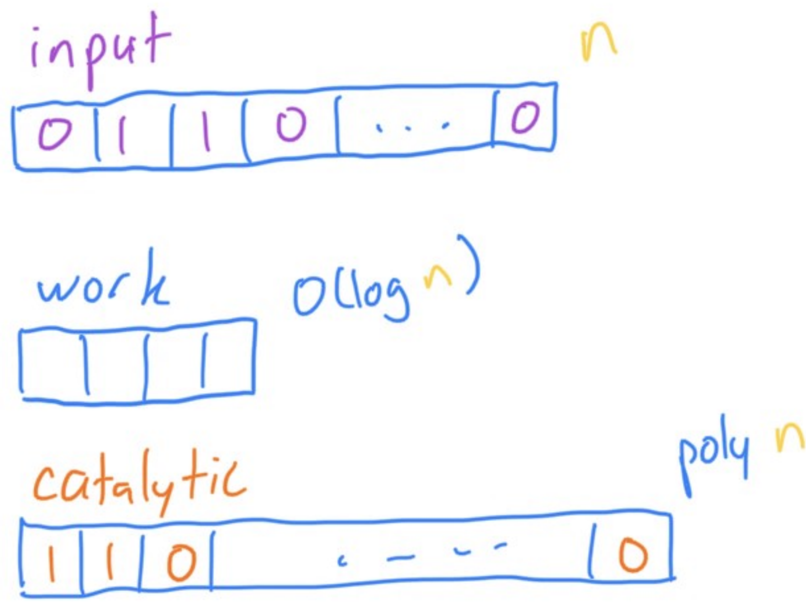
CATALYTIC COMPUTING

CL



Catalytic computing

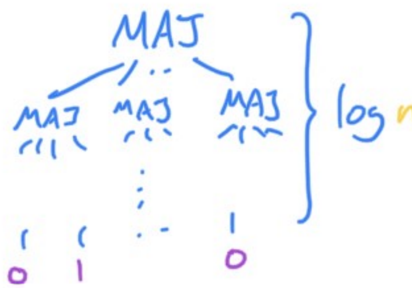
CL



Q: what can CL do that L can't?

Catalytic computing

TC^1

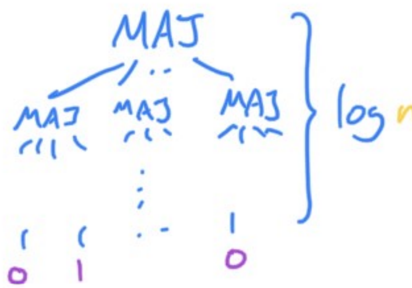


$$NL \subseteq TC^1 \subseteq NC^2$$

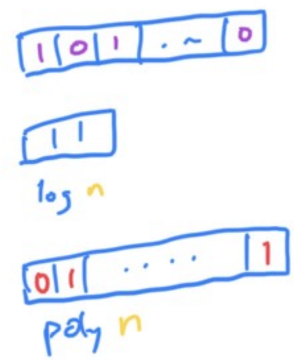
$$NC^1 \subseteq L \subseteq NL \subseteq NC^2 \subseteq P$$

Catalytic computing

TC^1



CL



$$TC^1 \subseteq CL$$

$$NL \subseteq TC^1 \subseteq NC^2$$

$$NC^1 \subseteq L \subseteq NL \subseteq NC^2 \subseteq P$$

CATALYTIC COMPUTING

LEMMA: $\forall g \in \mathcal{C}$, $\exists P_g$ s.t.

$$P_g : \quad R_0 \leftarrow R_0 + V_g$$
$$R_i \leftarrow R_i \qquad \forall i \neq 0$$

gateset: $MAJ(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^{m} x_i \geq \frac{m}{2} \\ 0 & \text{o.w.} \end{cases}$

CATALYTIC COMPUTING

$$\mathrm{MAJ}(x) \equiv \sum_{k=\frac{m}{2}}^{m} \left[1 - \left(\sum_{i=1}^{m} x_i - k \right)^{p-1} \right] \bmod p$$

