Demystifying the border of depth-3 algebraic circuits

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Oxford-Warwick Complexity Seminar

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Algebraic Complexity Theory

Basic goal

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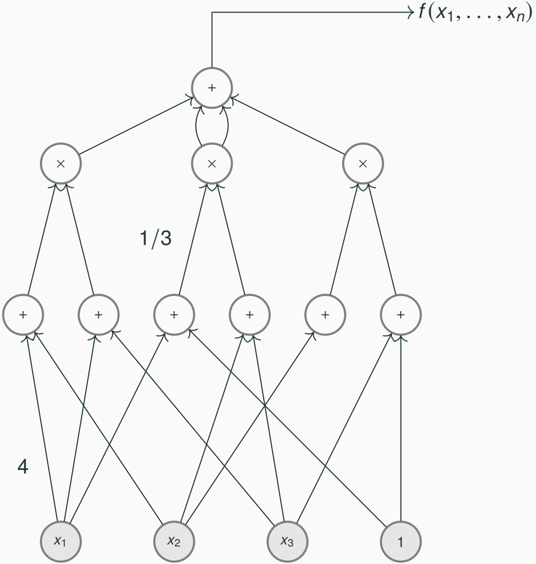
- [$\mathsf{P} \stackrel{?}{=} \mathsf{NP}$, Aronson 2011] calls GCT *"The String Theory of Computer Science"*.

Algebraic circuits



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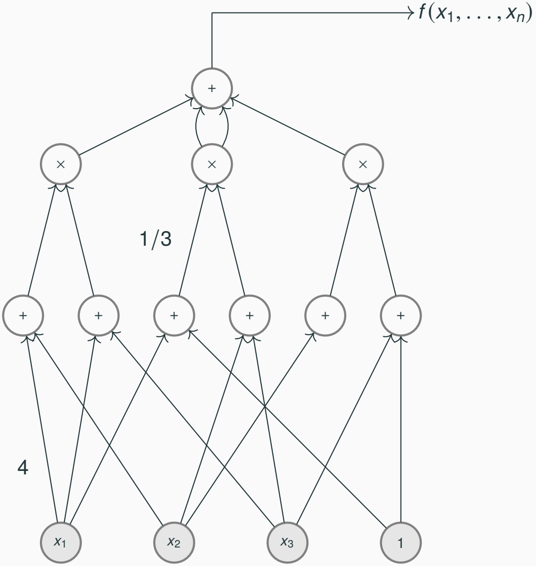

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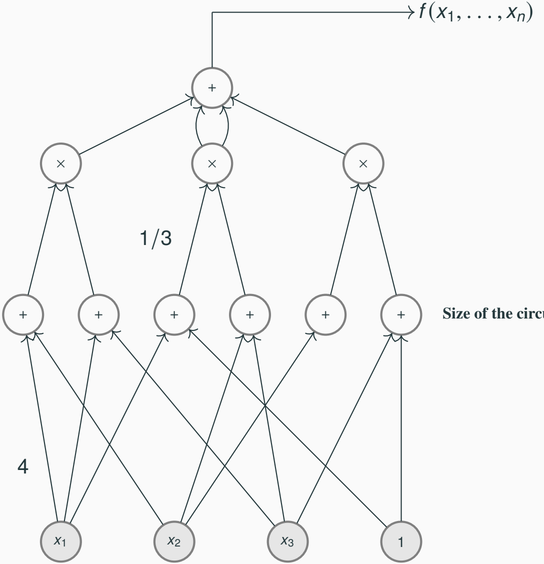
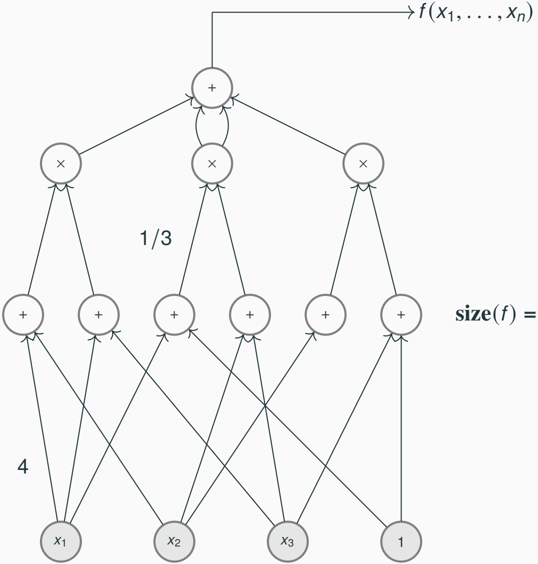


