Automating Tree-Like Resolution in Time $n^{o(\log n)}$ Is ETH-Hard

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Resolution proof system

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Given UNSAT CNF formula $\boldsymbol{F}$: $(\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z)$

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clauses/axioms

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clauses/axioms

Resolution rule: $$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

Refutation: Derivation of empty clause $\perp$

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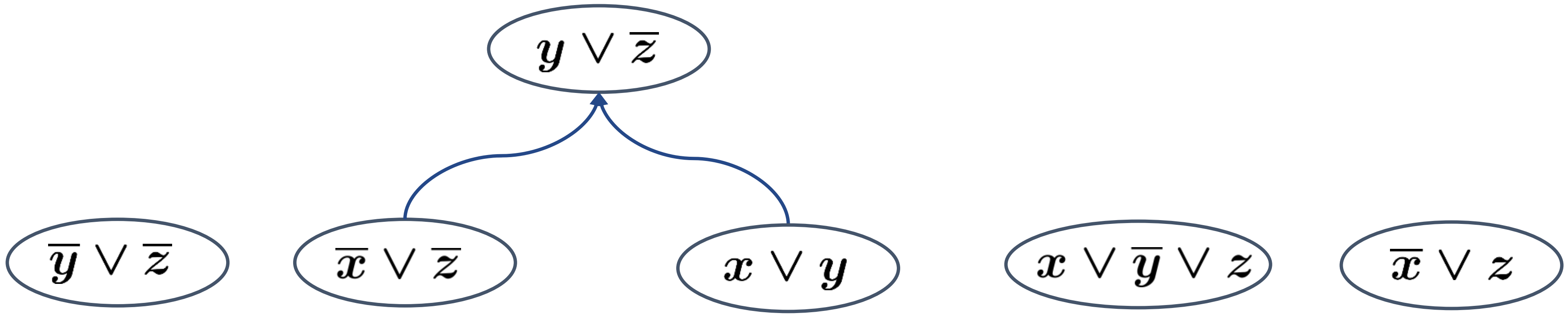

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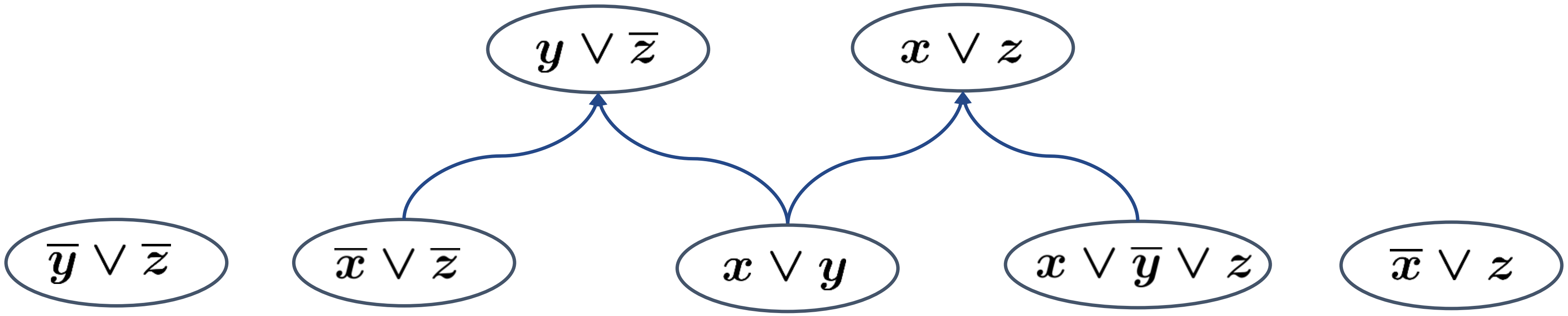

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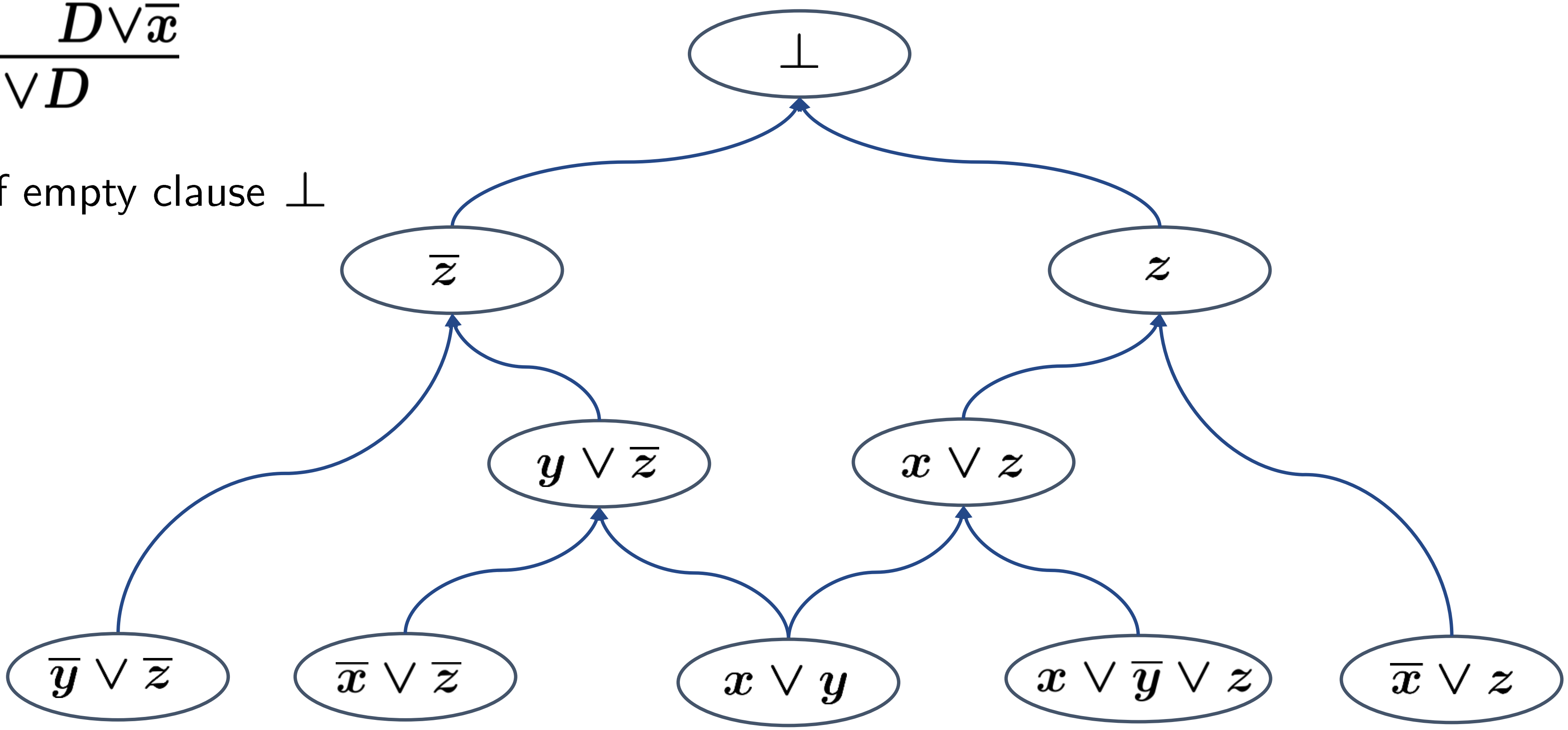

