On One-way Functions and Kolmogorov Complexity

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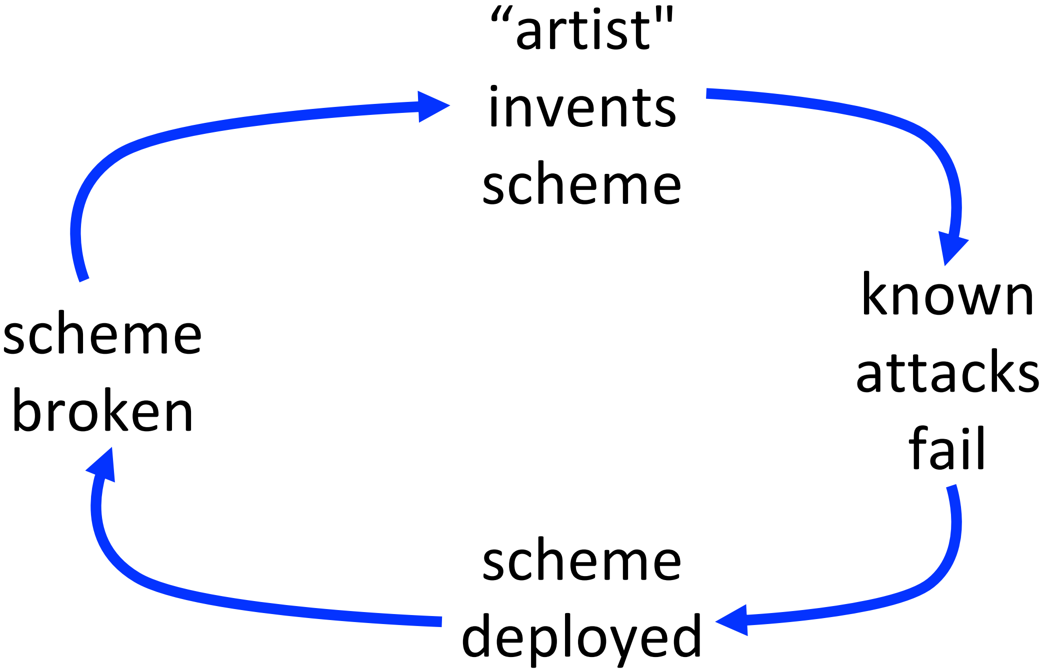
The “Dark Ages” Crypto Cycle

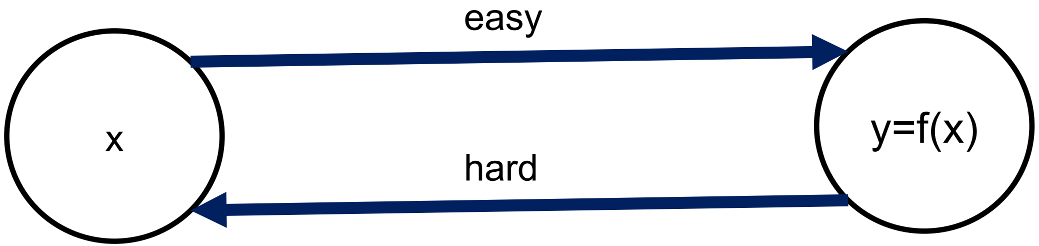
(the last 2000 years)



One-way Functions (OWF) [Diffie-Hellman'76]

A function **f** that is

- **Easy to compute**: can be computed in poly time
- **Hard to invert**: no PPT can invert it



Ex [Factoring]: use x to pick to 2 random "large" primes p,q, and output y = p* q

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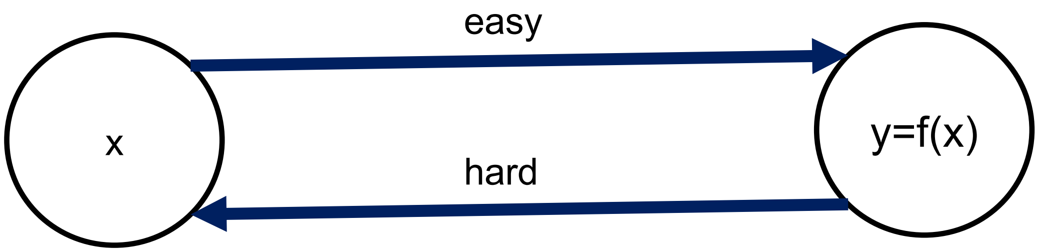