CATALYTIC COMPUTING

$$\mathrm{MAJ}(x) \equiv \underbrace{\sum_{k=\frac{m}{2}}^{m} \left[1 - \Big(\sum_{i=1}^{m} x_i - k \Big)^{p-1} \right]}_{P_\Sigma} \bmod p$$

P_Σ $\qquad P_{\wedge p-1}$

CATALYTIC COMPUTING

$$MAJ(x) \equiv \sum_{k=\frac{m}{2}}^{m} \left[1 - \left(\sum_{i=1}^{m} x_i - k \right)^{p-1} \right] \bmod p$$

P_{Σ} $\qquad$ $P_{\wedge p-1}$

Efficiency: poly n registers

$O(1)$ recursive calls $\square$

$$\mathrm{SPACE}(f, z) = \mathrm{SPACE}(f) + |z| \,?$$

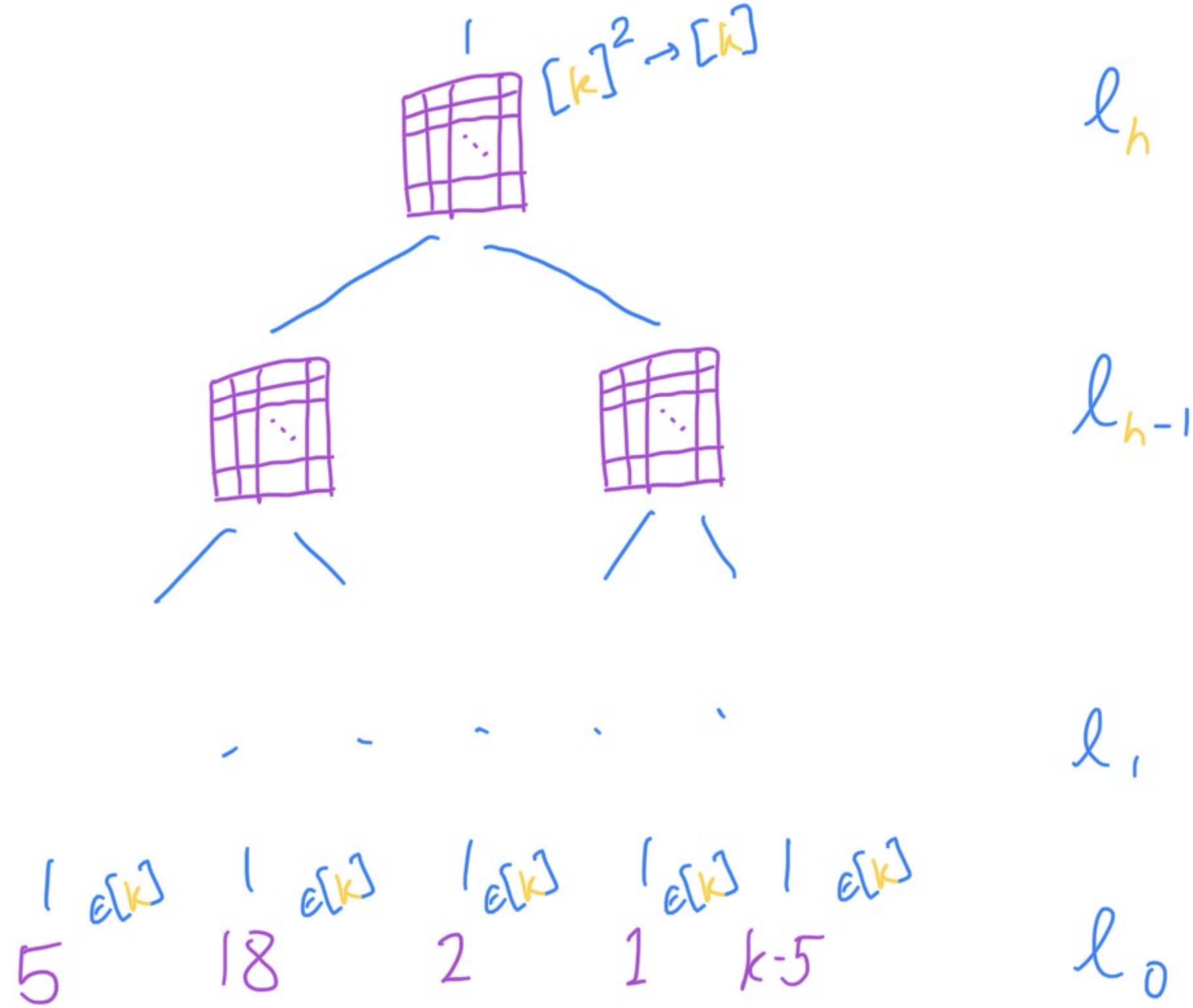
TEP UPPER BOUNDS

<u>THEOREM</u> [CM'20, 21]: $TEP_{k,h}$ can be solved in space $O(h \log k / \log h)$

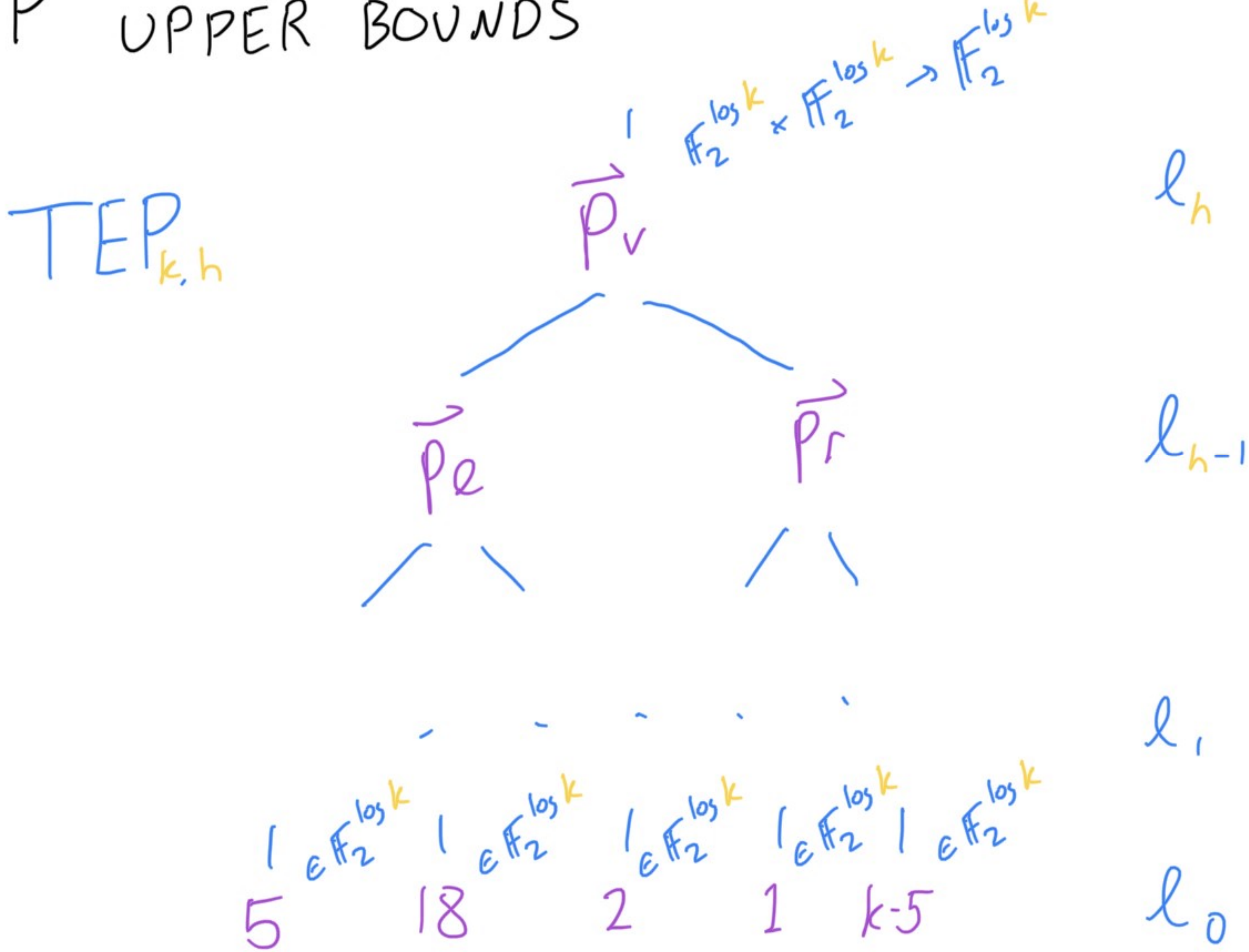
$\left(= O(\log^2 n / \log \log n)\right)$