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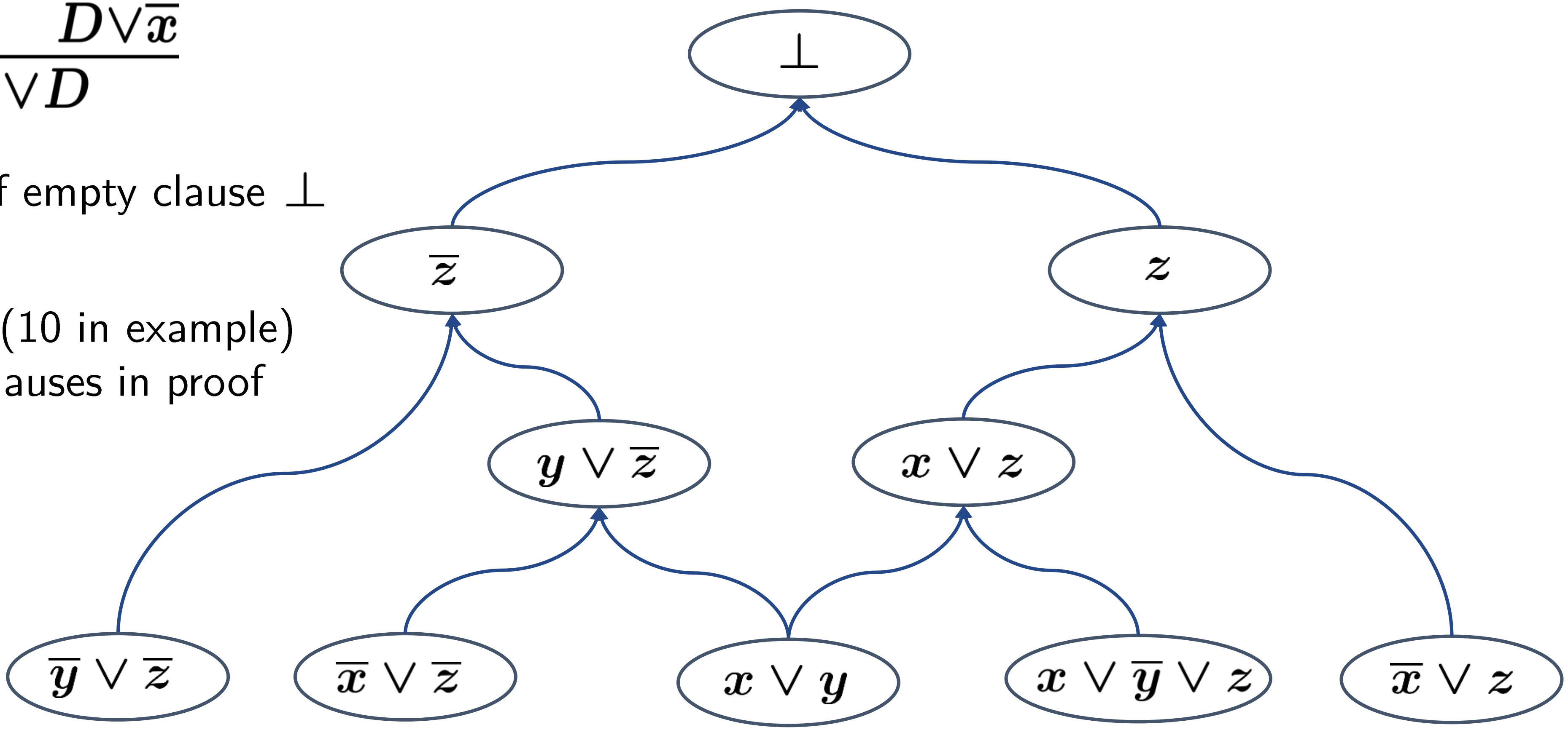
Width: *max* # literals/clauses in proof



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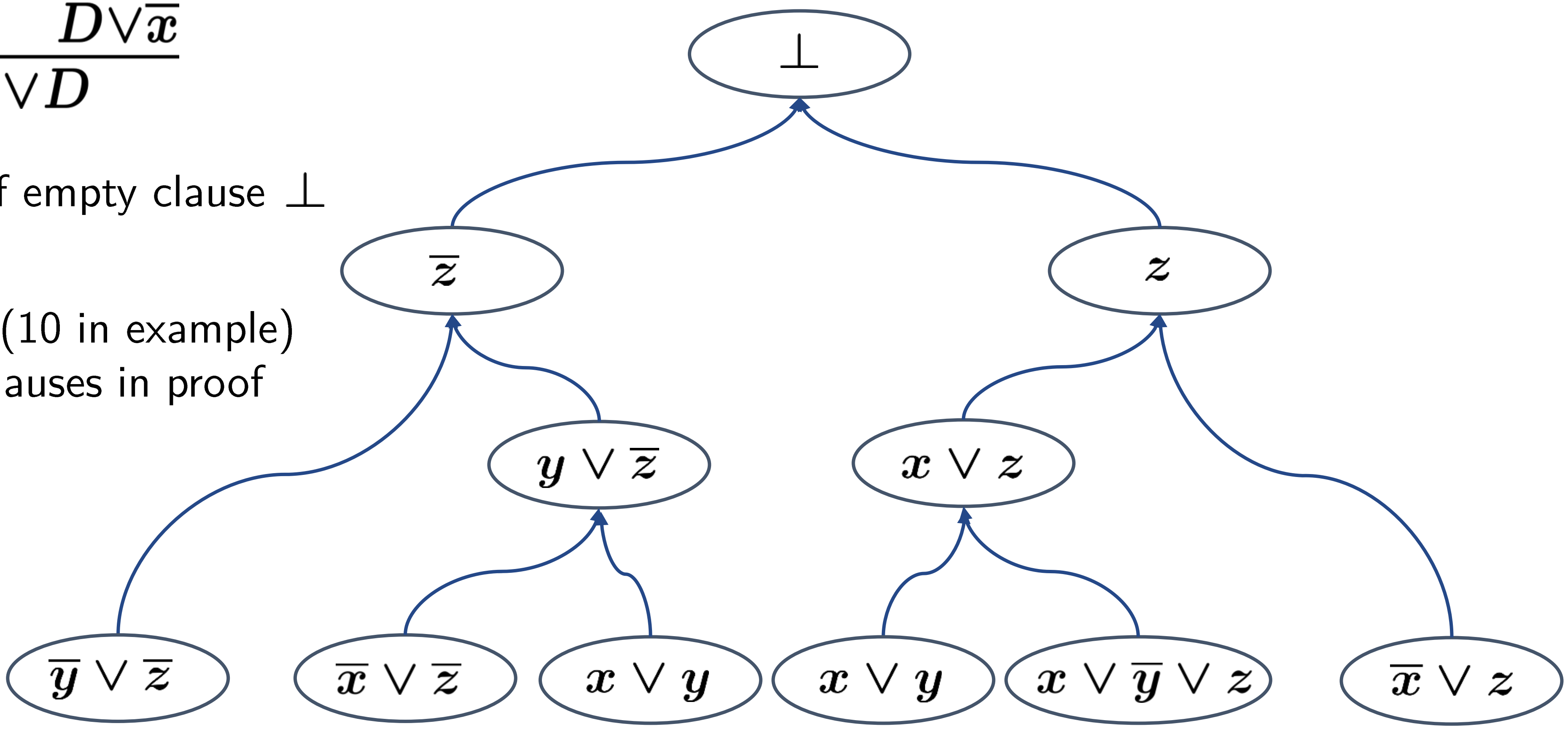