Definition 2.1. *Let* $f : \{0,1\}^* \to \{0,1\}^*$ *be a polynomial-time computable function.* f *is said to be a* one-way function (OWF) *if for every* PPT *algorithm* $\mathcal{A}$, *there exists a negligible function* μ *such that for all* $n \in \mathbb{N}$,

$$\Pr[x \leftarrow \{0,1\}^n; y = f(x) : A(1^n, y) \in f^{-1}(f(x))] \leq \mu(n)$$

One-way Functions (OWF) [Diffie-Hellman'76]

A function **f** that is
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OWF both necessary [IL'89] and sufficient for:
- Private-key encryption [GM84,HILL99]
- Pseudorandom generators [HILL99]
- Digital signatures [Rompel90]
- Authentication schemes [FS90]
- Pseudorandom functions [GGM84]
- Commitment schemes [Naor90]
- Coin-tossing [Blum'84]
- ZK proofs [GMW89]
- …

Not included:
public-key encryption, OT, obfuscation

Whether OWF exists is the most important problem in Cryptography

OWF v.s NP Hardness

Observation: OWF => NP ∉ BPP

“Holy grail” [DH’76]

Prove: NP ∉ BPP => OWF

Lots of **partial** BB “separations”: [Bra’79],[AGGM’06],[P’07],[MX’10]

In the absence of the holy-grail…

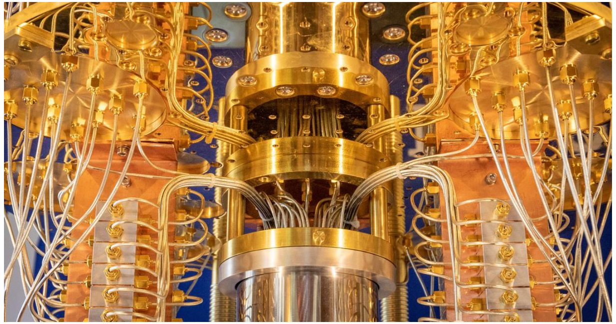
Discrete Logarithm Problem [DH'76]

Factoring [RSA'83]

Lattice Problems [Ajtai'96]

DES,
SHA,
AES…

QUANTUM COMPUTERS



So far, not broken…but for how long?
"Cryptographers seldom sleep well" - Micali'88

Have we really escaped from the "crypto cycle"?

In the absence of the holy-grail…

Discrete Logarithm Problem [DH'76]

Factoring [RSA'83]

Lattice Problems [Ajtai'96]

DES,
SHA,
AES…

Central question: Does there exist some **natural average-case hard problem** (a "mother problem") that **characterizes existence of OWF?**

Main Theorem

For every polynomial $t(n)>1.1n$:

OWFs exist iff **t-bounded Kolmogorov-complexity** is mildly hard-on-average

Kolmogorov Complexity [Sol'64,Kol'68,Cha'69]

Which of the following strings is more "random":

- 123123123123123231
- 173054459347394037

K(x) = length of the shortest program that outputs **x**

Formally, we fix a universal TM U, and are looking for the length of the shortest program Π = (M,w) s.t. U(M,w) = x

Lots of amazing applications (e.g., Godel's incompleteness theorem)
But **uncomputable**.

Time-Bounded Kolmogorov Complexity

Which of the following strings is more "random":

- 123123123123123123123
- 173054445934739403 7

$K(x)$ = length of the shortest program that outputs x

$K^t(x)$ = length of the shortest program that outputs x within time $t(|x|)$

Can K^t be **efficiently computed** when $t(n)$ is a polynomial?

- Studied in the Soviet Union since 60s [Kol'68,T'84]
- Independently by Hartmanis [83], Sipser [83], Ko [86]
- Closely related to **MCSP** (Minimum Circuit Size Problem) [T'84,KC'00]

Average-case Hardness of K^t

Frequential version [60's, T'84]
Does ∃ algorithm that computes **$K^t(x)$ for a "large" fraction of x's**?

Observation [60's, T'84]: **K^t** can be approximated within d log n w.p $1-1/n^d$
Proof: simply output n.

