A KRW-like theorem for Strong Composition

Or Meir

Outline

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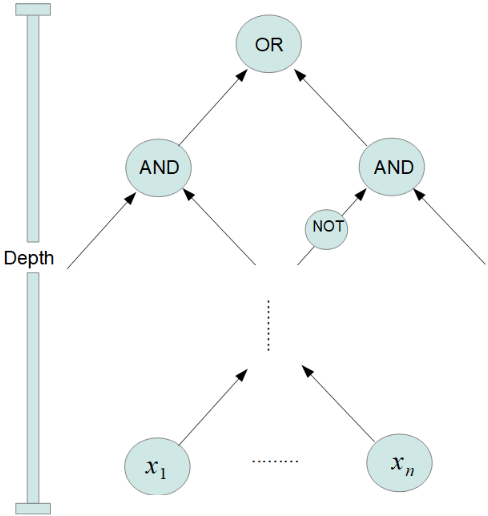
Circuit depth



Fan-in 2: Every gate has at most 2 incoming wires.

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- State of the art: $D(f) \geq (3 - o(1)) \cdot \log n$ [H93, T14].

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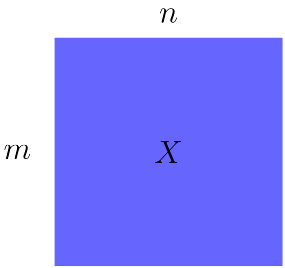

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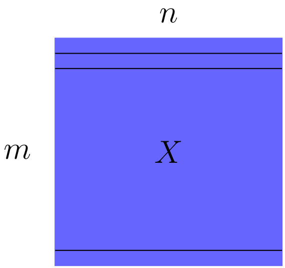

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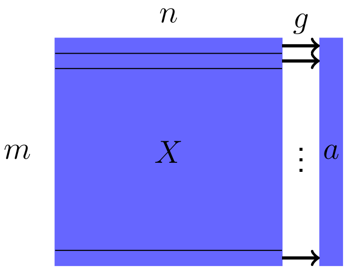

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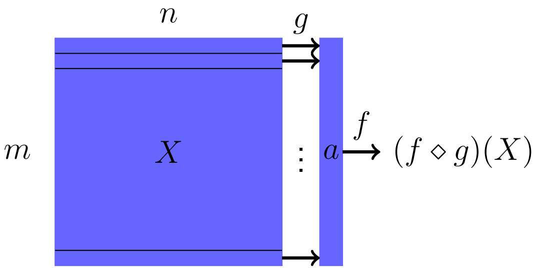

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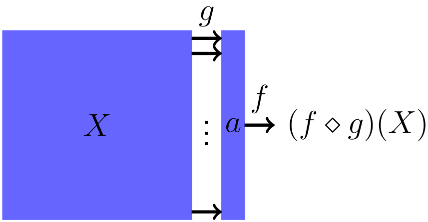

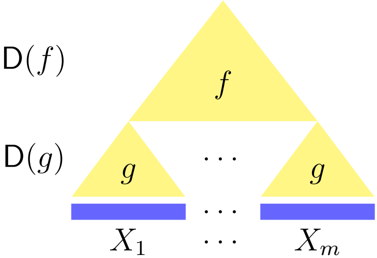

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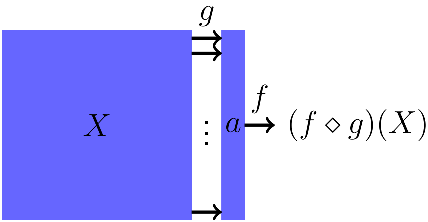

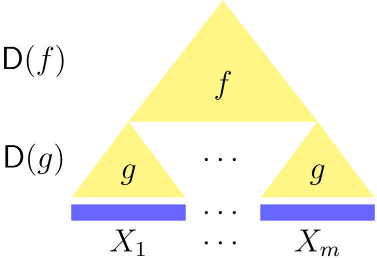

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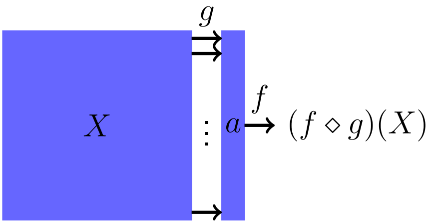