Automatability [BPR'97]

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Proof system $\mathcal{P}$ is *automatable* in time $f(n)$
if $\exists$ algorithm $\mathcal{A}$ that given unsat CNF $\boldsymbol{F}$
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size of smallest $\mathcal{P}$-refutation of $\boldsymbol{F}$
plus the size of $\boldsymbol{F}$

Atserias-Müller ‘19

If resolution is automatable:

1. in time $\mathbf{poly}(n)$ then $\mathsf{NP} \subseteq \mathsf{P}$
2. in time $\mathbf{quasipoly}(n)$ then $\mathsf{NP} \subseteq \mathsf{QP}$
3. in time $\mathbf{subexp}(n)$ then $\mathsf{NP} \subseteq \mathsf{SUBEXP}$

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Generalizations

1. Cutting planes [GKMP'20]
2. Res($\boldsymbol{k}$) [Gar'20]
3. Algebraic proof systems (NS, PC, SA) [dRGNPRS'21]

Tree-like resolution is automatable in time $n^{O(\log n)}$ [BP'96]

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Given UNSAT CNF formula $\boldsymbol{F}$

Suppose $\boldsymbol{s}$ is known

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Given UNSAT CNF formula F

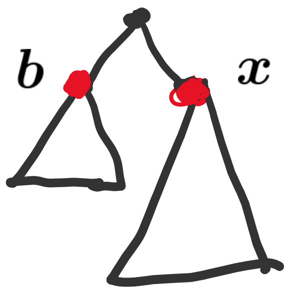
Suppose s is known

R^* : size of smallest tree-like resolution refutation

$\exists\, x$ s.t. $R^*(F|_{x=0}) \leq s/2$ or $R^*(F|_{x=1}) \leq s/2$

$\perp$

$x = b$ $\quad$ $x = 1 - b$



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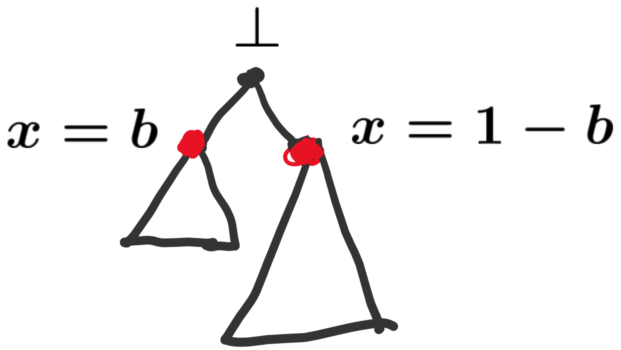

$$T(n, s) = 2n \cdot T(n-1, s/2) + T(n-1, s) + O(1)$$

$$T(n, s) = s^{O(\log n)}$$

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2. in time $n^{O(\log^{1/7-\epsilon} \log n)}$ then ETH is false [MPW'19]

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Theorem 1

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Main Theorem

$\exists$ algorithm that given n-variate F outputs $\mathcal{A}(F)$ in time $2^{O(\sqrt{n})}$ s.t.

1. F is SAT $\Rightarrow R^*(\mathcal{A}(F)) \leq 2^{O(\sqrt{n})}$
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Want to decide if 3-CNF F is SAT or UNSAT

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$\mathcal{A}(F)$: # var $2^{O(\sqrt{n})}$ and if F is SAT $N = 2^{O(\sqrt{n})}$

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$\mathcal{A}(F)$: # var $2^{O(\sqrt{n})}$ and if F is SAT $N = 2^{O(\sqrt{n})}$

Run $\mathbb{A}$ on $\mathcal{A}(F)$ for $f(2^{O(\sqrt{n})}) = 2^{o(n)}$ steps

obs. $o(\log N) = o(\sqrt{n})$

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$\mathbb{A}$ returns refutation $\Leftrightarrow$ F is SAT

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Atserias-Müller

Automating Resolution is NP-Hard – Simplified [dRGNPRS'21]

Atserias-Müller '19

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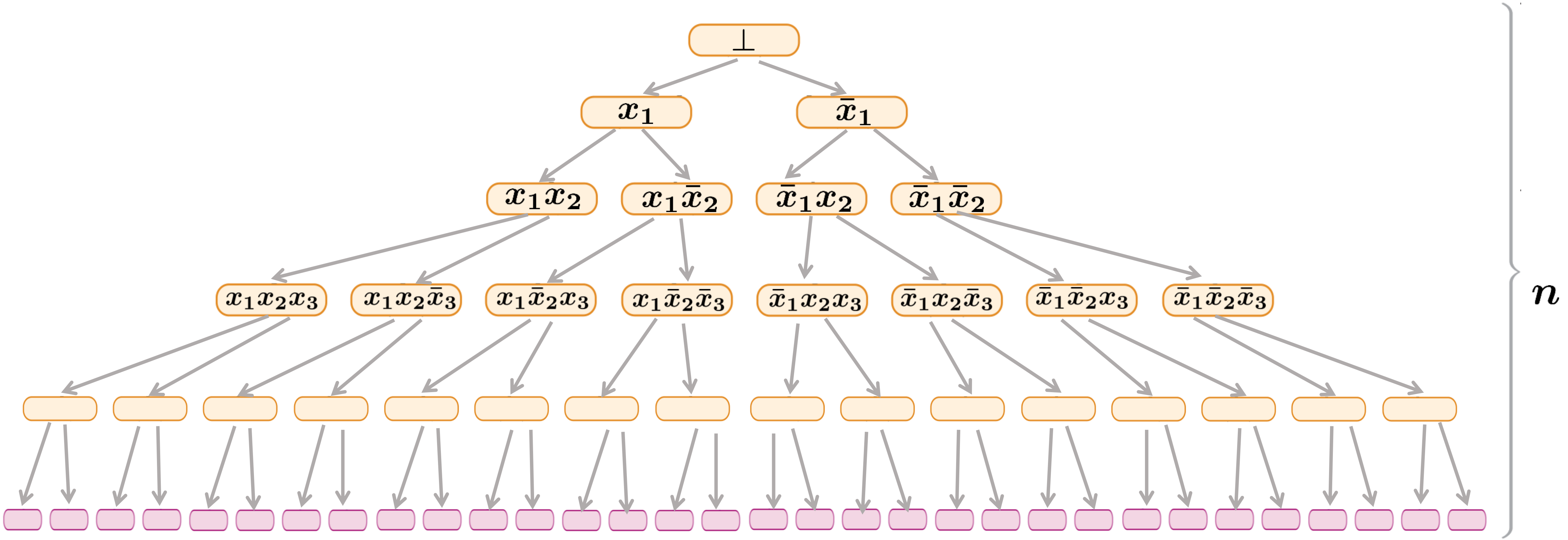
"Universal refutation" (complete Binary tree)