Def: **K^t** is **mildly-HOA** if there exists a polynomial p, such that no PPT heuristic H can compute **K^t** w.p 1-1/p(n) over random strings x for inf many n.

Def: **K^t** is **mildly-HOA to c-approximate** if there exists a polynomial p, such that no PPT heuristic H can c-approximate **K^t** w.p 1-1/p(n) over random strings x for inf many n.

Main Theorem

The following are equivalent:

1. **OWFs** exist
2. ∃ poly $t(n)>0$, s.t. **K^t is mildly-HOA**.
3. ∀ $c>0$, $\varepsilon>0$, poly $t(n)>(1+\varepsilon)\, n$,
   K^t is mildly-HOA to ($c\log n$)-approx.

Main Theorem

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3. ∀ $c>0$, $\varepsilon>0$, poly $t(n)>(1+\varepsilon)\,n$,
   K^t is mildly-HOA to (clog n)-approx.

Corr: For all poly $t(n)>(1+\varepsilon)n$,
OWFs exists iff K^t is mildly hard-on-average

Corr: For all $c>0$, $\varepsilon>0$, poly $t(n)>(1+\varepsilon)\,n$,
K^t is mildly hard-on-average to (clog n)-approx iff K^t is mildly hard-on-average.

Earlier Connections between OWF and K^t

- [RR'97,KC00,ABK+06]: OWF $\Longrightarrow$ exists poly t s.t K^t is ***worst-case*** hard
  - converse direction not known
  - this will be our starting point to showing OWF $\Longrightarrow$ K^t is **HOA**

- [Santhanam'19]: Under a new conjecture, MCSP is "errorless-HOA" iff OWF exists
  - as mentioned, MCSP is closely related to K^t
  - in contrast, our results are unconditional.

Main Theorem

The following are equivalent:

1. **OWFs** exist
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3. ∀ $c>0$, $\varepsilon>0$, poly $t(n)>(1+\varepsilon)\, n$,
   K^t is mildly-HOA to ($c\log n$)-approx.

Proof: (2) => (1) => (3)

Today: just sketch (1) <=> (2)

Theorem 1

Assume there exists some poly t(n)>0, s.t. **K^t is mildly-HOA**.
Then OWFs exist.

Theorem 2

Assume OWFs exists.
Then there exists some poly t(n)>0 s.t. **K^t is mildly-HOA**.

Theorem 1

Assume there exists some poly t(n)>0, s.t. **K^t is mildly-HOA**.
Then OWFs exist.

Weak OWF: "mild-HOA version" of a OWF:
efficient function f s.t. no PPT can invert f w.p. **1-1/p(n)**
for inf many n, for some poly p(n)>0.

Lemma [Yao'82]. If a Weak OWF exists, then a OWF exists.

So, we just need to construct a weak OWF.

Let c be a constant so that $K^t(x) < |x|+c$ for all x

Define **f(Π’,i)** where |**Π**’| = n, |i| = log (n+c) as follows:

- Let **Π** = first i bits of **Π**’ (i.e., truncate **Π**’ to i bits).
- Let y = output of **Π** after t(n) steps.
- Output i||y.

Assume for contradiction that f is not a Weak OWF.
Then, for every inverse polynomial δ, there exists a PPT **attacker A** that inverts f w.p **1-** δ**.**

We construct a **heuristic H** (using A) that **computes K^t w.p. 1-** δ **O(n),** which concludes that K^t is not mildly HOA, a contradiction.

Heuristic H(y) proceeds as follows given $x \in \{0,1\}^n$:

- For i = 1 to n+c
  - Run A(i||y) -> Π and check if Π outputs y within t(n) steps
- Output the smallest i for which the check passed.

Intuitively, if A succeeds with VERY high probability, then it should also succeed with high probability conditioned on length i, for EVERY $i \in [n+c]$

But: the problem is that H is feeding A the **wrong distribution** over y's.

In OWF experiment
(where A works):

i ← **$U_{\log(n+c)}$**
y ← output of a random program of length i

In the emulation by H in K^t experiment
(where we need to ***prove*** that A works):

i ← **$K^t(y)$**
y ← **U_n**

No reason to believe that the output of a random program will be close to uniform!