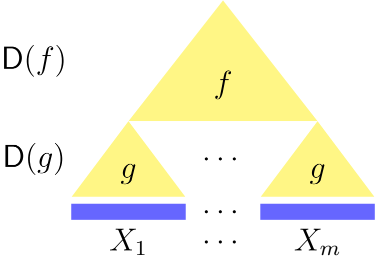

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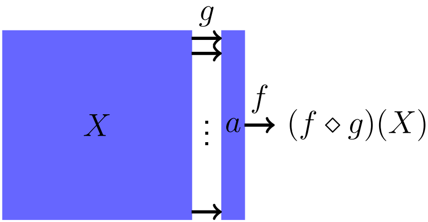

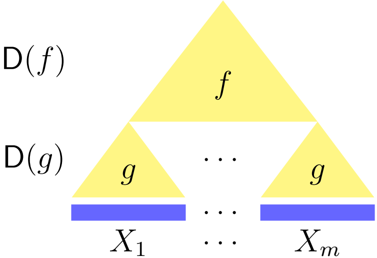

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- Special cases: [EIRS91, H93, HW93, GMWW14, DM16, KM18, dRMNPR20, FMT21].

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- [MS21]: proved such a result for $U \diamond g$.
  - $U =$ the universal relation.

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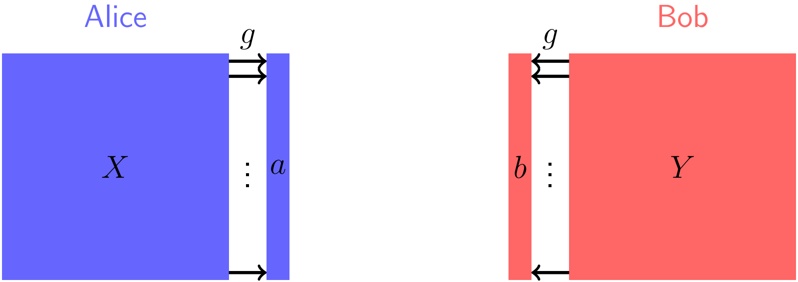

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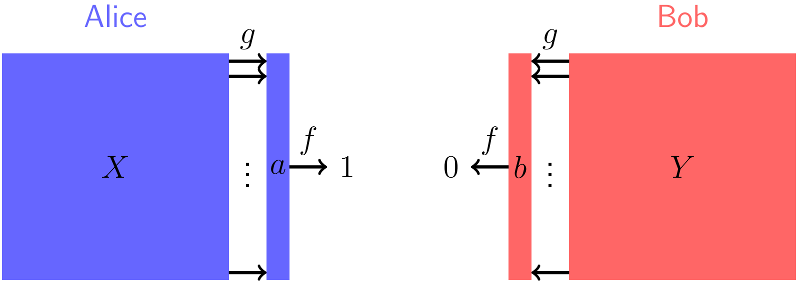

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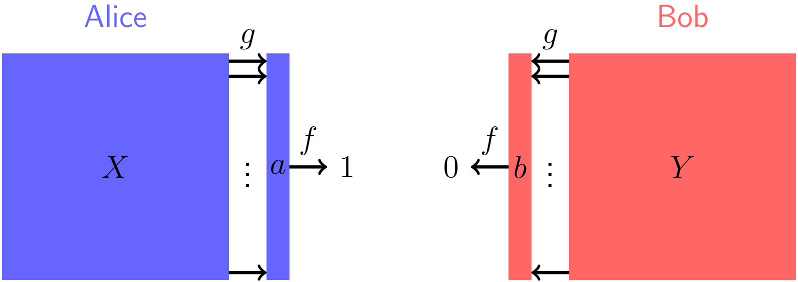