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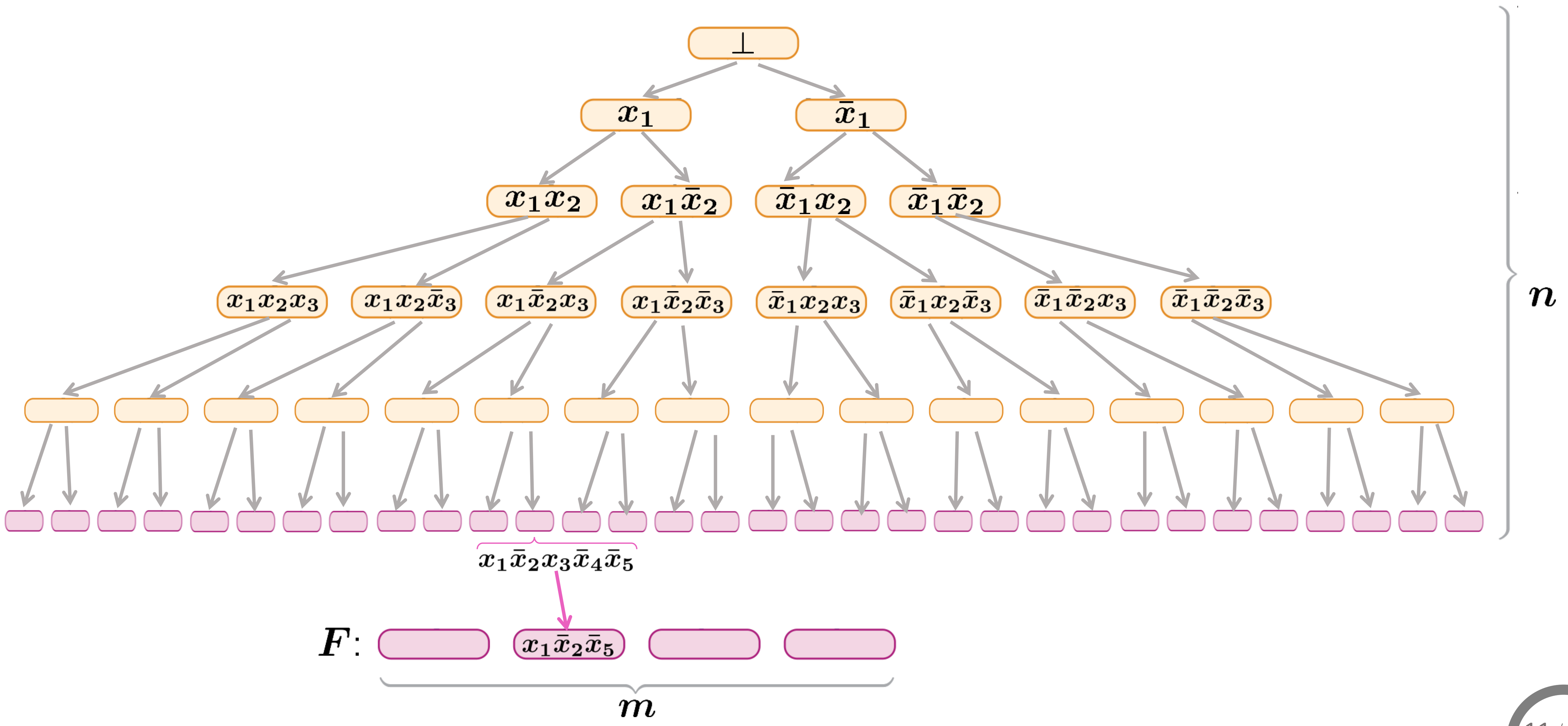
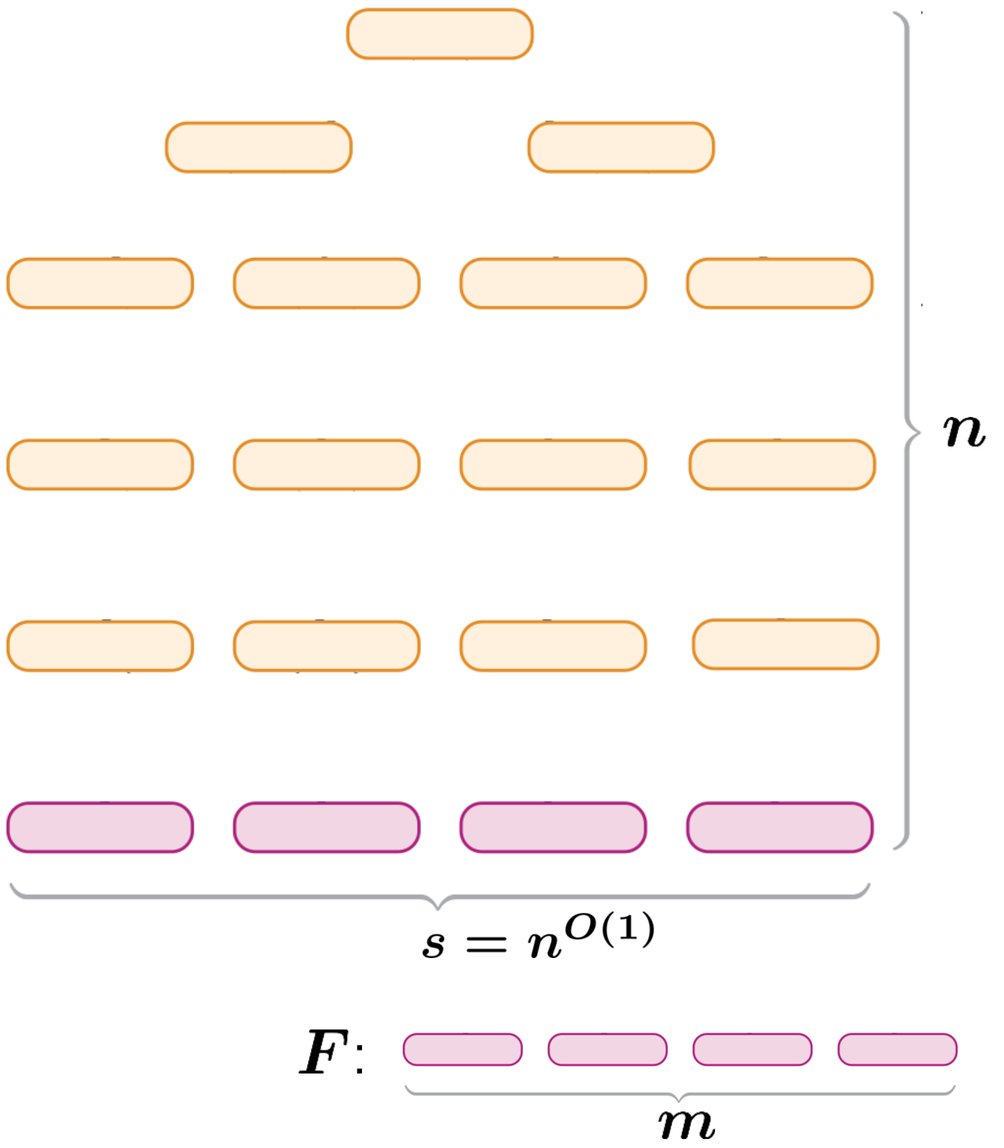


$\mathrm{Ref}(F)$

Encodes "F has short resolution refutation"