But: using a counting argument, we can show that they are not too far in **relative distance**

In OWF experiment
(where A works):

i $\leftarrow$ $\mathbf{U_{log(n+c)}}$
y $\leftarrow$ output of a random program of length i

In the emulation by H in K^t experiment
(where we need to ***prove*** that A works):

i $\leftarrow$ $\mathbf{K^t(y)}$
y $\leftarrow$ $\mathbf{U_n}$

Key idea:

- Assume for simplicity that **A** is deterministic.
- Consider some string **y** on which **H** fails. **y** has prob mass $\mathbf{2^{-n}}$ in the K^t exp.
- For **H(y)** to fail, **A(w||y)** must fail where $\mathbf{w = K^t(y)}$.
- But the pair **w||y** is sampled in the OWF exp w.p

$$1/(n+c) * 2^{-w} > 1/(n+c) * 2^{-n+c} > 1/O(n)\ 2^{-n}$$

- So, if H fails w.p. ε, A must fail w.p > ε /O(n) ≤ δ
- **Thus. H fails w.p ε ≤ δ O(n)**

Theorem 1

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High-level Idea [KC'00,ABK+'06]:

- Use OWF f to construct a **PRG G:$\{0,1\}^n$ -> $\{0,1\}^{2n}$** [HILL'99] (output of $G(U_n)$ is indistinguishable from U_{2n} by PPT observers)

- Use algorithm H for computing K^t to distinguish output of PRG from random, where t = running time of G, which yields a contradiction.

Uniform
$y \leftarrow U_{2n}$

$K^t(y) > 2n\text{-}O(\log n)$ w.h.p

Pseudorandom
$y \leftarrow G(U_n)$

$K^t(y) < n+O(1)$ w.p 1

So any algorithm H that computes K^t can break the PRG.

Important:

- Only works if **H computes K^t w.p 1.**
- if H is just a heuristic (that works w.p 1-neg), then we have no guarantees: H can fail on all pseudorandom strings, as they have tiny probability mass!

Entropy-preserving PRG (EP-PRG)

Efficiently computable function $G:\{0,1\}^n \to \{0,1\}^{n+c \log n}$

- **Pseudorandomness:** $G(U_n)$ indistinguishable from $U_{n+c \log n}$
- **Entropy-preserving:** $G(U_n)$ has Shannon entropy $n-O(\log n)$

Lemma: EP-PRG with running time t implies K^t is mildly-HOA

Uniform
$y \leftarrow U_{n+O(\log n)}$

$K^t(y) > n+O(\log n)$ **w.h.p**

Pseudorandom
$y \leftarrow G(U_n)$

$K^t(y) < n+O(1)$ **w.p 1**

If G is an EP-PRG, then **H(y) < n + O(1)** w.p O(1)/n^2 given pseudo random samples

Idea:

- If Shannon entropy is n – O(log n), then using an averaging argument, there exists a set S of strings in the support of $G(U_n)$, s.t.
  - **for every $y \in S$, $\Pr[G(U_n) = y] < 2^{-(n-O(\log n))}$**
  - **Pr[S] > 1/n**
- That is, conditioned on S, the **relative distance from uniform** is small, and we can use the same argument as for Thm 1 to argue that H's failure probability will be small.

Constructing EP-PRG

Good News: GL'89 construction of a PRG from a **OWP** f is entropy preserving.

$$G(s,r) = \underbrace{r, f(s)}_{\text{Entropy } n}, GL(s,r)$$

Bad News:

- HILL' 99 construction of a PRG from **OWF** is not entropy preserving (as far as we can tell)
- Don't know how to obtain an EP-PRG from OWF…

Need to relax the notion of an EP-PRG.

Entropy-preserving PRG (EP-PRG)

Efficiently computable function $G:\{0,1\}^n \to \{0,1\}^{n+c \log n}$

- **Pseudorandomness:** $G(U_n)$ indistinguishable from $U_{n+c \log n}$
- **Entropy-preserving:** $G(U_n)$ has Shannon entropy $n-O(\log n)$

Conditionally Entropy-preserving PRG (condEP-PRG)

Efficiently computable function $G:\{0,1\}^n \to \{0,1\}^{n+c \log n}$

- **Pseudorandomness:** $G(U_n \mid E)$ indistinguishable from $U_{n+c \log n}$
- **Entropy-preserving:** $G(U_n \mid E)$ has Shannon entropy $n-O(\log n)$

For some event **E**

Lemma: condEP-PRG with running time t implies K^t is mildly-HOA

Same proof as before works.