TEP UPPER BOUNDS

$\mathrm{TEP}_{k,h}$



TEP UPPER BOUNDS



TEP UPPER BOUNDS

<u>LEMMA</u>: $\forall g \in \mathcal{C}$, $\exists P_g$ s.t.

$$P_g : \quad R_0 \leftarrow R_0 + v_g$$

$$R_i \leftarrow R_i \quad \forall i \neq 0$$

gateset: $\vec{p}(\vec{y}, \vec{z}) : \mathbb{F}_2^{\log k} \times \mathbb{F}_2^{\log k} \to \mathbb{F}_2^{\log k}$

TEP UPPER BOUNDS

$$\vec{p}(\vec{y}, \vec{z}) : \mathbb{F}_2^{\log k} \times \mathbb{F}_2^{\log k} \to \mathbb{F}_2^{\log k}$$

Efficiency: $3 \log k$ registers over $\mathbb{F}_2$

$\to$ space $= 3 \log k$

TEP UPPER BOUNDS

$$\vec{p}(\vec{y}, \vec{z}) : \mathbb{F}_2^{\log k} \times \mathbb{F}_2^{\log k} \to \mathbb{F}_2^{\log k}$$

Efficiency: $3 \log k$ registers over $\mathbb{F}_2$

k^2 recusive calls

$$\to \text{space} = 3 \log k + \log (k^2)^h = h \log k \quad \cancel{\square}$$

TEP UPPER BOUNDS

$$\vec{p}(\vec{y}, \vec{z}) : \mathbb{F}_2^{\log k} \times \mathbb{F}_2^{\log k} \to \mathbb{F}_2^{\log k}$$

Efficiency:

$3 \log k$ registers over $\mathbb{F}_2$

$2^{\deg(p)}$ recusive calls

TEP upper bounds

FEWER RECURSIVE CALLS $2^{o(\deg(p))}$

↓

FASTER TEP ALGORITHMS $\left(2^{o(\log k)}\right)^h$

↓

SMALLER SPACE TEP $o(h \log k)$

alternative facts ahead

TEP lower bounds are a hoax

Warning

Not Peer Reviewed

OFFICIALLY

I do my own research

Free Speech zone

TEP UPPER BOUNDS

THEOREM [CM'23]: $TEP_{k,h}$ can be solved in space $O(h + \log k) \cdot \log\log k$

