The KRW Conjecture

Past, Present, and Future

Or Meir

Depth complexity

- Let $f : \{0,1\}^n \to \{0,1\}$.
- The depth complexity $\mathsf{D}(f)$ is the depth of the shallowest circuit for f.
- Captures the complexity of parallel computation.

Depth complexity

- Let $f : \{0,1\}^n \to \{0,1\}$.
- The depth complexity $\mathsf{D}(f)$ is the depth of the shallowest circuit for f.
- Captures the complexity of parallel computation.
- We only consider circuits with fan-in 2.

Depth complexity

- Let $f : \{0,1\}^n \to \{0,1\}$.
- The depth complexity $\mathsf{D}(f)$ is the depth of the shallowest circuit for f.
- Captures the complexity of parallel computation.
- We only consider circuits with fan-in 2.
- Major frontier: Explicit f with $\mathsf{D}(f) = \omega(\log n)$.
- a.k.a. $\mathbf{P} \neq \mathbf{NC}^1$.

Depth complexity

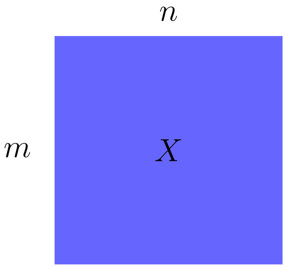
- Let $f : \{0,1\}^n \to \{0,1\}$.
- The depth complexity $\mathsf{D}(f)$ is the depth of the shallowest circuit for f.
- Captures the complexity of parallel computation.
- We only consider circuits with fan-in 2.
- Major frontier: Explicit f with $\mathsf{D}(f) = \omega(\log n)$.
- a.k.a. $\mathbf{P} \neq \mathbf{NC}^1$.
- State of the art: $D(f) \geq (3 - o(1)) \cdot \log n$ [Håstad 98, Tal 14].

Composition

- [Karchmer-Raz-Wigderson-91]:
  We need to understand composition.

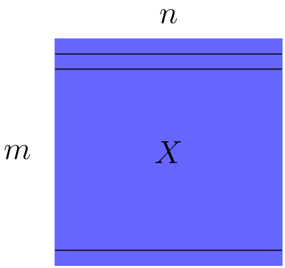
Composition

- [Karchmer-Raz-Wigderson-91]:
  We need to understand composition.
- Let $f : \{0,1\}^m \to \{0,1\}$, $g : \{0,1\}^n \to \{0,1\}$.
- The composition $f \diamond g : \{0,1\}^{m \times n} \to \{0,1\}$ is



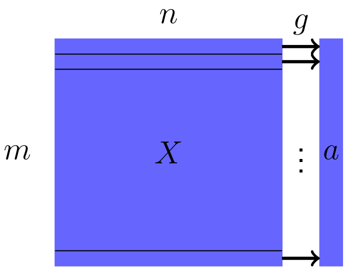
Composition

- [Karchmer-Raz-Wigderson-91]:
  We need to understand composition.
- Let $f : \{0,1\}^m \to \{0,1\}$, $g : \{0,1\}^n \to \{0,1\}$.
- The composition $f \diamond g : \{0,1\}^{m \times n} \to \{0,1\}$ is



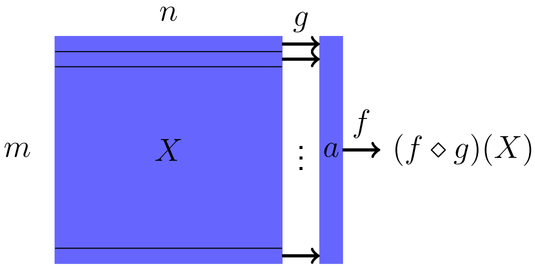
Composition

- [Karchmer-Raz-Wigderson-91]:
  We need to understand composition.
- Let $f : \{0,1\}^m \to \{0,1\}$, $g : \{0,1\}^n \to \{0,1\}$.
- The composition $f \diamond g : \{0,1\}^{m \times n} \to \{0,1\}$ is

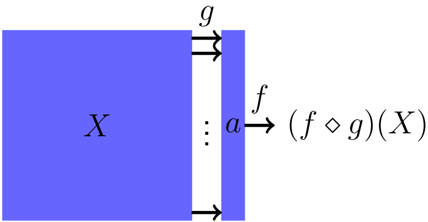


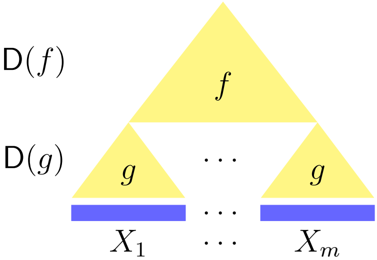
Composition

- [Karchmer-Raz-Wigderson-91]:
  We need to understand composition.
- Let $f : \{0,1\}^m \to \{0,1\}$, $g : \{0,1\}^n \to \{0,1\}$.
- The composition $f \diamond g : \{0,1\}^{m \times n} \to \{0,1\}$ is



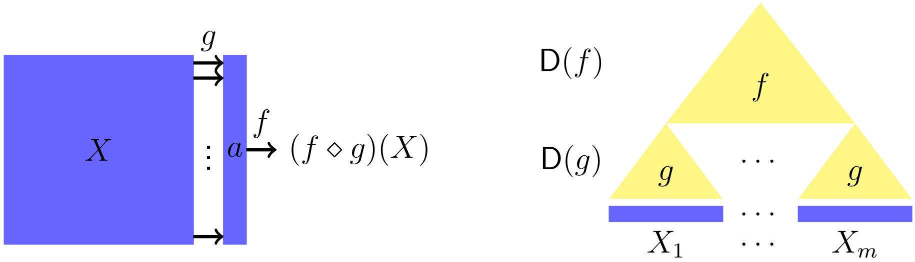
The KRW conjecture





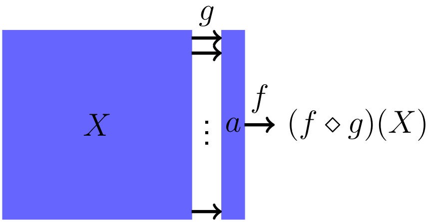
- Clearly, $\mathsf{D}(f \diamond g) \leq \mathsf{D}(f) + \mathsf{D}(g)$.

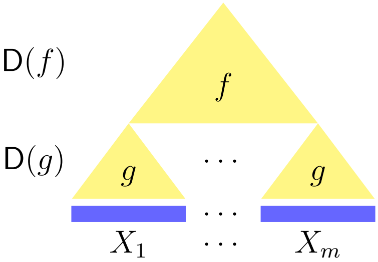
The KRW conjecture



- Clearly, $\mathsf{D}(f \diamond g) \le \mathsf{D}(f) + \mathsf{D}(g)$.
- KRW conjecture: $\forall f, g: \;\mathsf{D}(f \diamond g) \approx \mathsf{D}(f) + \mathsf{D}(g)$.

The KRW conjecture





- Clearly, $\mathsf{D}(f \diamond g) \leq \mathsf{D}(f) + \mathsf{D}(g)$.
- KRW conjecture: $\forall f, g:\ \mathsf{D}(f \diamond g) \approx \mathsf{D}(f) + \mathsf{D}(g)$.
- Theorem [KRW91]: the conjecture implies that $\mathbf{P} \neq \mathbf{NC}^1$.

Outline

1. Known results
2. A research agenda
3. A new work

Outline

Karchmer-Wigderson relations

- Relate $\mathsf{D}(f)$ to complexity of a communication problem KW_f.

Karchmer-Wigderson relations

- Relate $\mathsf{D}(f)$ to complexity of a communication problem KW_f.
- The KW relation KW_f is defined as follows:
  - Alice gets $x \in f^{-1}(1)$.
  - Bob gets $y \in f^{-1}(0)$.
  - Clearly, $x \neq y$, so $\exists i$ s.t. $x_i \neq y_i$.
  - Want to find such i.