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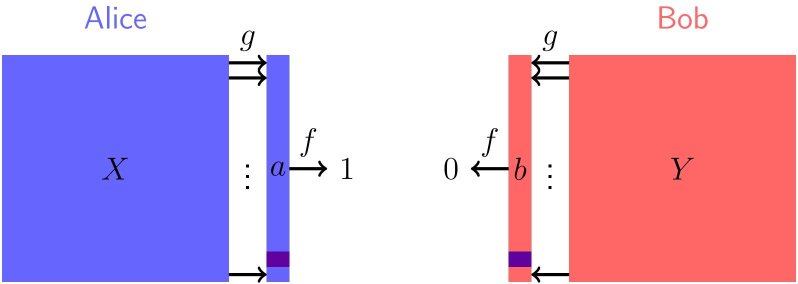

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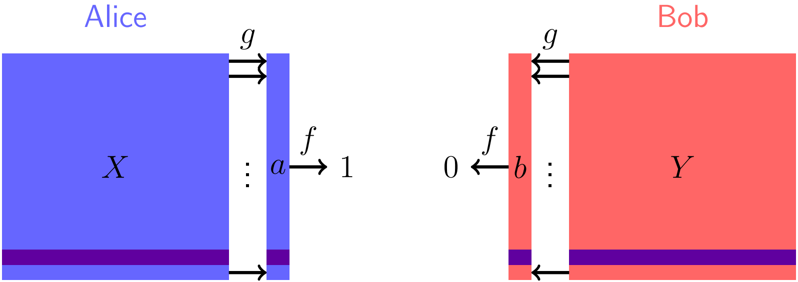

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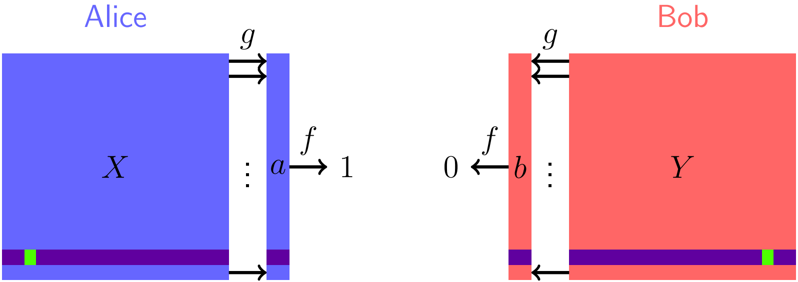

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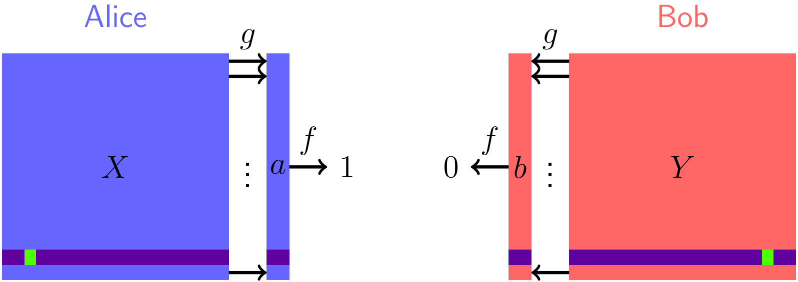

- KRW conjecture: the obvious protocol is essentially optimal.