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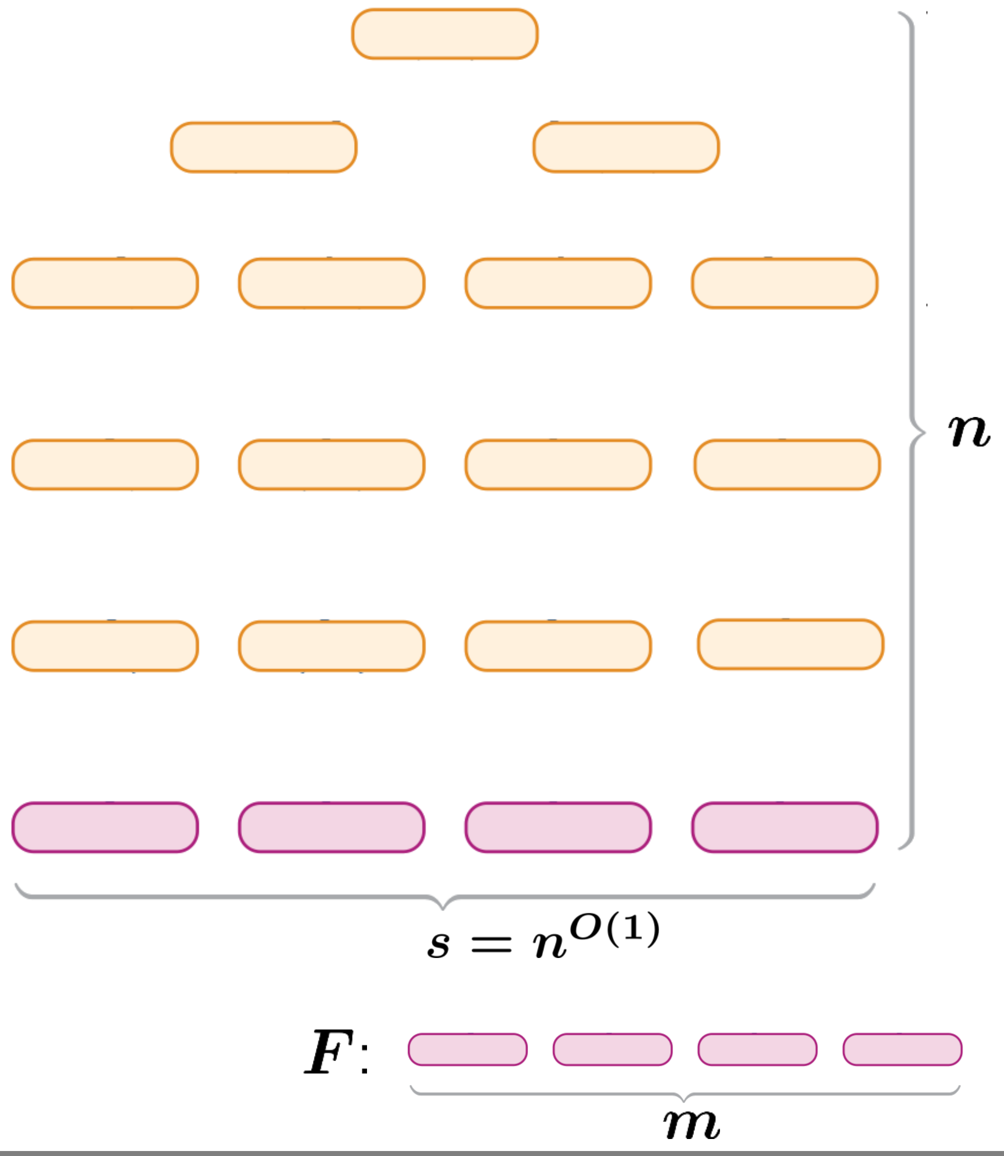
Encodes "F has short $\overbrace{\text{ resolution}}^{\text{structured}}$ refutation"



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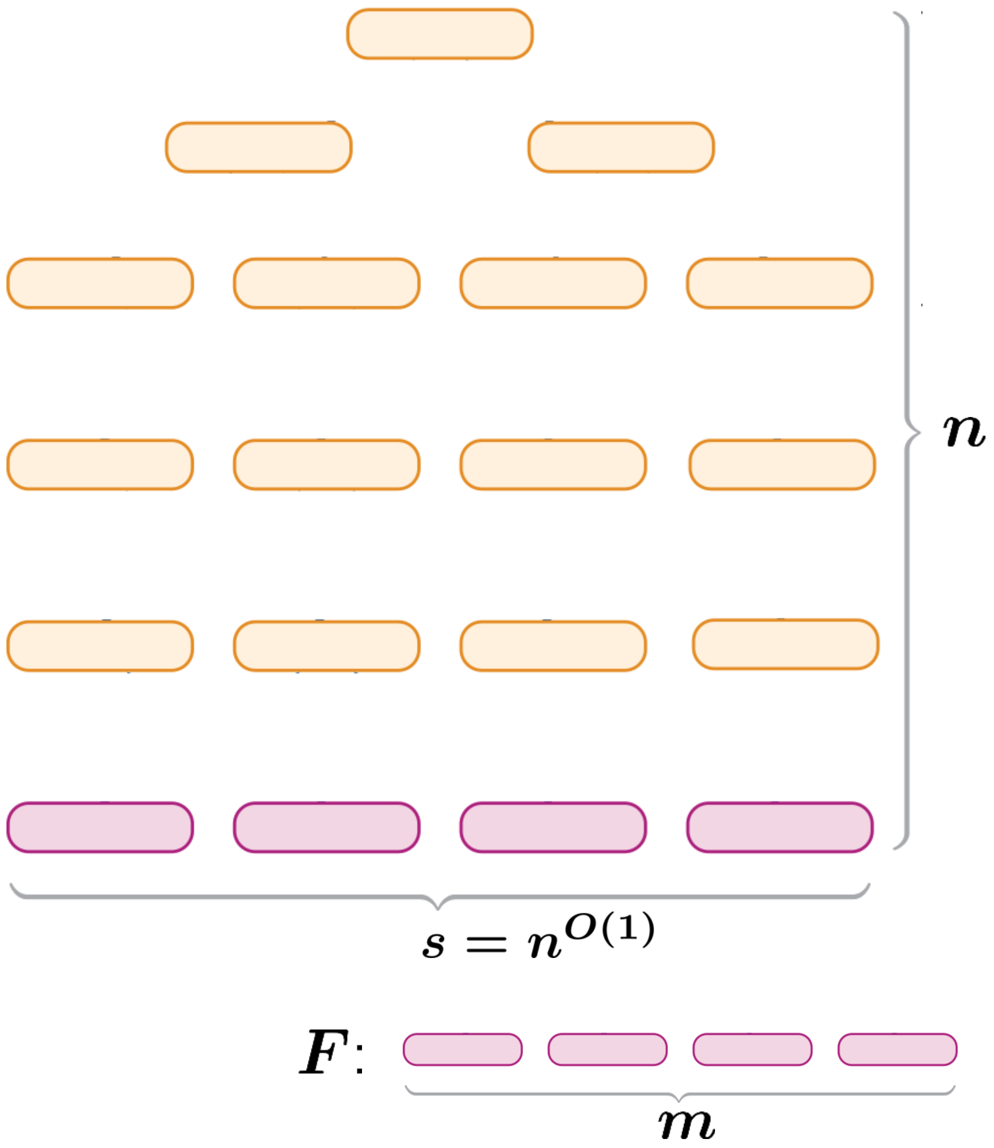
Encodes "F has $\underbrace{\text{short resolution}}$ refutation"
structured

Variables for each block B:



$\mathrm{Ref}(F)$

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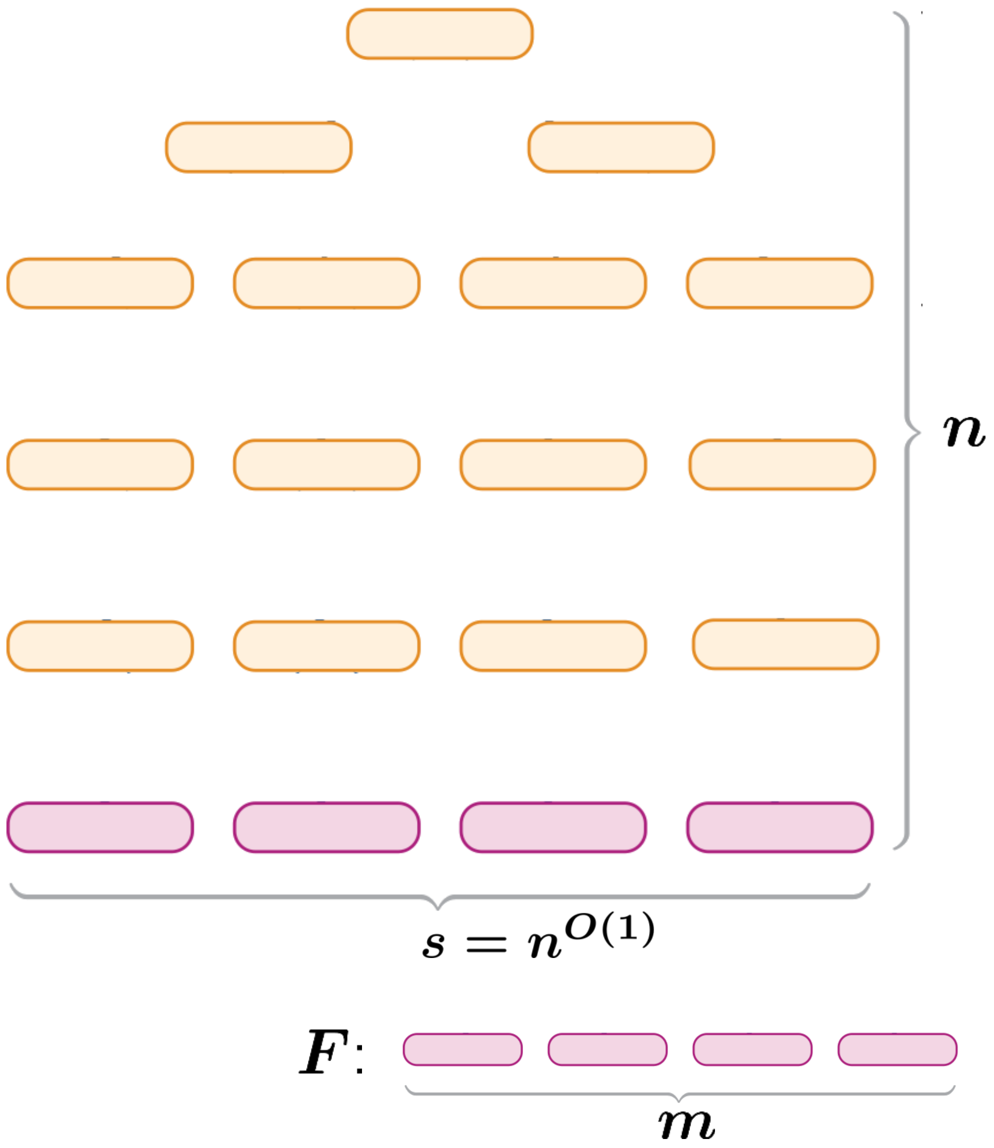
Variables for each block B:

- $2n$ variables (1 per literal): indicates clause



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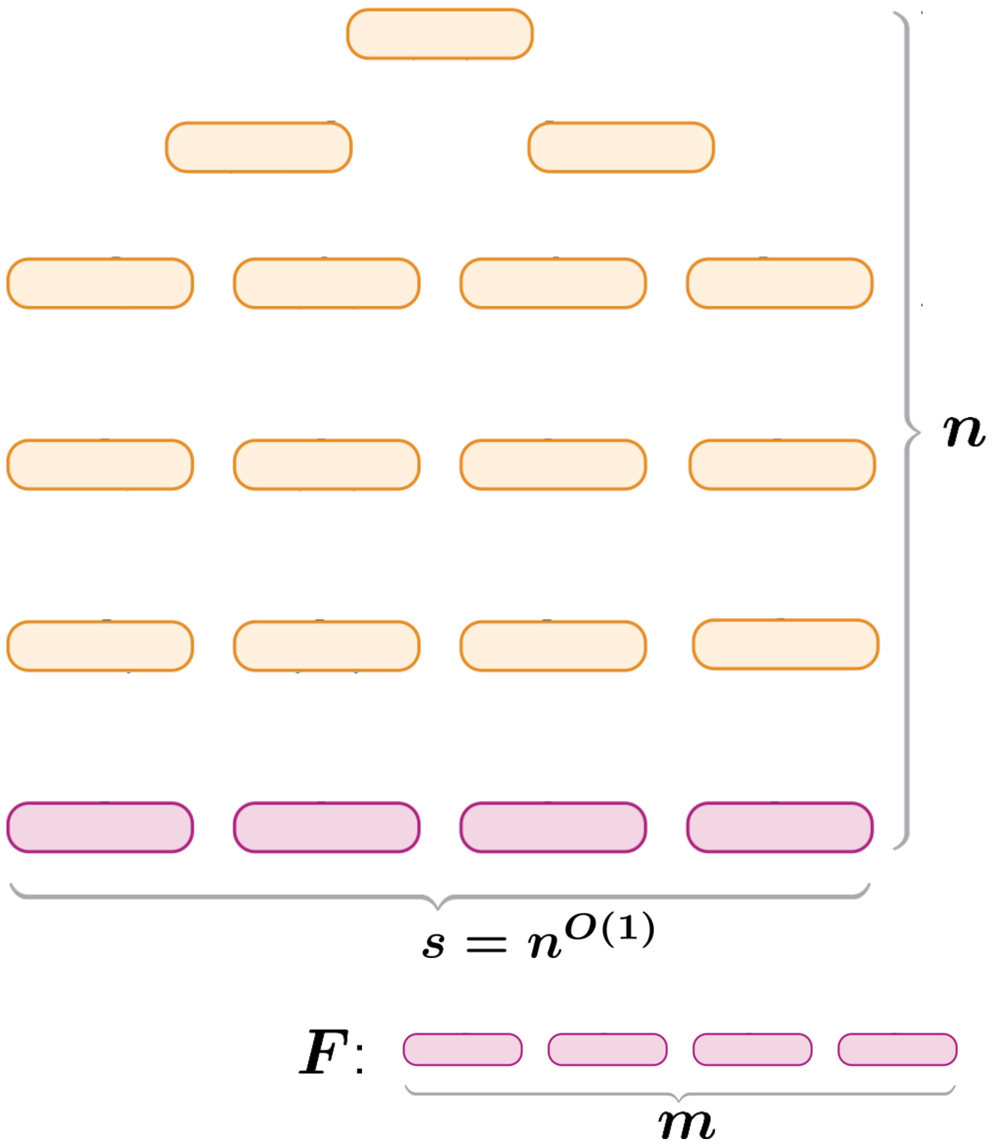