Karchmer-Wigderson relations

- Relate $\mathsf{D}(f)$ to complexity of a communication problem KW_f.
- The KW relation KW_f is defined as follows:
  - Alice gets $x \in f^{-1}(1)$.
  - Bob gets $y \in f^{-1}(0)$.
  - Clearly, $x \neq y$, so $\exists i$ s.t. $x_i \neq y_i$.
  - Want to find such i.
- Theorem [Karchmer-Wigderson-88]: $\mathsf{D}(f) = \mathsf{C}(KW_f)$.

Karchmer-Wigderson relations

- Relate $\mathsf{D}(f)$ to complexity of a communication problem KW_f.
- The KW relation KW_f is defined as follows:
  - Alice gets $x \in f^{-1}(1)$.
  - Bob gets $y \in f^{-1}(0)$.
  - Clearly, $x \neq y$, so $\exists i$ s.t. $x_i \neq y_i$.
  - Want to find such i.
- Theorem [Karchmer-Wigderson-88]: $\mathsf{D}(f) = \mathsf{C}(KW_f)$.
- Only deterministic protocols!

Karchmer-Wigderson relations

- Relate $\mathsf{D}(f)$ to complexity of a communication problem KW_f.
- The KW relation KW_f is defined as follows:
  - Alice gets $x \in f^{-1}(1)$.
  - Bob gets $y \in f^{-1}(0)$.
  - Clearly, $x \neq y$, so $\exists i$ s.t. $x_i \neq y_i$.
  - Want to find such i.
- Theorem [Karchmer-Wigderson-88]: $\mathsf{D}(f) = \mathsf{C}(KW_f)$.
- Only deterministic protocols!
- KRW conjecture: $\mathsf{C}(KW_{f \diamond g}) \approx \mathsf{C}(KW_f) + \mathsf{C}(KW_g)$.

Karchmer-Wigderson relations

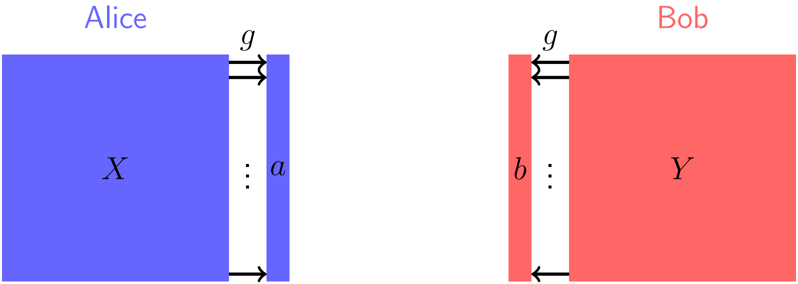
- Relate $\mathsf{D}(f)$ to complexity of a communication problem KW_f.
- The KW relation KW_f is defined as follows:
  - Alice gets $x \in f^{-1}(1)$.
  - Bob gets $y \in f^{-1}(0)$.
  - Clearly, $x \neq y$, so $\exists i$ s.t. $x_i \neq y_i$.
  - Want to find such i.
- Theorem [Karchmer-Wigderson-88]: $\mathsf{D}(f) = \mathsf{C}(KW_f)$.
- Only deterministic protocols!
- KRW conjecture: $\mathsf{C}(KW_{f \diamond g}) \approx \mathsf{C}(KW_f) + \mathsf{C}(KW_g)$.
- Notation: Denote $KW_{f \diamond g}$ by $KW_f \diamond KW_g$.

KRW and KW

- Can we use KW games to attack the KRW conjecture?
- What does $KW_f \diamond KW_g$ look like?
- Recall: $f \diamond g$ maps $\{0,1\}^{m \times n}$ to $\{0,1\}$.

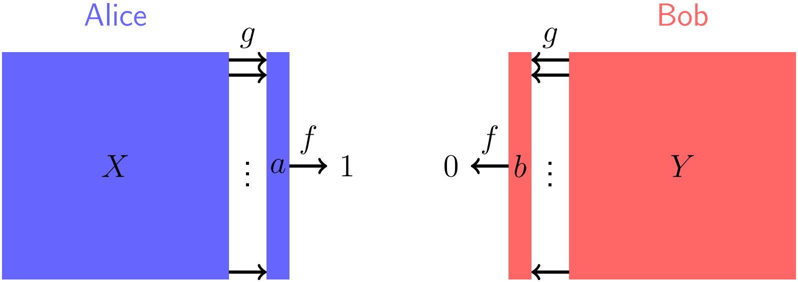
KRW and KW

- Can we use KW games to attack the KRW conjecture?
- What does $KW_f \diamond KW_g$ look like?
- Recall: $f \diamond g$ maps $\{0,1\}^{m \times n}$ to $\{0,1\}$.



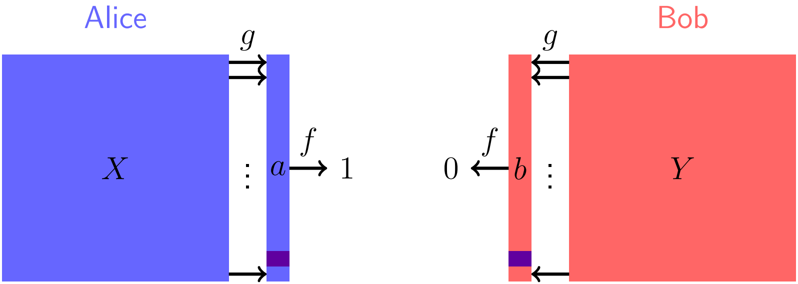
KRW and KW

- Can we use KW games to attack the KRW conjecture?
- What does $KW_f \diamond KW_g$ look like?
- Recall: $f \diamond g$ maps $\{0,1\}^{m \times n}$ to $\{0,1\}$.



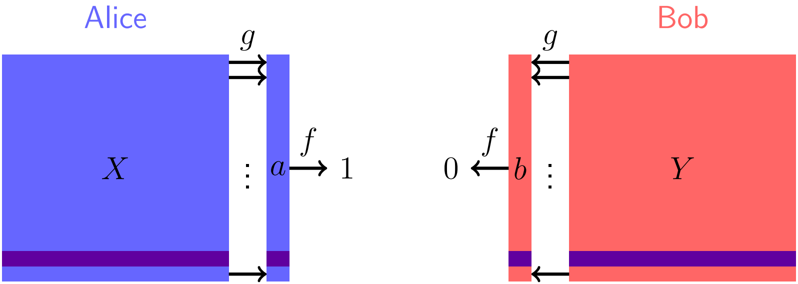
KRW and KW

- Can we use KW games to attack the KRW conjecture?
- What does $KW_f \diamond KW_g$ look like?
- Recall: $f \diamond g$ maps $\{0,1\}^{m \times n}$ to $\{0,1\}$.



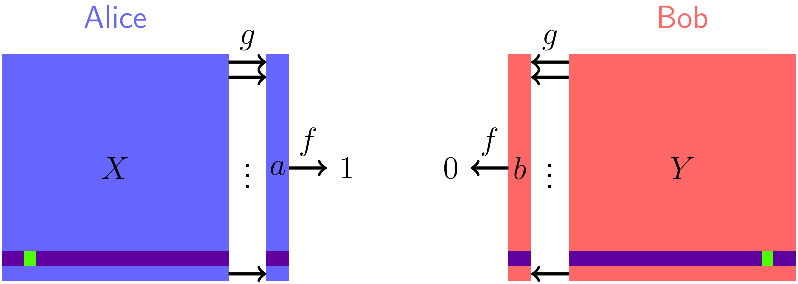
KRW and KW

- Can we use KW games to attack the KRW conjecture?
- What does $KW_f \diamond KW_g$ look like?
- Recall: $f \diamond g$ maps $\{0,1\}^{m \times n}$ to $\{0,1\}$.