Size of the circuit = number of nodes + edges



size(f) = min size of the circuit computing f

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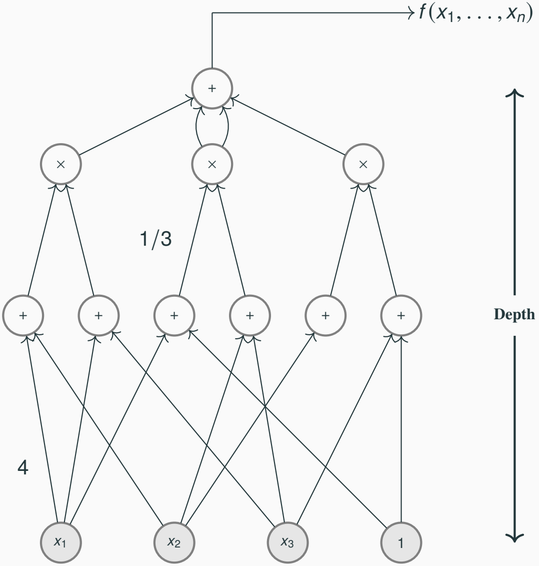

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The class VP is defined as the set of all sequences of polynomials $(f_n(x_1, \ldots, x_n))_{n \geq 1}$ such that $\mathrm{size}(f_n), \mathsf{deg}(f_n)$ are both bounded by n^c for some constant c.

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- E.g. $\mathrm{dc}(x_1 \cdots x_n) = n$, since

$$x_1 \cdots x_n = \det \begin{pmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & x_n \end{pmatrix} .$$

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- Connections: Linear algebra, Volume, counting planar matchings.

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$\mathsf{VBP} \neq \mathsf{VNP}$ & $\mathsf{VP} \neq \mathsf{VNP}$. Equivalently, $\mathsf{dc}(\mathsf{perm}_n)$ and $\mathsf{size}(\mathsf{perm}_n)$ are both $n^{\omega(1)}$.

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  - Assuming GRH (Generalized Riemann hypothesis), the results hold over $\mathbb{C}$ as well.

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- Can there be 'algebraic natural proofs' to prove $\mathsf{VP} \neq \mathsf{VNP}$? Some answers: [Chatterjee-Kumar-Ramya-Saptharishi-Tengse 2020, Kumar-Ramya-Saptharishi-Tengse 2020].

Border Complexity and GCT

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- $\mathsf{WR}(h) \leq r$ is denoted as $h \in \Sigma^{[r]} \wedge \Sigma$ (homogeneous *depth-3 diagonal* circuits).

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- X_1 is a *closed* set. If there are three sequences (a_n, b_n, c_n) such that $a_n \to a, b_n \to b, c_n \to c$, i.e. limits exist, such that $(a_n, b_n, c_n) \in X_1$, then $(a, b, c) \in X_1$.

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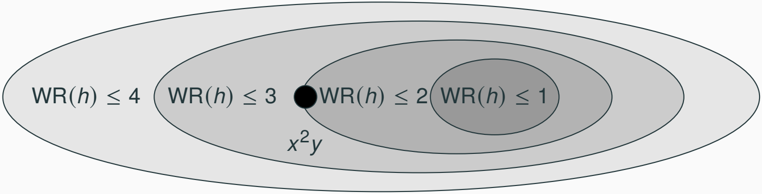

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- On to proving lower bounds: To show $\overline{\mathsf{WR}}(p) > n$, for some p, it suffices to show that $p \notin X_n$, i.e. find a *continuous* function f that vanishes on X_n but not on p.

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  Border tensor rank is *directly* related to the matrix multiplication exponent ω [Bini 1980, Coppersmith-Winograd 1990].

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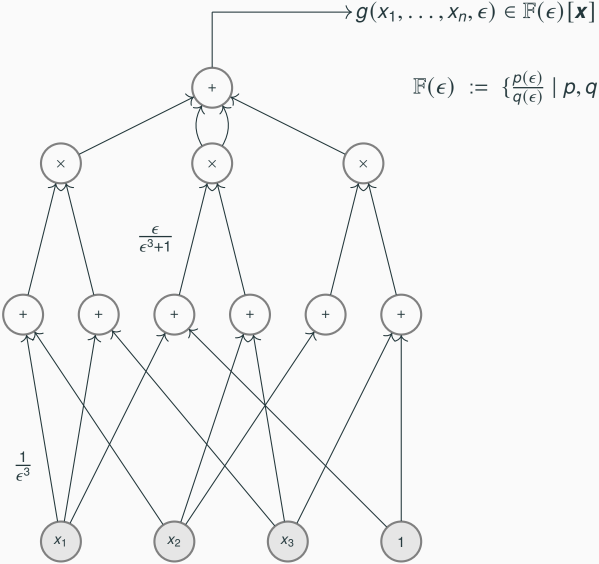
- Yes! Via *'approximative circuits'*.



$$\mathbb{F}(\epsilon) := \{\tfrac{p(\epsilon)}{q(\epsilon)} \mid p, q \in \mathbb{F}[\epsilon], q(\epsilon) \neq 0\}$$

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- Summary: g_0 is really something **non-trivial** and being 'approximated' by the circuit since $\lim_{\epsilon \to 0} g(\boldsymbol{x}, \epsilon) = g_0$.