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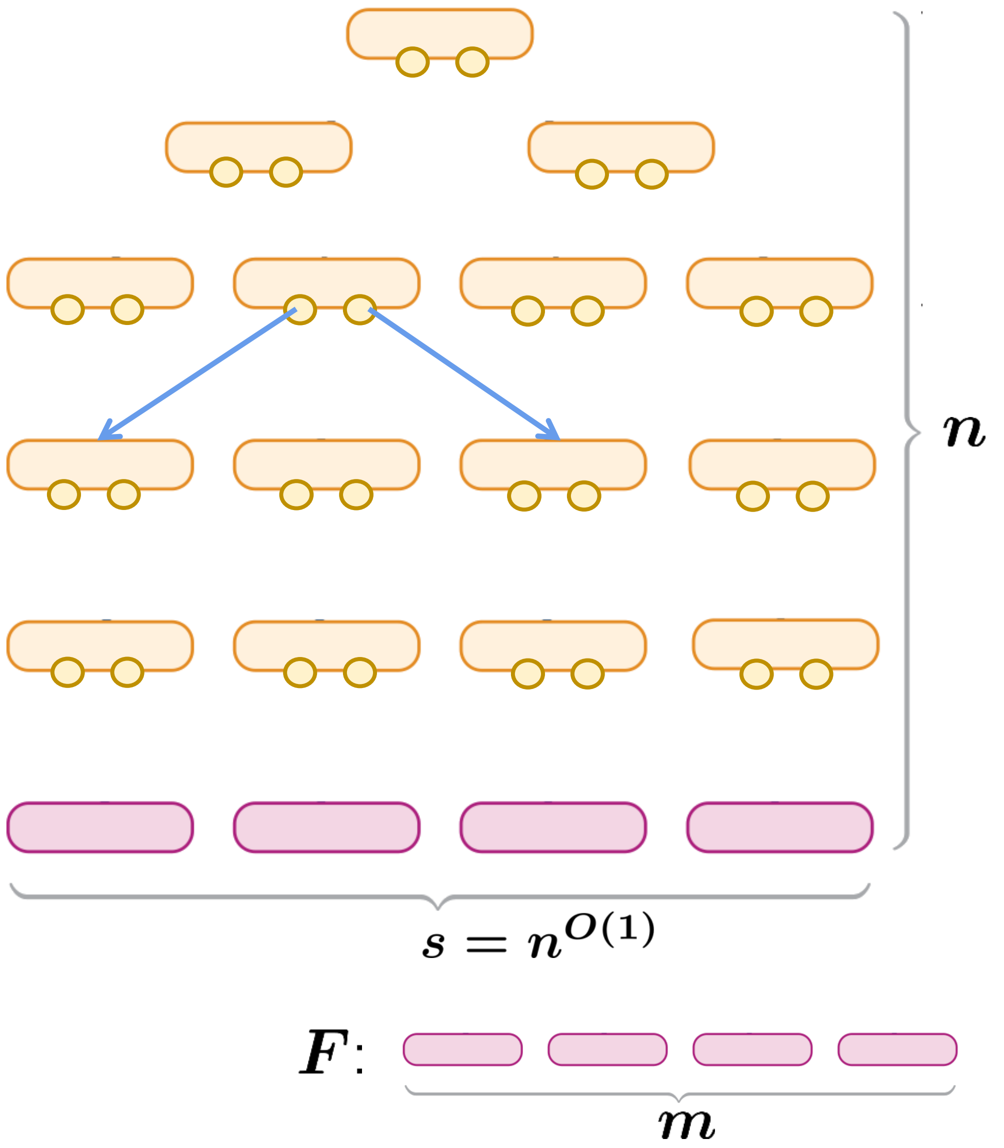

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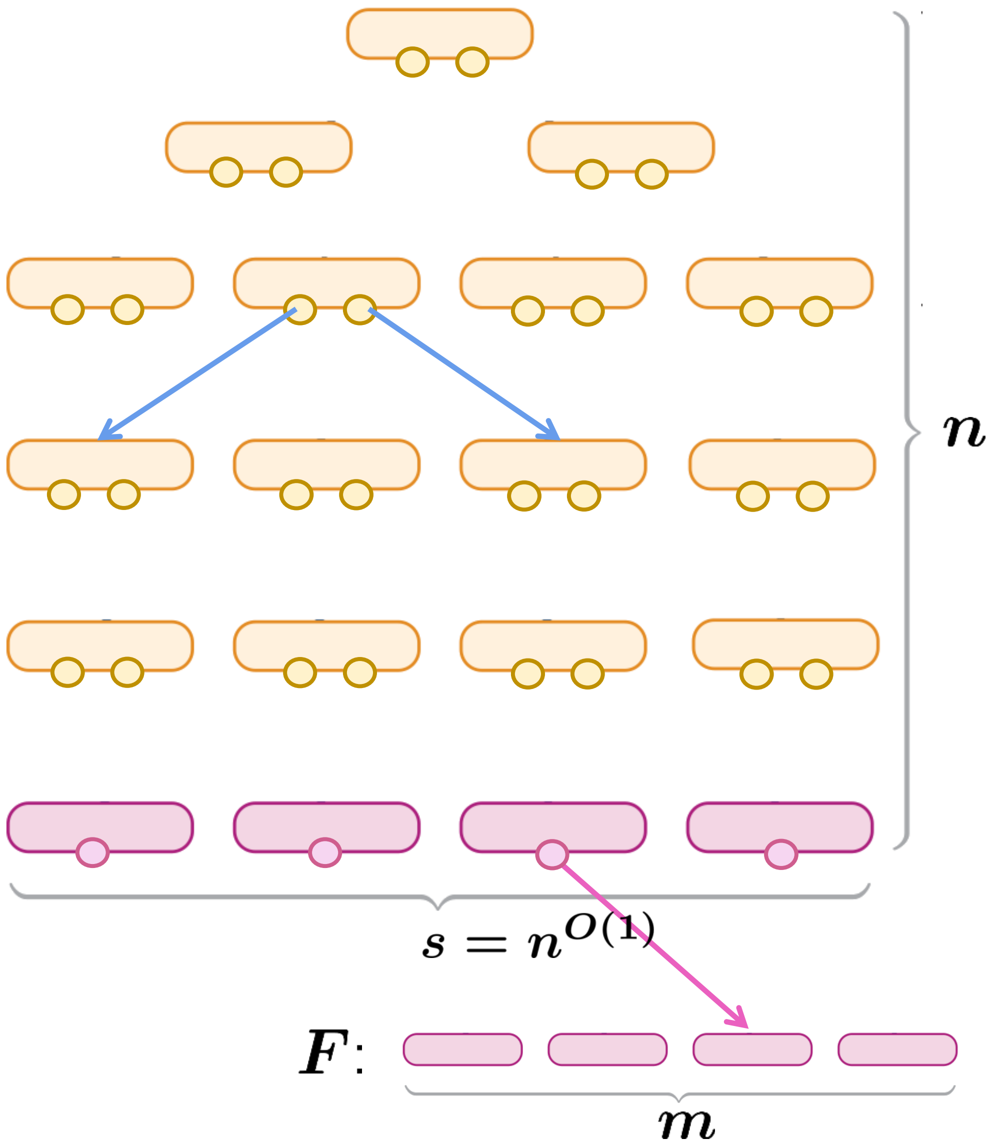