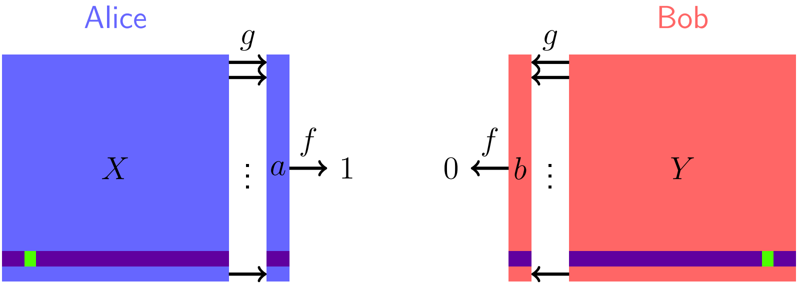
KRW and KW

- Can we use KW games to attack the KRW conjecture?
- What does $KW_f \diamond KW_g$ look like?
- Recall: $f \diamond g$ maps $\{0,1\}^{m \times n}$ to $\{0,1\}$.



KRW and KW

- Can we use KW games to attack the KRW conjecture?
- What does $KW_f \diamond KW_g$ look like?
- Recall: $f \diamond g$ maps $\{0,1\}^{m \times n}$ to $\{0,1\}$.



- Hence, $\mathsf{C}(KW_f \diamond KW_g) \leq \mathsf{C}(KW_f) + \mathsf{C}(KW_g)$.
- KRW conjecture: the obvious protocol is essentially optimal.

The universal relation

- The KRW conjecture is hard.
- [KRW91] suggested a starting point.

The universal relation

- The KRW conjecture is hard.
- [KRW91] suggested a starting point.
- The universal relation U_n is:
  - Alice gets $x \in \{0,1\}^n$.
  - Bob gets $y \in \{0,1\}^n$.
  - $x \neq y$.
  - Wish to find i s.t. $x_i \neq y_i$.

The universal relation

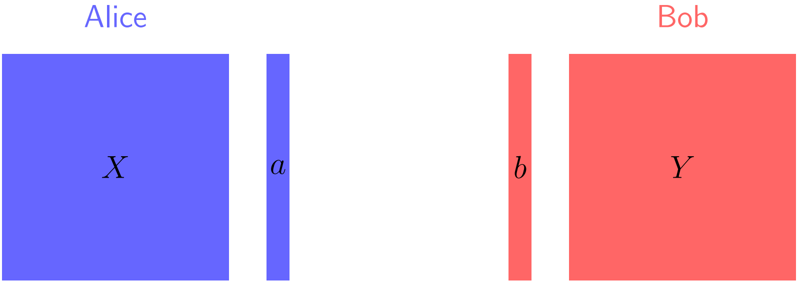
- The KRW conjecture is hard.
- [KRW91] suggested a starting point.
- The universal relation U_n is:
  - Alice gets $x \in \{0,1\}^n$.
  - Bob gets $y \in \{0,1\}^n$.
  - $x \neq y$.
  - Wish to find i s.t. $x_i \neq y_i$.
- Easy to prove: $\mathsf{C}(\mathrm{U}_n) \geq n$.

The universal relation

- The KRW conjecture is hard.
- [KRW91] suggested a starting point.
- The universal relation U_n is:
  - Alice gets $x \in \{0,1\}^n$.
  - Bob gets $y \in \{0,1\}^n$.
  - $x \neq y$.
  - Wish to find i s.t. $x_i \neq y_i$.
- Easy to prove: $\mathsf{C}(\mathrm{U}_n) \geq n$.
- [KRW91] suggested to study $\mathrm{U}_m \diamond \mathrm{U}_n$.

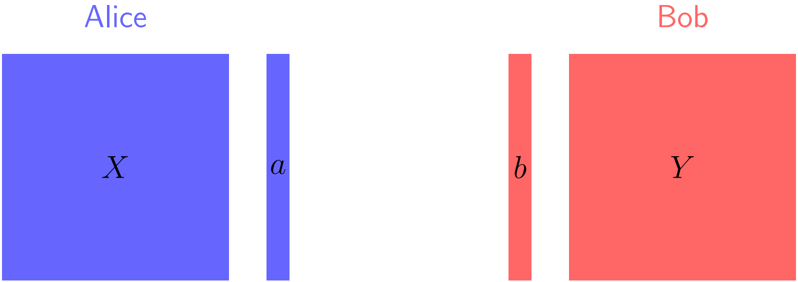
The composition of the universal relation

- [KRW91] suggested to study the composition $\mathrm{U}_m \diamond \mathrm{U}_n$.



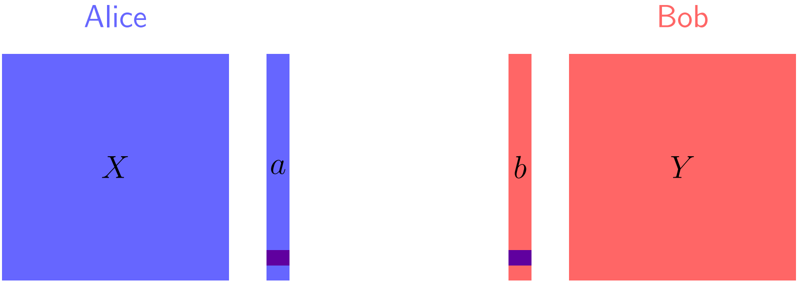
The composition of the universal relation

- [KRW91] suggested to study the composition $\mathrm{U}_m \diamond \mathrm{U}_n$.
- $a \neq b$.
- If $a_i \neq b_i$ then $X_i \neq Y_i$.



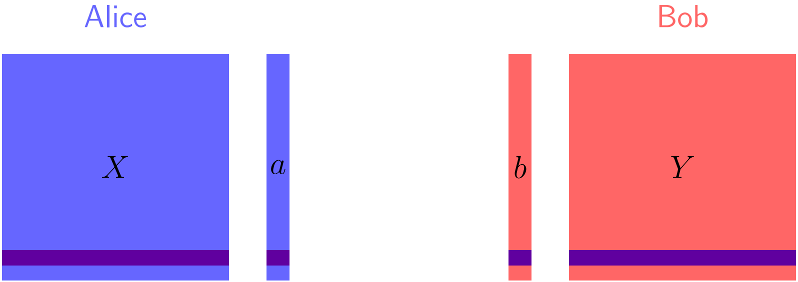
The composition of the universal relation

- [KRW91] suggested to study the composition $\mathrm{U}_m \diamond \mathrm{U}_n$.
- $a \neq b$.
- If $a_i \neq b_i$ then $X_i \neq Y_i$.



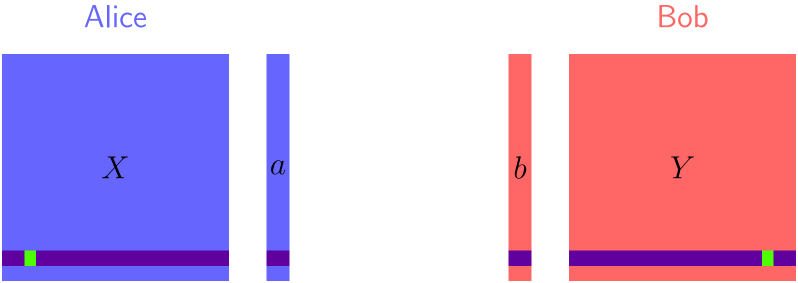
The composition of the universal relation

- [KRW91] suggested to study the composition $\mathrm{U}_m \diamond \mathrm{U}_n$.
- $a \neq b$.
- If $a_i \neq b_i$ then $X_i \neq Y_i$.



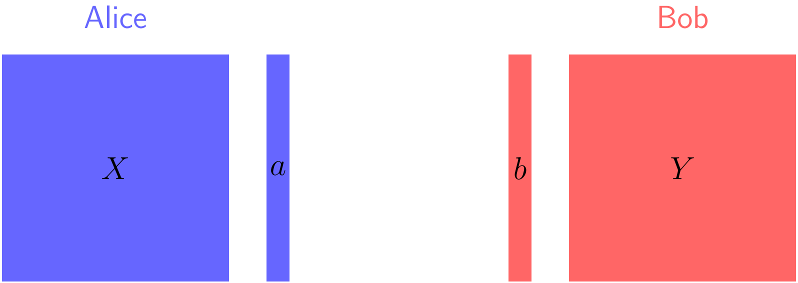
The composition of the universal relation

- [KRW91] suggested to study the composition $\mathrm{U}_m \diamond \mathrm{U}_n$.
- $a \neq b$.
- If $a_i \neq b_i$ then $X_i \neq Y_i$.