$(= O(\log n \cdot \log\log n))$

TEP UPPER BOUNDS

$\vec{p}_v(\vec{l}, \vec{r})$: $O(deg(p))$ recusive calls

$$\text{space} = \log\Big[\underbrace{O(\log k)^{h}}_{\text{runtime}}\Big]$$

TEP UPPER BOUNDS

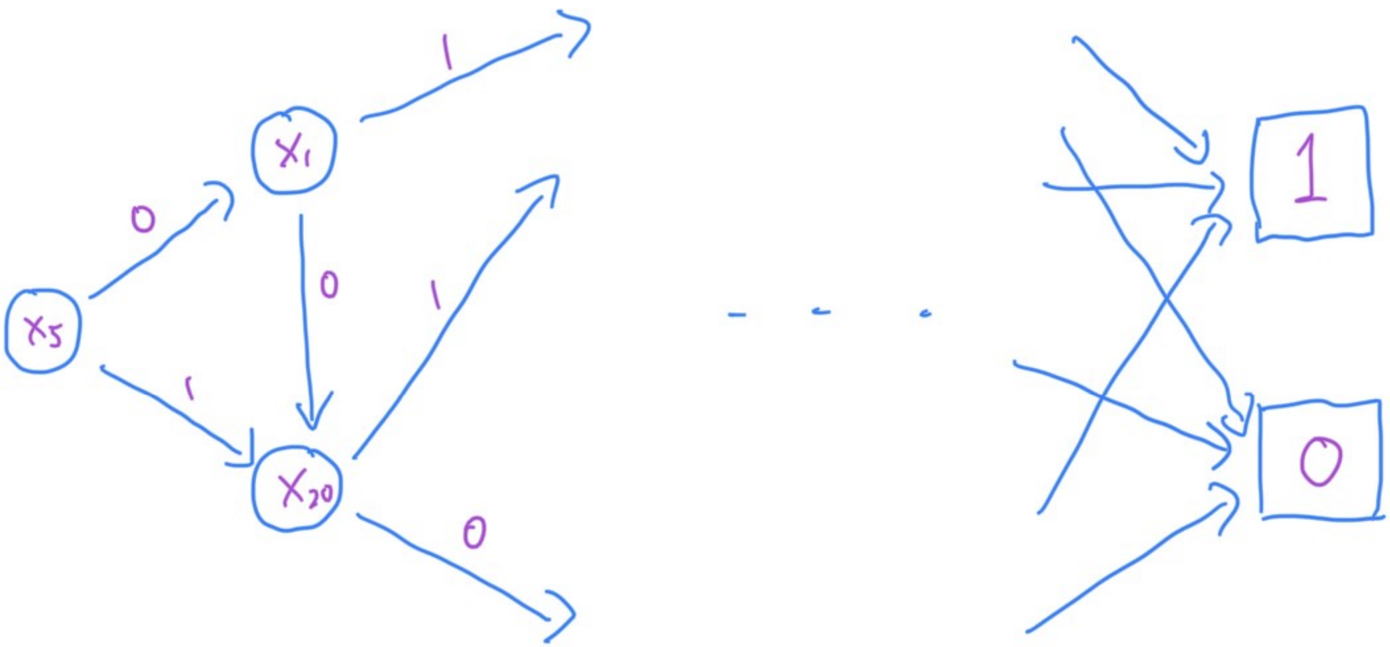
$\vec{p}_v(\vec{\ell}, \vec{r})$: $O(\deg(p))$ recusive calls

$3 \log k$ registers over $\mathbb{F}_{O(\deg(p))}$

$$\text{space} = \underbrace{\log\left[O(\log k)^h\right]}_{\text{runtime}} + \underbrace{(3 \log k)}_{\text{\# reg.}} \cdot \underbrace{\log\left[O(\log k)\right]}_{\text{size per reg.}}$$