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- Both det and perm have 'nice' **symmetries**.
- Symmetry-characterization **avoids** the Razborov–Rudich barrier: *Very few* functions are symmetry-characterized, so symmetry-characterization violates the largeness criterion!

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- Upper bounds and lower bounds are *dual* to each other.
- Further potential applications in identity testing and understanding its 'robustness'.

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- **No**. E.g. the *Inner Product* polynomial $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = x_1y_1 + \ldots + x_{k+1}y_{k+1}$ **cannot** be written as a $\Sigma^{[k]}\Pi^{[d]}\Sigma$ circuit, *regardless* of the product fan-in d!

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Border depth-3 fan-in 2 circuits are 'universal' [Kumar 2020]

Let P be *any* homogeneous n-variate degree d polynomial. Then, $P \in \overline{\Sigma^{[2]}\Pi^{[D]}\Sigma}$, where $D := \exp(n, d)$.

De-bordering $\overline{\Sigma^{[2]}\Pi\Sigma}$ circuits

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Remark. The result holds if one replaces the top-fanin-2 by arbitrary constant k.

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- We devise a technique called DiDIL - **Di**vide, **D**erive, **I**nterpolate with **L**imit.

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$$
\begin{aligned}
f + \epsilon \cdot S =&\ T_1 + T_2 \\
\implies \Phi(f) + \epsilon \cdot \Phi(S) =&\ \Phi(T_1) + \Phi(T_2) \\
\implies \Phi(f)/\tilde{T}_2 + \epsilon \cdot \Phi(S)/\tilde{T}_2 =&\ \epsilon^{a_2} + \Phi(T_1)/\tilde{T}_2 \\
\implies \partial_z\left(\Phi(f)/\tilde{T}_2\right) + \epsilon \cdot \partial_z\left(\Phi(S)/\tilde{T}_2\right) =&\ \partial_z\left(\Phi(T_1)/\tilde{T}_2\right) =: g_1\ . \qquad (1)
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$$\begin{aligned} f + \epsilon \cdot S &= T_1 + T_2 \\ \implies \Phi(f) + \epsilon \cdot \Phi(S) &= \Phi(T_1) + \Phi(T_2) \\ \implies \Phi(f)/\tilde{T}_2 + \epsilon \cdot \Phi(S)/\tilde{T}_2 &= \epsilon^{a_2} + \Phi(T_1)/\tilde{T}_2 \\ \implies \partial_z\left(\Phi(f)/\tilde{T}_2\right) + \epsilon \cdot \partial_z\left(\Phi(S)/\tilde{T}_2\right) &= \partial_z\left(\Phi(T_1)/\tilde{T}_2\right) =: g_1 \,. \end{aligned} \tag{1}$$

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- Here Σ means just a linear polynomial ℓ.

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- Suffices to compute $\lim_{\epsilon \to 0} g_1 \mod z^d$.

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$$\begin{aligned}\mathrm{dlog}(A - zB) &= -\frac{B}{A\left(1 - z \cdot B/A\right)} \\ &= -\frac{B}{A} \cdot \sum_{j=0}^{d-1} \left(\frac{z \cdot B}{A}\right)^j \mod z^d \\ &\in \Sigma \wedge \Sigma \,.\end{aligned}$$

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Thus,

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- Thus, $1/t_2 \cdot \partial_z(\Phi(f)) = \lim_{\epsilon \to 0} g_1 = \mathsf{ABP}$.
- Thus, $\Phi(f)/t_2 = \mathsf{ABP} \implies \Phi(f) = \mathsf{ABP} \implies f = \mathsf{ABP}$.

Derandomizing border depth-3 circuits

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  - Primality Testing [Agrawal-Kayal-Saxena'04].

Border PIT

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Border hitting set

$\mathcal{H}$ is a hitting set for a class $\overline{\mathcal{C}}$, if $g(x, \epsilon) \in \mathcal{C}_{\mathbb{F}(\epsilon)}$ approximates a non-zero polynomial $h(\boldsymbol{x})$, then $\exists \boldsymbol{a} \in \mathcal{H}$ such that $g(\boldsymbol{a}, \epsilon) \notin \epsilon \cdot \mathbb{F}[\epsilon]$, i.e. $h(\boldsymbol{a}) \neq 0$.

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Theorem 2 (Derandomizing polynomal-sized depth-3 top-fanin-*k* circuits) [Dutta-Dwivedi-Saxena 2021]

There exists an explicit quasipolynomial-time ($s^{O(\log\log s)}$) hitting set for size-s $\overline{\Sigma^{[k]}\Pi\Sigma}$ circuits, for any constant k.

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Thank you & stay safe!