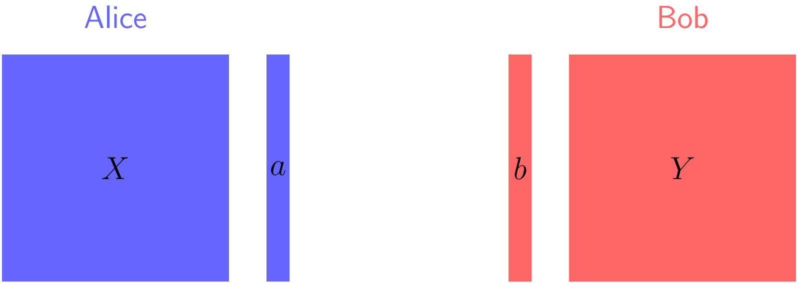
The composition of the universal relation

- Goal: $\mathsf{C}(\mathrm{U}_m \diamond \mathrm{U}_n) \approx \mathsf{C}(\mathrm{U}_m) + \mathsf{C}(\mathrm{U}_n) \geq m + n$.



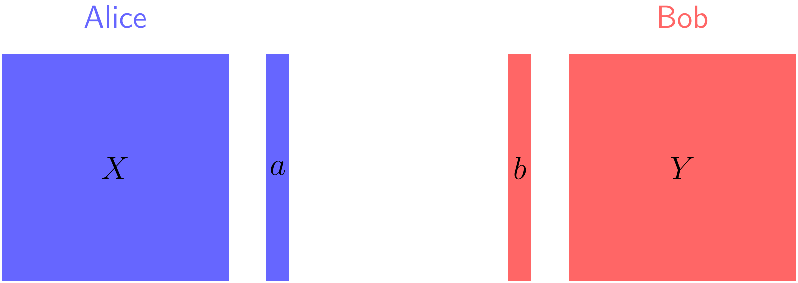
The composition of the universal relation

- Goal: $\mathsf{C}(\mathrm{U}_m \diamond \mathrm{U}_n) \approx \mathsf{C}(\mathrm{U}_m) + \mathsf{C}(\mathrm{U}_n) \geq m + n$.
- Challenge was met by [Edmonds-Impagliazzo-Rudich-S'gall-91].



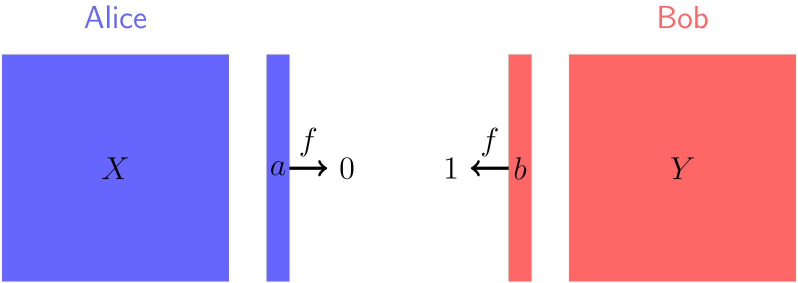
The composition of the universal relation

- Goal: $\mathsf{C}(\mathrm{U}_m \diamond \mathrm{U}_n) \approx \mathsf{C}(\mathrm{U}_m) + \mathsf{C}(\mathrm{U}_n) \geq m + n$.
- Challenge was met by [Edmonds-Impagliazzo-Rudich-S'gall-91].
- Alternative proof obtained by [Håstad-Wigderson-93].



Composing a function and the universal relation

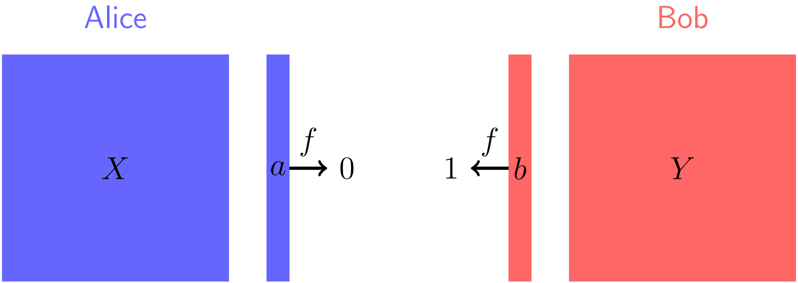
- An analog of KRW conjecture for $KW_f \diamond \mathrm{U}_n$ for any f. [Gavinsky-M-Weinstein-Wigderson-14]



- If $a_i \neq b_i$ then $X_j \neq Y_j$.
- The obvious protocol works.

Composing a function and the universal relation

- An analog of KRW conjecture for $KW_f \diamond \mathrm{U}_n$ for any f. [Gavinsky-M-Weinstein-Wigderson-14]
- Quantative improvement by [Koroth-M-18].



- If $a_i \neq b_i$ then $X_j \neq Y_j$.
- The obvious protocol works.

Composing any function and parity

- [Dinur-M-16]: (Re-)proved KRW conjecture for $f \diamond \bigoplus_n$.

Composing any function and parity

- [Dinur-M-16]: (Re-)proved KRW conjecture for $f \diamond \bigoplus_n$.
- Actually, this case was already implicit in [Håstad 98].

Composing any function and parity

- [Dinur-M-16]: (Re-)proved KRW conjecture for $f \diamond \bigoplus_n$.
- Actually, this case was already implicit in [Håstad 98].
- However, our proof was very different, and more in line with the other works on the KRW conjecture.

Composing any function and parity

- [Dinur-M-16]: (Re-)proved KRW conjecture for $f \diamond \bigoplus_n$.
- Actually, this case was already implicit in [Håstad 98].
- However, our proof was very different, and more in line with the other works on the KRW conjecture.
- Upcoming work [Filmus-M-Tal]: Any inner function that has a tight Khrapchencko-like lower bound.

Outline

Where to now?

- KRW conjecture: need $f \diamond g$ for arbitrary f, g.

Where to now?

- KRW conjecture: need $f \diamond g$ for arbitrary f, g.
- Can deal with arbitrary f...
- but only with very specific choices of g.

Where to now?

- KRW conjecture: need $f \diamond g$ for arbitrary f, g.
- Can deal with arbitrary f...
- but only with very specific choices of g.
- Arbitrary g seems much harder.

Where to now?

- KRW conjecture: need $f \diamond g$ for arbitrary f, g.
- Can deal with arbitrary f...
- but only with very specific choices of g.
- Arbitrary g seems much harder.
- Maybe we do not need an arbitrary g to prove lower bounds?

Where to now?

- KRW conjecture: need $f \diamond g$ for arbitrary f, g.
- Can deal with arbitrary f...
- but only with very specific choices of g.
- Arbitrary g seems much harder.
- Maybe we do not need an arbitrary g to prove lower bounds?
- There is a "complete" relation MUX_n, s.t. it suffices to prove the KRW conjecture for $KW_f \diamond MUX_n$.

The multiplexor relation [EIRS91]

- The function g becomes part of the input.

The multiplexor relation [EIRS91]

- The function g becomes part of the input.
- Alice gets a function $g : \{0,1\}^n \to \{0,1\}$ and $x \in g^{-1}(1)$.
- Bob gets the same function g and $y \in g^{-1}(0)$.
- Want to find $i \in [n]$ such that $x_i \neq y_i$.