Constructing condEP-PRG from OWF

Lemma: OWF => cond EP-PRG

Proof:

- Use a variant of PRG from **regular OWF** from [HILL'99,Gol'01,YLW'15]
- Show that it satisfies our notion of a cond EP-PRG when using **any OWF.**

$$G(s,r1,r2,r3,i) = \underbrace{r1,r2,r3, [Ext_{r1}(s)]_{i-O(\log n)} [Ext_{r2}(f(s))]_{n-i-O(\log n)}}_{\text{Shannon Entropy } n - O(\log n)} \; GL(s,r3)$$

Not a PRG. Not EP.

But is a PRG and EP **conditioned** on the event that **(i,s) is "good"**

"good" : s has regularity r that is "common", i = r

Ensures that extractors work.

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Then OWFs exist.

Theorem 2

Assume OWFs exists.
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Main Theorem

For all ε>0, all poly $t(n)>(1+\varepsilon)n$
OWFs exist iff **K^t is mildly-HOA**.

First natural avg-case problem characterizing the feasibility of the basic tasks in Crypto
(i.e., private-key encryption, digital sigs, PRGs, PRFs, commitments, authentication, ZK…)

Recent Results on K^t and Friends

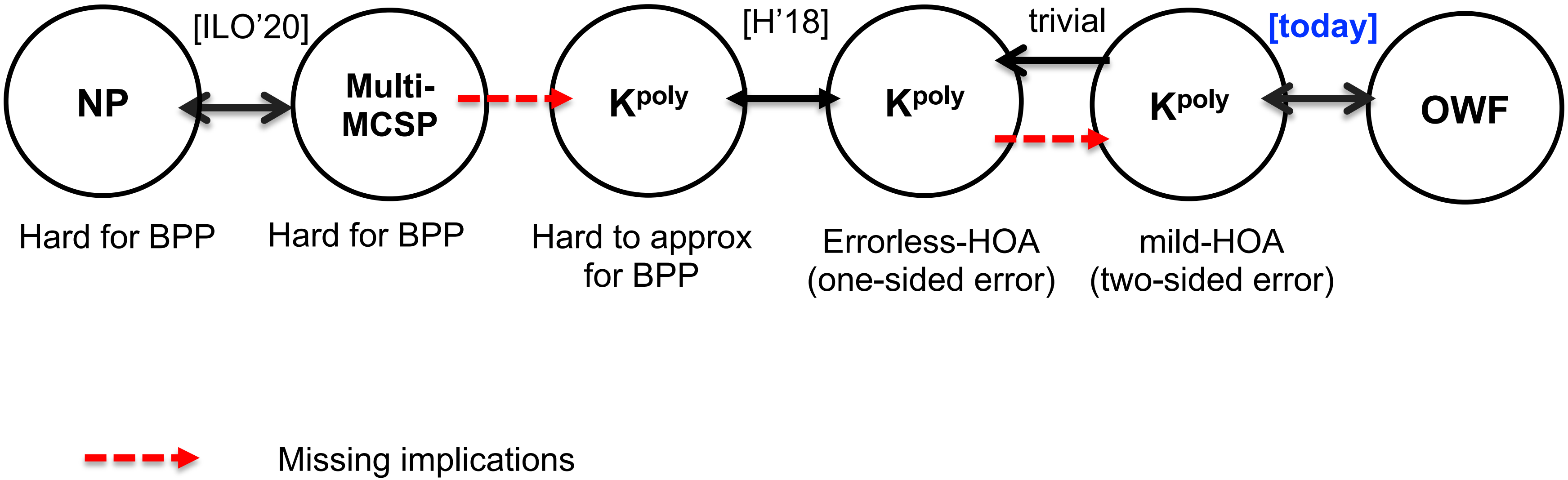
- [Hirahara'18]: presents a **worst-case to average-case reduction** for K^t:
  K^t is **errorless-HAO** if K^t is **worst-case** hard to approximate.
  Similar results indep. obtained by [Santhanam'19] w.r.t. a variant of MCSP.

  Our results to not extend to errorless-HAO…

- [Ilango-Loff-Oliviera'20]: **Multi-MCSP** is NP-Hard

- [Oliviera-Santhanam]: Hardness magnification for MCSP

Towards the "holy-grail"



Thank You