Outline

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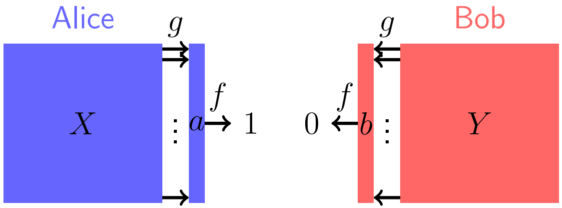

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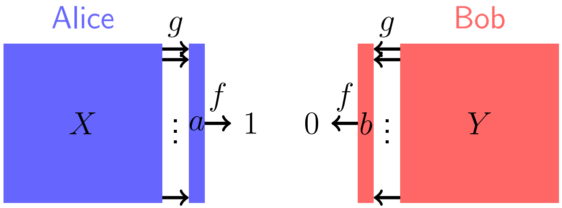

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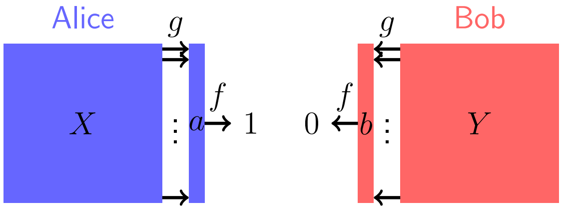

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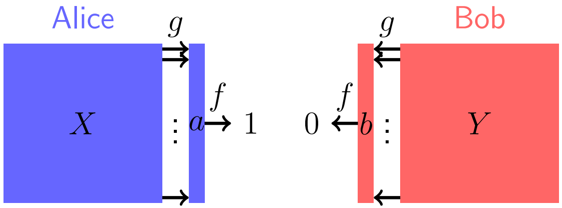

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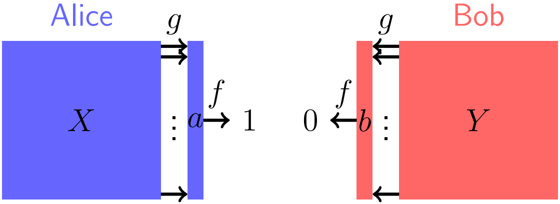

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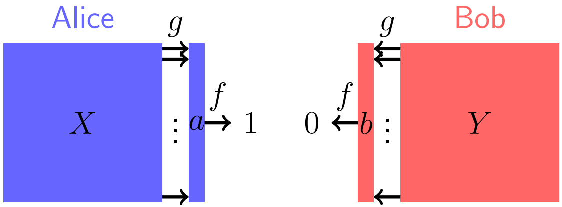

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In this work, we focus on the direct-sum problem.

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For every $f : \{0,1\}^m \to \{0,1\}$ and every $n \in \mathbb{N}$,
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- First improvement in depth lower bounds since [H93]!
- Insufficient for proving $\mathbf{P} \not\subseteq \mathbf{NC}^1$ due to $-0.96 \cdot m$.

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Multiplexor composition

- Fix a function $f : \{0,1\}^m \to \{0,1\}$ and $n \in \mathbb{N}$.
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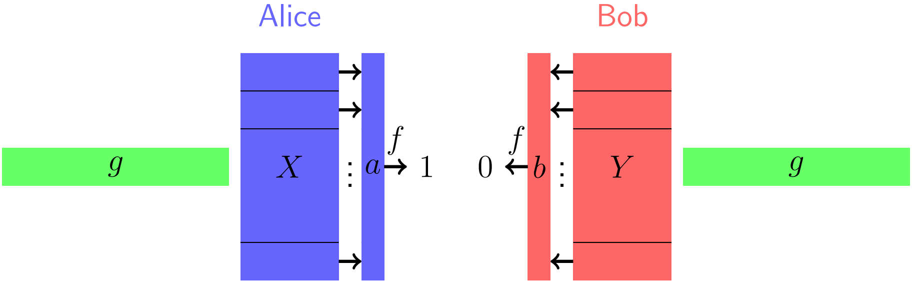
- Goal: $\exists\, g : \{0,1\}^n \to \{0,1\}$ s.t. $\mathsf{CC}(\mathit{KW}_f \circledast \mathit{KW}_g)$ is large.
- Define the composition $\mathit{KW}_f \circledast \mathit{MUX}_n$:



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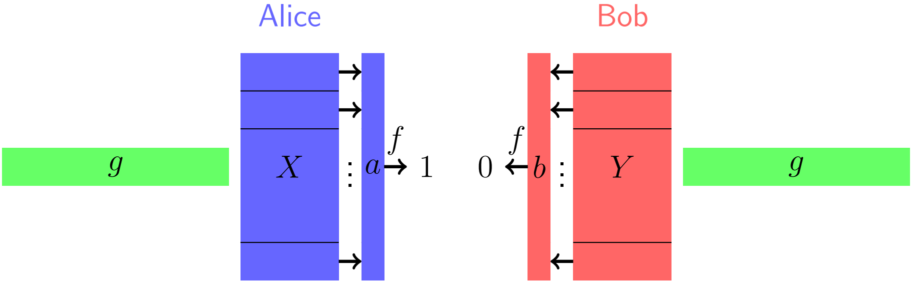

- Suffices [MS21]*: $\mathsf{CC}(KW_f \circledast MUX_n) > \mathsf{CC}(KW_f) + n -$ loss.

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- Fix a protocol Π for $KW_f \circledast MUX_n$.
- Roughly, we prove that:
  - as long as Π does not finish solving KW_f,
  - it cannot make progress on KW_g.

Proof strategy [EIRS91]

Structure theorem (informal)

Let π_1 be a partial transcript s.t.

- π_1 is still far from solving KW_f, and
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Then, after reaching π_1, the players must still communicate $\approx n$ more bits.

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- It is not hard to show that there exists such π_1 of length $\mathsf{CC}(KW_f) - \mathsf{loss}$.
- By applying the theorem, we get a lower bound of

$$\approx \mathsf{CC}(KW_f) + n - \mathsf{loss}.$$

Outline

Intersecting functions

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Definition

We say that $g_1, g_2 : \{0,1\}^n \to \{0,1\}$ intersect iff

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If $\exists$ a set $\mathcal{V}$ of functions s.t. $\forall$ distinct $g_1, g_2 \in \mathcal{V}$ intersect, then the players must send $\gtrapprox \log\log|\mathcal{V}|$ more bits after reaching π_1.

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- To use lemma, need to construct $\mathcal{V}$ s.t. $|\mathcal{V}| \approx 2^{2^n}$.
- Difficulty: need that every two functions in $\mathcal{V}$ intersect.

A graph-theoretic perspective

Definition

The characteristic graph $\mathcal{G}_{\pi_1}$ satisfies:

- The vertices are all functions $g : \{0,1\}^n \to \{0,1\}$.
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Lemma (this work)

The players must send $\gtrapprox \log \log \chi(\mathcal{G}_{\pi_1})$ more bits
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- or vice versa.

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- Warm-up: prove that there exist $X \in \mathcal{X}(g_1)$ and $Y \in \mathcal{Y}(g_2)$ that are equal on $\geq \alpha \cdot m$ rows.

A simpler combinatorial question

- Let Σ be a finite alphabet.
- Let $\mathcal{X}, \mathcal{Y} \subseteq \Sigma^m$ be sets of strings of density $\geq 2^{-\varepsilon \cdot m}$ (for some $\varepsilon > 0$).

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- Idea: choose I such that $\mathcal{X}|_I$ and $\mathcal{Y}|_I$ are "prefix-thick sets".

Prefix-thick sets

Definition

We say that $\mathcal{X} \subseteq \Sigma^m$ is prefix thick iff for every prefix w of $\mathcal{X}$ of length $< m$, there exist more than $\frac{|\Sigma|}{2}$ symbols σ such that $w \circ \sigma$ is a prefix of $\mathcal{X}$.

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Observation

If $\mathcal{X}$ and $\mathcal{Y}$ are prefix-thick subsets of Σ^m, then $\mathcal{X} \cap \mathcal{Y} \neq \emptyset$.

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- Proof: Easy corollary of a result of [ST14] about discrete dynamical systems.
- Can be viewed as a generalization of the Sauer-Shelah lemma to large alphabets.

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- and hence get the desired lower bound on communication complexity.

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Thank you!