The multiplexor relation [EIRS91]

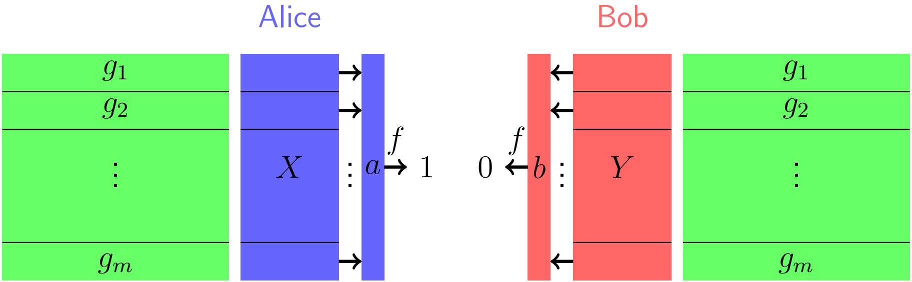
- The function g becomes part of the input.
- Alice gets a function $g : \{0,1\}^n \to \{0,1\}$ and $x \in g^{-1}(1)$.
- Bob gets the same function g and $y \in g^{-1}(0)$.
- Want to find $i \in [n]$ such that $x_i \neq y_i$.
- Like KW relation of the address function, but with promise.

The multiplexor relation [EIRS91]

- The function g becomes part of the input.
- Alice gets a function $g : \{0,1\}^n \to \{0,1\}$ and $x \in g^{-1}(1)$.
- Bob gets the same function g and $y \in g^{-1}(0)$.
- Want to find $i \in [n]$ such that $x_i \neq y_i$.
- Like KW relation of the address function, but with promise.
- Easy to prove: $\mathsf{C}(MUX_n) = \Omega(n)$.

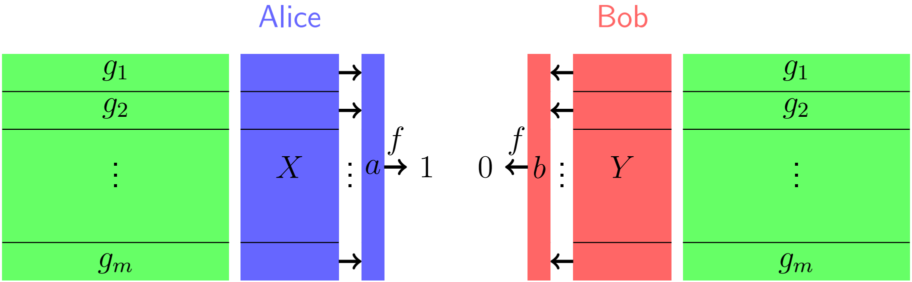
Multiplexor composition

- Given f, define the composition $KW_f \diamond MUX_n$:



Multiplexor composition

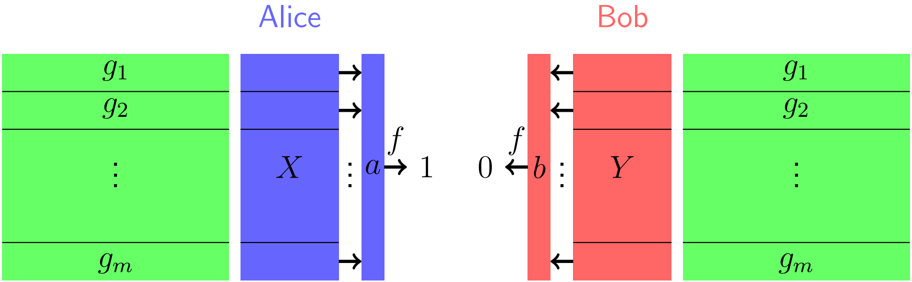
- Given f, define the composition $KW_f \diamond MUX_n$:



- Conjecture: $\mathsf{C}(KW_f \diamond MUX_n) \gtrapprox \mathsf{C}(KW_f) + \Omega(n)$.

Multiplexor composition

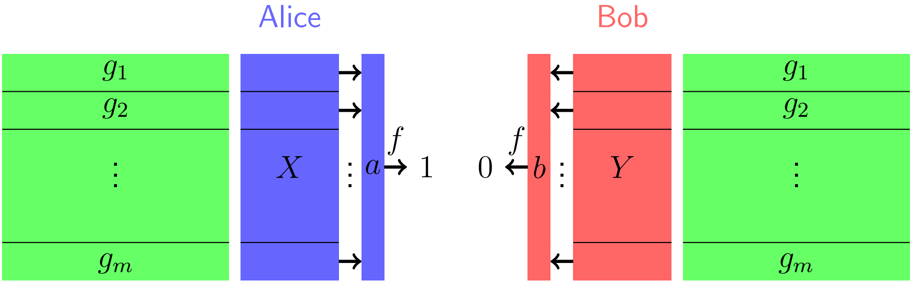
- Given f, define the composition $KW_f \diamond MUX_n$:



- Conjecture: $\mathsf{C}(KW_f \diamond MUX_n) \gtrapprox \mathsf{C}(KW_f) + \Omega(n)$.
- Theorem [M-19]: Conjecture* implies that $\mathbf{P} \neq \mathbf{NC}^1$.

Multiplexor composition

- Given f, define the composition $KW_f \diamond MUX_n$:



- Conjecture: $\mathsf{C}(KW_f \diamond MUX_n) \gtrapprox \mathsf{C}(KW_f) + \Omega(n)$.
- Theorem [M-19]: Conjecture* implies that $\mathbf{P} \neq \mathbf{NC}^1$.
  - Caveat: Should consider alternating protocols.

Research agenda

- Known: $KW_f \diamond \mathrm{U}_n$ and $KW_f \diamond KW_{\bigoplus_n}$.
- We wish to replace U_n and $KW_{\bigoplus_n}$ with MUX_n.

Research agenda

- Known: $KW_f \diamond \mathrm{U}_n$ and $KW_f \diamond KW_{\bigoplus_n}$.
- We wish to replace U_n and $KW_{\bigoplus_n}$ with MUX_n.
- Previous results work by taking a proof of a lower bound for KW_g and incorporating it in a lower bound for $KW_f \diamond KW_g$.

Research agenda

- Known: $KW_f \diamond \mathrm{U}_n$ and $KW_f \diamond KW_{\bigoplus_n}$.
- We wish to replace U_n and $KW_{\bigoplus_n}$ with MUX_n.
- Previous results work by taking a proof of a lower bound for KW_g and incorporating it in a lower bound for $KW_f \diamond KW_g$.
- We would like to do it for MUX_n.

Research agenda

- Known: $KW_f \diamond \mathrm{U}_n$ and $KW_f \diamond KW_{\bigoplus_n}$.
- We wish to replace U_n and $KW_{\bigoplus_n}$ with MUX_n.
- Previous results work by taking a proof of a lower bound for KW_g and incorporating it in a lower bound for $KW_f \diamond KW_g$.
- We would like to do it for MUX_n.
- Difficulty:
  - the lower bound proof for MUX_n is much more complicated,
  - so it is hard to incorporate it in a lower bound for $KW_f \diamond MUX_n$.

Research agenda

- We need to extend our techniques to handle more complicated lower bound proofs.

Research agenda

- We need to extend our techniques to handle more complicated lower bound proofs.

Intermediate goal

Prove the conjecture for KW_g's that are as complex as possible.

Research agenda

- We need to extend our techniques to handle more complicated lower bound proofs.

Intermediate goal

Prove the conjecture for KW_g's that are as complex as possible.

- Hopefully, this will allow us to develop our techniques further, until they are strong enough to deal with MUX_n.

Research agenda

- We need to extend our techniques to handle more complicated lower bound proofs.

Intermediate goal

Prove the conjecture for KW_g's that are as complex as possible.

- Hopefully, this will allow us to develop our techniques further, until they are strong enough to deal with MUX_n.
- Ideally, we should prove the conjecture for KW_g's that are

Research agenda

- We need to extend our techniques to handle more complicated lower bound proofs.

Intermediate goal

Prove the conjecture for KW_g's that are as complex as possible.

- Hopefully, this will allow us to develop our techniques further, until they are strong enough to deal with MUX_n.
- Ideally, we should prove the conjecture for KW_g's that are
  - more interesting than U_n and $KW_{\oplus_n}$,

Research agenda

- We need to extend our techniques to handle more complicated lower bound proofs.

Intermediate goal

Prove the conjecture for KW_g's that are as complex as possible.

- Hopefully, this will allow us to develop our techniques further, until they are strong enough to deal with MUX_n.
- Ideally, we should prove the conjecture for KW_g's that are
    - more interesting than U_n and $KW_{\oplus_n}$,
    - and less complicated than MUX_n.