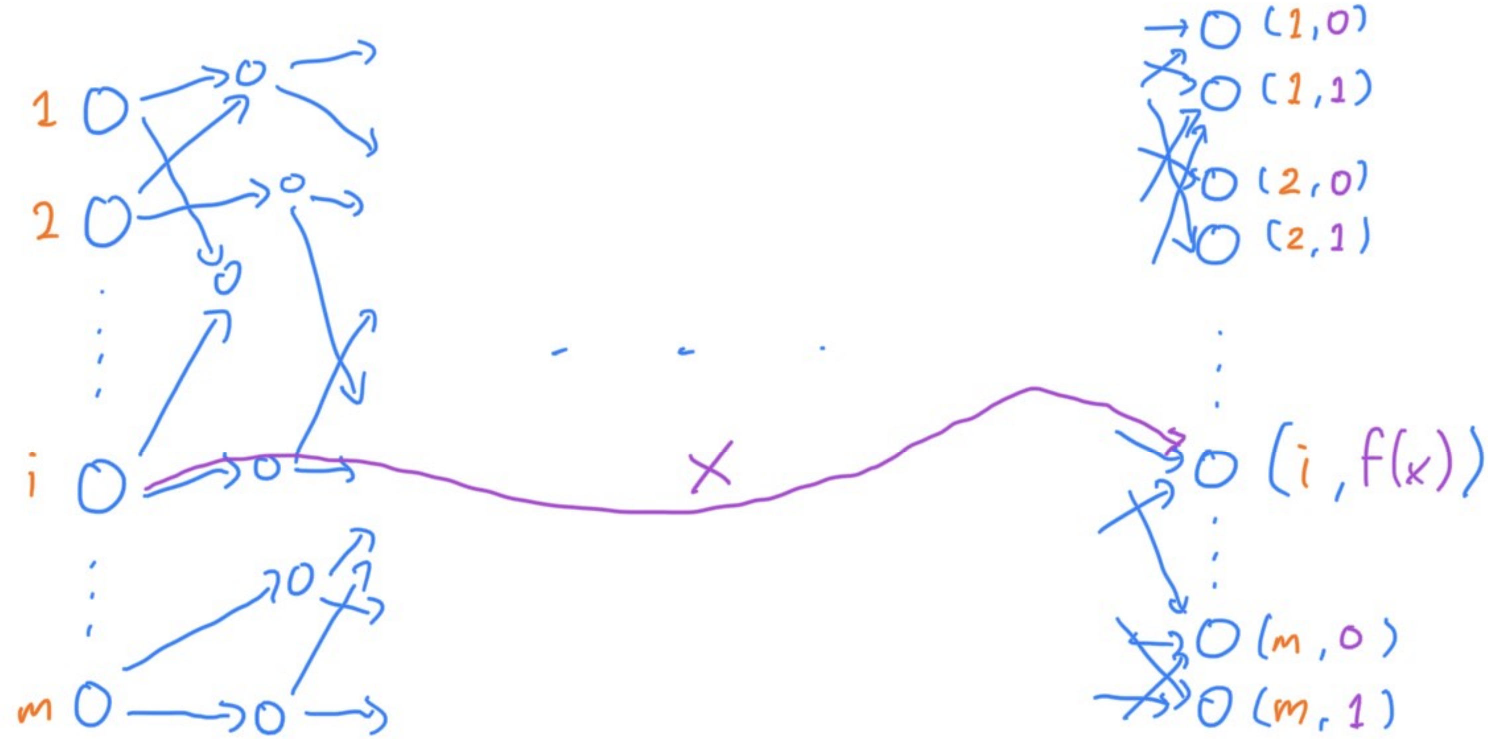
$$= O(h + \log k) \cdot \log\log k$$

AMORTIZED BPs



size s $\equiv$ non-uniform space $\log s$

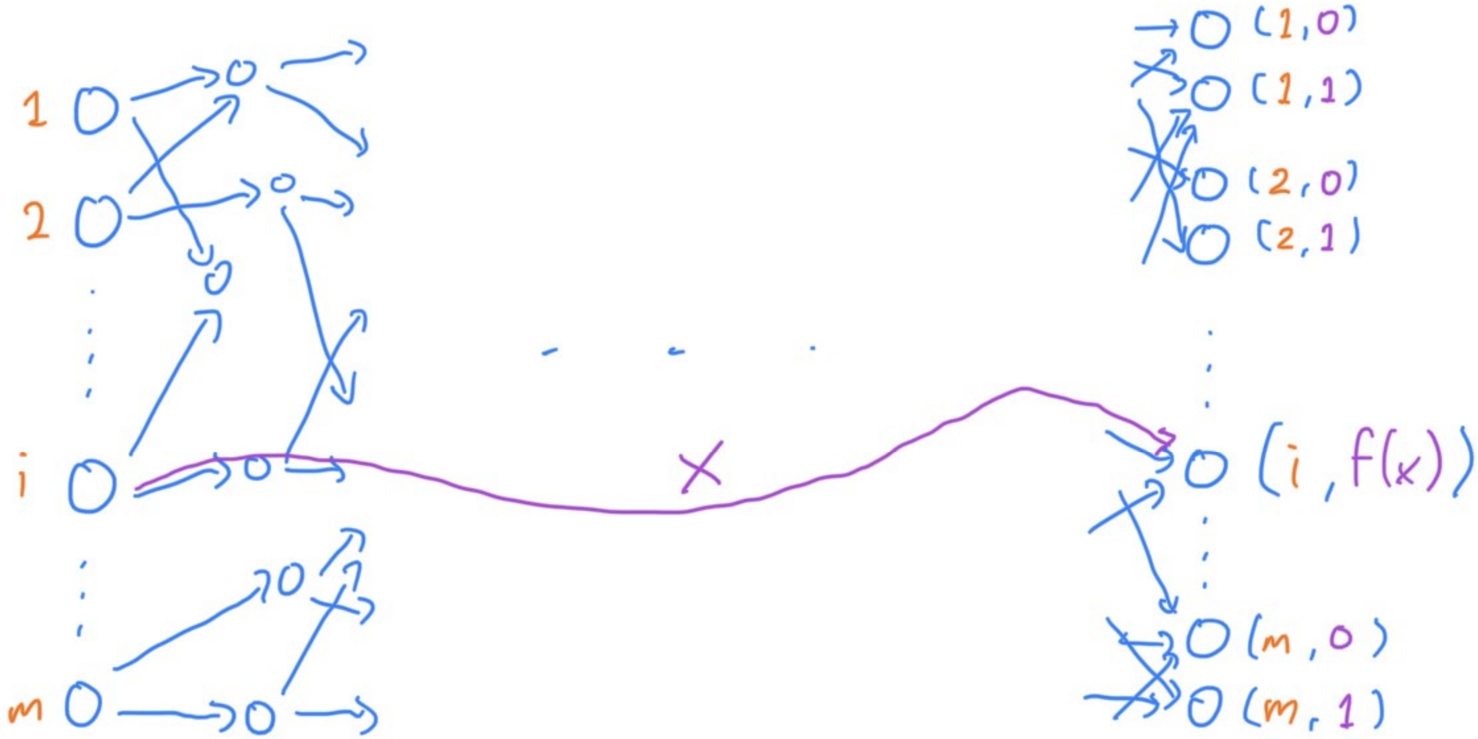
AMORTIZED BPs



catalytic $\log m$

size mS $\equiv$ non-uniform space $\log S$

AMORTIZED BPs



size ms $\equiv$ amortized$_m$ size s

AMORTIZED BPs

THEOREM [P'17]: $s = O(n)$ $m = 2^{2^n}$

AMORTIZED BPs

THEOREM [P'17]: $s = O(n)$ $\quad m = 2^{2^n}$

THEOREM [CM'22]: $s = O_\varepsilon(n)$ $\quad m = 2^{2^{\varepsilon n}}$

AMORTIZED BPs

THEOREM [P'17]: $s = O(n)$ $m = 2^{2^{n}}$

THEOREM [CM'22]: $s = O_{\varepsilon}(n)$ $m = 2^{2^{\varepsilon n}}$

THEOREM [CM'23]: $s = n^{2+\varepsilon}$ $m = 2^{O_{\varepsilon}(n)}$

KRW AND SPACE

<u>CONJECTURE</u> [KRW'95]: $TEP_{d,h} \notin NC^1$

KRW and Space

<u>Conjecture</u> [KRW'95]: $TEP_{d,h} \notin NC^1$

<u>Theorem</u> [CM'23]: $KRW \rightarrow NC^1 \neq L$

KRW AND SPACE

1. KRW gives a very sharp separation between complexity classes

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2. KRW $\rightarrow$ quasipoly (uniform) separation between formulas and branching programs

KRW AND SPACE

1. KRW gives a very sharp separation between complexity classes

2. KRW $\rightarrow$ quasipoly (uniform) separation between formulas and branching programs

3. formally easier to show STCONN $\notin NC^1$ than TEP $\notin NC^1$

WHAT NOW?

- TEP still not in L yet!

WHAT NOW?

- TEP still not in L yet!
- what is TEP complete for?

WHAT NOW?

- TEP still not in L yet!
- what is TEP complete for?
- other things to do with catalytic?

SHAMELESS PLUGS

RESULTS [CM'23]: ECCC (soon!)

(longer talk to be posted on release)

SHAMELESS PLUGS

RESULTS [CM'23] : ECCC (soon!)

(longer talk to be posted on release)

SURVEY [M'23] : B. EATCS

$\rightarrow$ ECCC

suggestions appreciated!

THANKS!