Research agenda

- We need to extend our techniques to handle more complicated lower bound proofs.

Intermediate goal

Prove the conjecture for KW_g's that are as complex as possible.

- Hopefully, this will allow us to develop our techniques further, until they are strong enough to deal with MUX_n.
- Ideally, we should prove the conjecture for KW_g's that are
  - more interesting than U_n and $KW_{\oplus_n}$,
  - and less complicated than MUX_n.
- Problem: There are not many such KW_g's...

Outline

Moving to a different setting

- Problem: Not many KW relations are well-understood.

Moving to a different setting

- Problem: Not many KW relations are well-understood.
- Way to move forward: KW relations are much better understood in the monotone setting.

Moving to a different setting

- Problem: Not many KW relations are well-understood.
- Way to move forward: KW relations are much better understood in the monotone setting.
- In a new joint work with de Rezende, Nordström, Pitassi, and Robere, we consider

Moving to a different setting

- Problem: Not many KW relations are well-understood.
- Way to move forward: KW relations are much better understood in the monotone setting.
- In a new joint work with de Rezende, Nordström, Pitassi, and Robere, we consider
  - The monotone KRW conjecture.

Moving to a different setting

- Problem: Not many KW relations are well-understood.
- Way to move forward: KW relations are much better understood in the monotone setting.
- In a new joint work with de Rezende, Nordström, Pitassi, and Robere, we consider
  - The monotone KRW conjecture.
  - A new semi-monotone KRW conjecture.

Moving to a different setting

- Problem: Not many KW relations are well-understood.
- Way to move forward: KW relations are much better understood in the monotone setting.
- In a new joint work with de Rezende, Nordström, Pitassi, and Robere, we consider
  - The monotone KRW conjecture.
  - A new semi-monotone KRW conjecture.
- We prove these conjectures for inner functions g whose lower bound is obtained by lifting theorems.

Moving to a different setting

- Problem: Not many KW relations are well-understood.
- Way to move forward: KW relations are much better understood in the monotone setting.
- In a new joint work with de Rezende, Nordström, Pitassi, and Robere, we consider
  - The monotone KRW conjecture.
  - A new semi-monotone KRW conjecture.
- We prove these conjectures for inner functions g whose lower bound is obtained by lifting theorems.
- Much richer family than what was known before.

Monotone KW relations

- mD(f) — depth complexity by monotone circuits.

Monotone KW relations

- mD(f) — depth complexity by monotone circuits.
- Monotone KW relation mKW_f:
  - Alice gets $x \in f^{-1}(1)$, Bob gets $y \in f^{-1}(0)$.
  - Want to find i such that $x_i > y_i$.

Monotone KW relations

- $\mathsf{mD}(f)$ — depth complexity by monotone circuits.
- Monotone KW relation mKW_f:
  - Alice gets $x \in f^{-1}(1)$, Bob gets $y \in f^{-1}(0)$.
  - Want to find i such that $x_i > y_i$.
  - Such i must exist since f is monotone.

Monotone KW relations

- $\mathsf{mD}(f)$ — depth complexity by monotone circuits.
- Monotone KW relation mKW_f:
  - Alice gets $x \in f^{-1}(1)$, Bob gets $y \in f^{-1}(0)$.
  - Want to find i such that $x_i > y_i$.
  - Such i must exist since f is monotone.
- Theorem [KW88]: $\mathsf{mD}(f) = \mathsf{C}(mKW_f)$.

Monotone KW relations

- mD(f) — depth complexity by monotone circuits.
- Monotone KW relation mKW_f:
  - Alice gets $x \in f^{-1}(1)$, Bob gets $y \in f^{-1}(0)$.
  - Want to find i such that $x_i > y_i$.
  - Such i must exist since f is monotone.
- Theorem [KW88]: mD(f) = C(mKW_f).
- Used to prove lower bound on: stConnectivity [KW88], Clique [HG91, RW92], Matching [RW92].

Monotone KW relations

- mD(f) — depth complexity by monotone circuits.
- Monotone KW relation mKW_f:
  - Alice gets $x \in f^{-1}(1)$, Bob gets $y \in f^{-1}(0)$.
  - Want to find i such that $x_i > y_i$.
  - Such i must exist since f is monotone.
- Theorem [KW88]: mD$(f) =$ C(mKW_f).
- Used to prove lower bound on: stConnectivity [KW88], Clique [HG91, RW92], Matching [RW92].
- Used to separate the monotone **NC** hierarchy using the Generation function [RM97].

Lifting

- Lifting: a technique for proving lower bounds in communication complexity.

Lifting

- Lifting: a technique for proving lower bounds in communication complexity.
- Many applications: [RM97, SZ09, S11, GP14, GPW15, GJ16, GLMWZ16, RNV16, RPRC16, CKLM18, GGKS18, HHL18,...].

Lifting

- Lifting: a technique for proving lower bounds in communication complexity.
- Many applications: [RM97, SZ09, S11, GP14, GPW15, GJ16, GLMWZ16, RNV16, RPRC16, CKLM18, GGKS18, HHL18,...].
- Proves lower bounds on lifted problems.

Lifting

- Lifting: a technique for proving lower bounds in communication complexity.
- Many applications: [RM97, SZ09, S11, GP14, GPW15, GJ16, GLMWZ16, RNV16, RPRC16, CKLM18, GGKS18, HHL18,...].
- Proves lower bounds on lifted problems.
- We use two lifting theorems:

Lifting

- Lifting: a technique for proving lower bounds in communication complexity.
- Many applications: [RM97, SZ09, S11, GP14, GPW15, GJ16, GLMWZ16, RNV16, RPRC16, CKLM18, GGKS18, HHL18,...].
- Proves lower bounds on lifted problems.
- We use two lifting theorems:
  - Lifting from query complexity [CFKMP19], following [RM97, CKLM17, WYY17].

Lifting

- Lifting: a technique for proving lower bounds in communication complexity.
- Many applications: [RM97, SZ09, S11, GP14, GPW15, GJ16, GLMWZ16, RNV16, RPRC16, CKLM18, GGKS18, HHL18,...].
- Proves lower bounds on lifted problems.
- We use two lifting theorems:
  - Lifting from query complexity [CFKMP19], following [RM97, CKLM17, WYY17].
  - Lifting from Nullstellensatz degree [dRMNPRV19].

Monotone KRW conjecture

- Monotone KRW: $\mathsf{mD}(f \diamond g) \approx \mathsf{mD}(f) + \mathsf{mD}(g)$.
- Alternatively, $\mathsf{C}(mKW_f \diamond mKW_g) \approx \mathsf{C}(mKW_f) + \mathsf{C}(mKW_g)$.

Monotone KRW conjecture

- Monotone KRW: $\mathsf{mD}(f \diamond g) \approx \mathsf{mD}(f) + \mathsf{mD}(g)$.
- Alternatively, $\mathsf{C}(mKW_f \diamond mKW_g) \approx \mathsf{C}(mKW_f) + \mathsf{C}(mKW_g)$.
- We prove it for:
  - Every monotone f.
  - Every monotone g whose lower bound is proved by the lifting theorem of [CFKMP19].

Monotone KRW conjecture

- Monotone KRW: $\mathsf{mD}(f \diamond g) \approx \mathsf{mD}(f) + \mathsf{mD}(g)$.
- Alternatively, $\mathsf{C}(mKW_f \diamond mKW_g) \approx \mathsf{C}(mKW_f) + \mathsf{C}(mKW_g)$.
- We prove it for:
  - Every monotone f.
  - Every monotone g whose lower bound is proved by the lifting theorem of [CFKMP19].
  - (i.e., there is a reduction from a lifted problem to mKW_g).

Monotone KRW conjecture

- Monotone KRW: $\mathsf{mD}(f \diamond g) \approx \mathsf{mD}(f) + \mathsf{mD}(g)$.
- Alternatively, $\mathsf{C}(mKW_f \diamond mKW_g) \approx \mathsf{C}(mKW_f) + \mathsf{C}(mKW_g)$.
- We prove it for:
  - Every monotone f.
  - Every monotone g whose lower bound is proved by the lifting theorem of [CFKMP19].
  - (i.e., there is a reduction from a lifted problem to mKW_g).
- This includes g being stConnectivity, Clique, or Generation.

Monotone KRW conjecture

- Monotone KRW: $\mathsf{mD}(f \diamond g) \approx \mathsf{mD}(f) + \mathsf{mD}(g)$.
- Alternatively, $\mathsf{C}(mKW_f \diamond mKW_g) \approx \mathsf{C}(mKW_f) + \mathsf{C}(mKW_g)$.
- We prove it for:
  - Every monotone f.
  - Every monotone g whose lower bound is proved by the lifting theorem of [CFKMP19].
  - (i.e., there is a reduction from a lifted problem to mKW_g).
- This includes g being stConnectivity, Clique, or Generation.
- Basically, every monotone g for which we have a lower bound, except Matching.

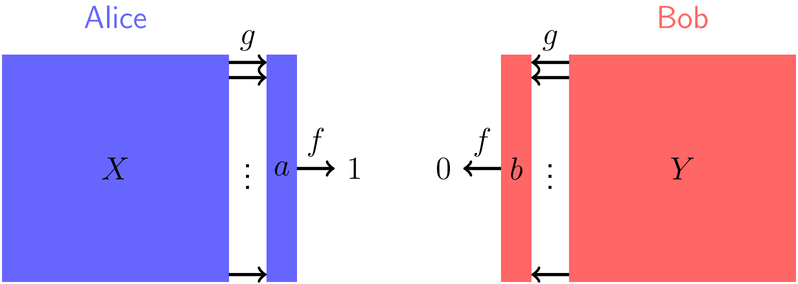
Semi-monotone composition

- We would like to go from monotone to non-monotone.

Semi-monotone composition

- We would like to go from monotone to non-monotone.
- We define a notion of "semi-monotone" composition:
  - f is non-monotone.
  - g is monotone.

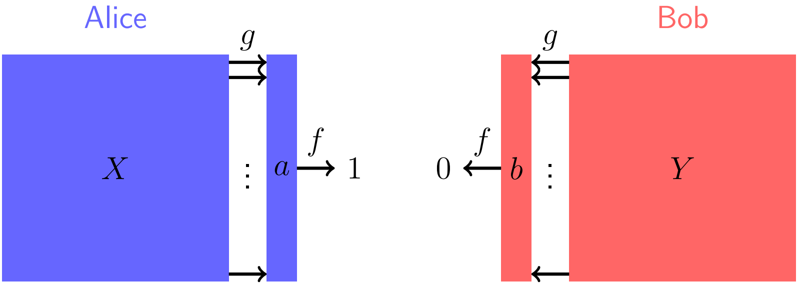
Semi-monotone composition



Recall: in the standard composition:

- Players can find row i s.t. $a_i \neq b_i$ by solving KW_f.
- On such row, they can solve KW_g.

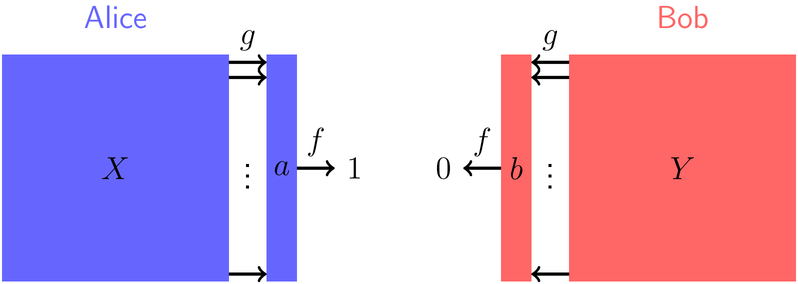
Semi-monotone composition



Recall: in the standard composition:

- Players can find row i s.t. $a_i \neq b_i$ by solving KW_f.
- On such row, they can solve KW_g.
- On other rows, they can try to solve the universal relation.

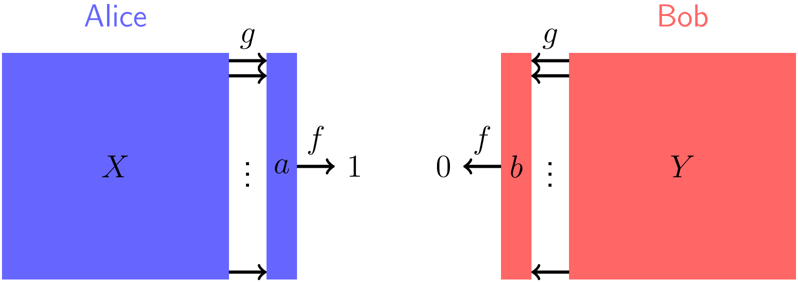
Semi-monotone composition



We define semi-monotone composition such that:

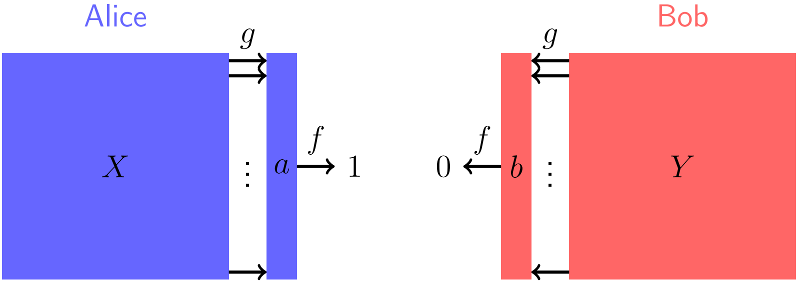
- Players can find row i s.t. $a_i \neq b_i$ by solving KW_f.
- On such row, they can solve mKW_g.
- On other rows, they can try to solve the universal relation.

Semi-monotone composition



Formally, in the semi-monotone relation $KW_f \diamond mKW_g$, Alice and Bob should output an entry (i, j) such that either

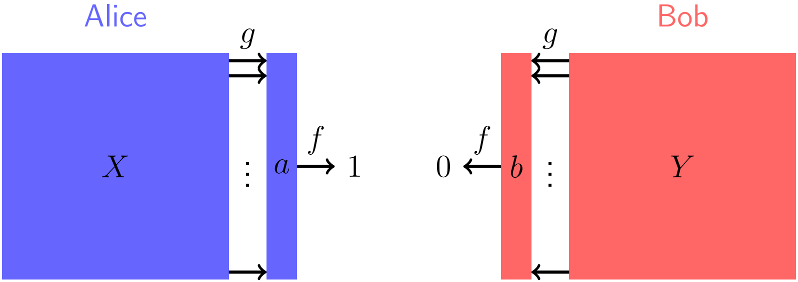
Semi-monotone composition



Formally, in the semi-monotone relation $KW_f \diamond mKW_g$, Alice and Bob should output an entry (i, j) such that either

- $a_i > b_i$ and $X_{i,j} > Y_{i,j}$, or

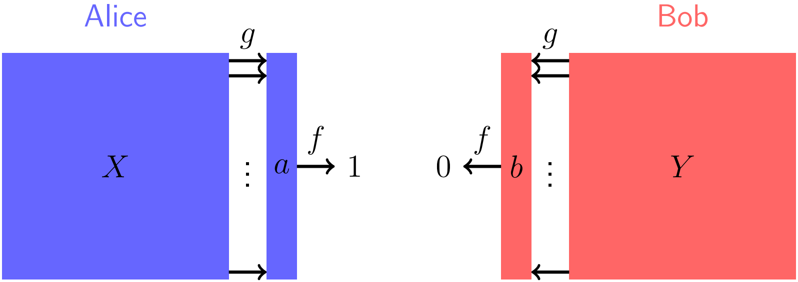
Semi-monotone composition



Formally, in the semi-monotone relation $KW_f \diamond mKW_g$, Alice and Bob should output an entry (i, j) such that either

- $a_i > b_i$ and $X_{i,j} > Y_{i,j}$, or
- $b_i > a_i$ and $Y_{i,j} > X_{i,j}$, or

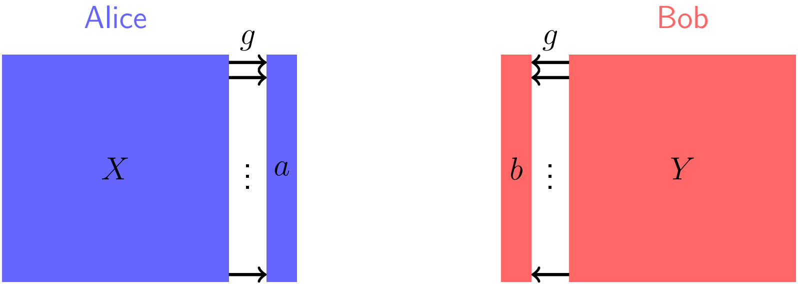
Semi-monotone composition



Formally, in the semi-monotone relation $KW_f \diamond mKW_g$, Alice and Bob should output an entry (i, j) such that either

- $a_i > b_i$ and $X_{i,j} > Y_{i,j}$, or
- $b_i > a_i$ and $Y_{i,j} > X_{i,j}$, or
- $a_i = b_i$ and $X_{i,j} \neq Y_{i,j}$.

Semi-monotone composition



The composition $\mathrm{U}_n \diamond mKW_g$ is defined similarly, but the promise

- $f(a) = 1$, $f(b) = 0$.

is replaced with the promise

- $a \neq b$.

Semi-monotone KRW conjecture

- Semi-monotone KRW:
  $\mathsf{C}(KW_f \diamond mKW_g) \approx \mathsf{C}(KW_f) + \mathsf{C}(mKW_g)$.

Semi-monotone KRW conjecture

- Semi-monotone KRW:
  $\mathsf{C}(KW_f \diamond mKW_g) \approx \mathsf{C}(KW_f) + \mathsf{C}(mKW_g)$.
- Simpler conjecture: $\mathsf{C}(\mathrm{U}_n \diamond mKW_g) \approx \mathsf{C}(\mathrm{U}_n) + \mathsf{C}(mKW_g)$.

Semi-monotone KRW conjecture

- Semi-monotone KRW:
  $\mathsf{C}(KW_f \diamond mKW_g) \approx \mathsf{C}(KW_f) + \mathsf{C}(mKW_g)$.
- Simpler conjecture: $\mathsf{C}(\mathrm{U}_n \diamond mKW_g) \approx \mathsf{C}(\mathrm{U}_n) + \mathsf{C}(mKW_g)$.
- We prove the latter for every monotone g whose lower bound is proved by the lifting theorem of [dRMNPRV19].

Semi-monotone KRW conjecture

- Semi-monotone KRW: $\mathsf{C}(KW_f \diamond mKW_g) \approx \mathsf{C}(KW_f) + \mathsf{C}(mKW_g)$.
- Simpler conjecture: $\mathsf{C}(\mathrm{U}_n \diamond mKW_g) \approx \mathsf{C}(\mathrm{U}_n) + \mathsf{C}(mKW_g)$.
- We prove the latter for every monotone g whose lower bound is proved by the lifting theorem of [dRMNPRV19].
- Again, this includes g being stConnectivity, Clique, or Generation, but not Matching.

Proof of the monotone result

By combining [KRW91] with [EIRS91], we show that it suffices to prove a lower bound on a simpler problem:

Proof of the monotone result

By combining [KRW91] with [EIRS91], we show that it suffices to prove a lower bound on a simpler problem:

- Given several instances of mKW_g, solve one of them...

Proof of the monotone result

By combining [KRW91] with [EIRS91], we show that it suffices to prove a lower bound on a simpler problem:

- Given several instances of mKW_g, solve one of them...
- assuming the players already know a few bits of information on each instance.

Proof of the monotone result

By combining [KRW91] with [EIRS91], we show that it suffices to prove a lower bound on a simpler problem:

- Given several instances of mKW_g, solve one of them...
- assuming the players already know a few bits of information on each instance.

We establish this lower bound by proving a generalization of the lifting theorem of [CFKMP19].

Proof of semi-monotone result

- The Razborov rank method: Proves a lower bound on a communication problem by exhibiting a matrix A such that

Proof of semi-monotone result

- The Razborov rank method: Proves a lower bound on a communication problem by exhibiting a matrix A such that
  - The rank of A is large.

Proof of semi-monotone result

- The Razborov rank method: Proves a lower bound on a communication problem by exhibiting a matrix A such that
  - The rank of A is large.
  - The rank of $A|_R$ is small for every monochromatic rectangle R.

Proof of semi-monotone result

- The Razborov rank method: Proves a lower bound on a communication problem by exhibiting a matrix A such that
  - The rank of A is large.
  - The rank of $A|_R$ is small for every monochromatic rectangle R.
- The lifting theorem of [dRMNPRV19] gives us such a matrix A for mKW_g.
- We use it to construct a matrix for $U_m \diamond mKW_g$.

Summary

- The KRW conjecture is a promising approach for proving $\mathbf{P} \neq \mathbf{NC}^1$.

Summary

- The KRW conjecture is a promising approach for proving $\mathbf{P} \neq \mathbf{NC}^1$.
- We know how to prove it when the inner function is
    - the universal relation,
    - the parity function (and related functions).

Summary

- The KRW conjecture is a promising approach for proving $\mathbf{P} \neq \mathbf{NC}^1$.
- We know how to prove it when the inner function is
  - the universal relation,
  - the parity function (and related functions).
- Possible approach to $\mathbf{P} \neq \mathbf{NC}^1$:
  - Prove the KRW conjecture when the inner function is the multiplexor relation.

Summary

- The KRW conjecture is a promising approach for proving $\mathbf{P} \neq \mathbf{NC}^1$.
- We know how to prove it when the inner function is
  - the universal relation,
  - the parity function (and related functions).
- Possible approach to $\mathbf{P} \neq \mathbf{NC}^1$:
  - Prove the KRW conjecture when the inner function is the multiplexor relation.
- New work: For every monotone g whose lower bound is shown by lifting, we prove

Summary

- The KRW conjecture is a promising approach for proving $\mathbf{P} \neq \mathbf{NC}^1$.
- We know how to prove it when the inner function is
  - the universal relation,
  - the parity function (and related functions).
- Possible approach to $\mathbf{P} \neq \mathbf{NC}^1$:
  - Prove the KRW conjecture when the inner function is the multiplexor relation.
- New work: For every monotone g whose lower bound is shown by lifting, we prove
  - The monotone KRW conjecture for $mKW_f \diamond mKW_g$ for every monotone f.

Summary

- The KRW conjecture is a promising approach for proving $\mathbf{P} \neq \mathbf{NC}^1$.
- We know how to prove it when the inner function is
  - the universal relation,
  - the parity function (and related functions).
- Possible approach to $\mathbf{P} \neq \mathbf{NC}^1$:
  - Prove the KRW conjecture when the inner function is the multiplexor relation.
- New work: For every monotone g whose lower bound is shown by lifting, we prove
  - The monotone KRW conjecture for $mKW_f \diamond mKW_g$ for every monotone f.
  - The semi-monotone KRW conjecture for $U_m \diamond mKW_g$.

Open problems

- The semi-monotone conjecture for $KW_f \diamond mKW_g$ for
  - every f, and
  - a rich family of g's.

Open problems

- The semi-monotone conjecture for $KW_f \diamond mKW_g$ for
  - every f, and
  - a rich family of g's.
- The monotone conjecture with $g = \textsc{Matching}$.

Open problems

- The semi-monotone conjecture for $KW_f \diamond mKW_g$ for
  - every f, and
  - a rich family of g's.
- The monotone conjecture with $g = \textsc{Matching}$.
- The monotone conjecture with $g =$monotone multiplexor.

Open problems

- The semi-monotone conjecture for $KW_f \diamond mKW_g$ for
  - every f, and
  - a rich family of g's.
- The monotone conjecture with $g = \textsc{Matching}$.
- The monotone conjecture with $g =$monotone multiplexor.
- The non-monotone conjecture for $U_m \diamond MUX_n$.

Thank you!