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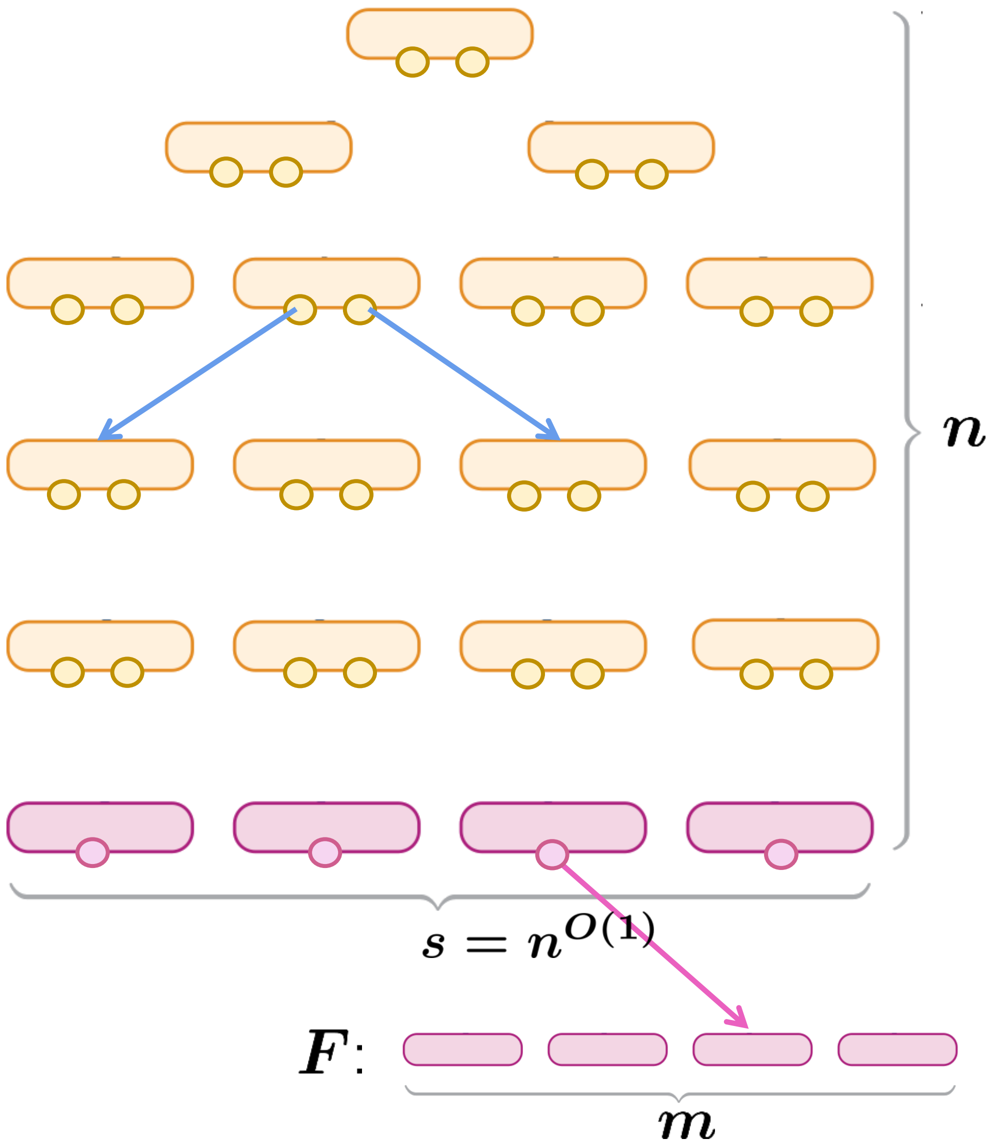
$\boxed{O(n^2 s) \text{ variables}}$



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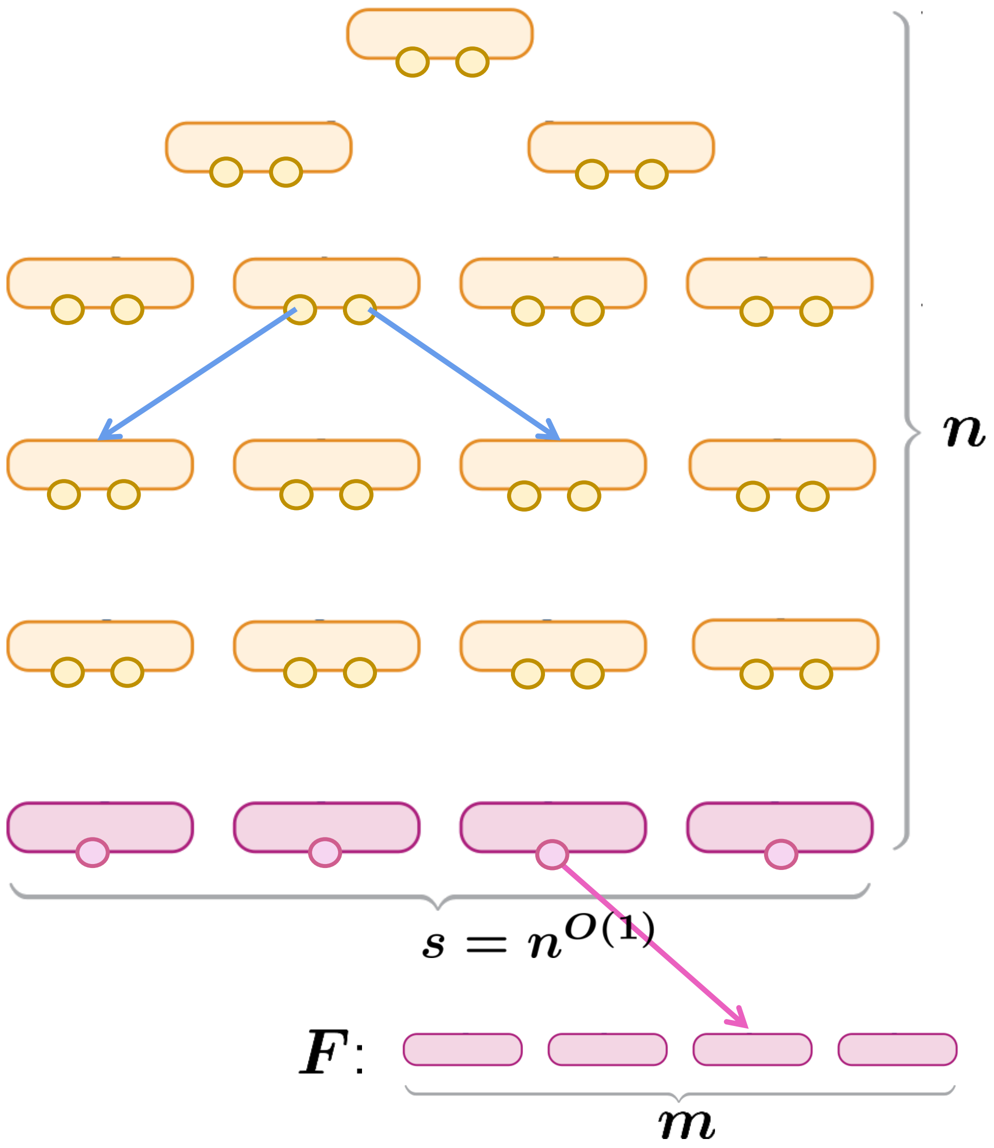
$\boxed{O(n^2 s) \text{ variables}}$

Axioms of $\mathbf{Ref}(F)$:

- Root: $\perp$ clause
- Derived: valid resolution step
- Axiom: weakening of axiom



$\mathbf{Ref}(F)$

structured

Encodes "F has short resolution refutation"

Variables for each block B:

- $2n$ variables (1 per literal): indicates clause
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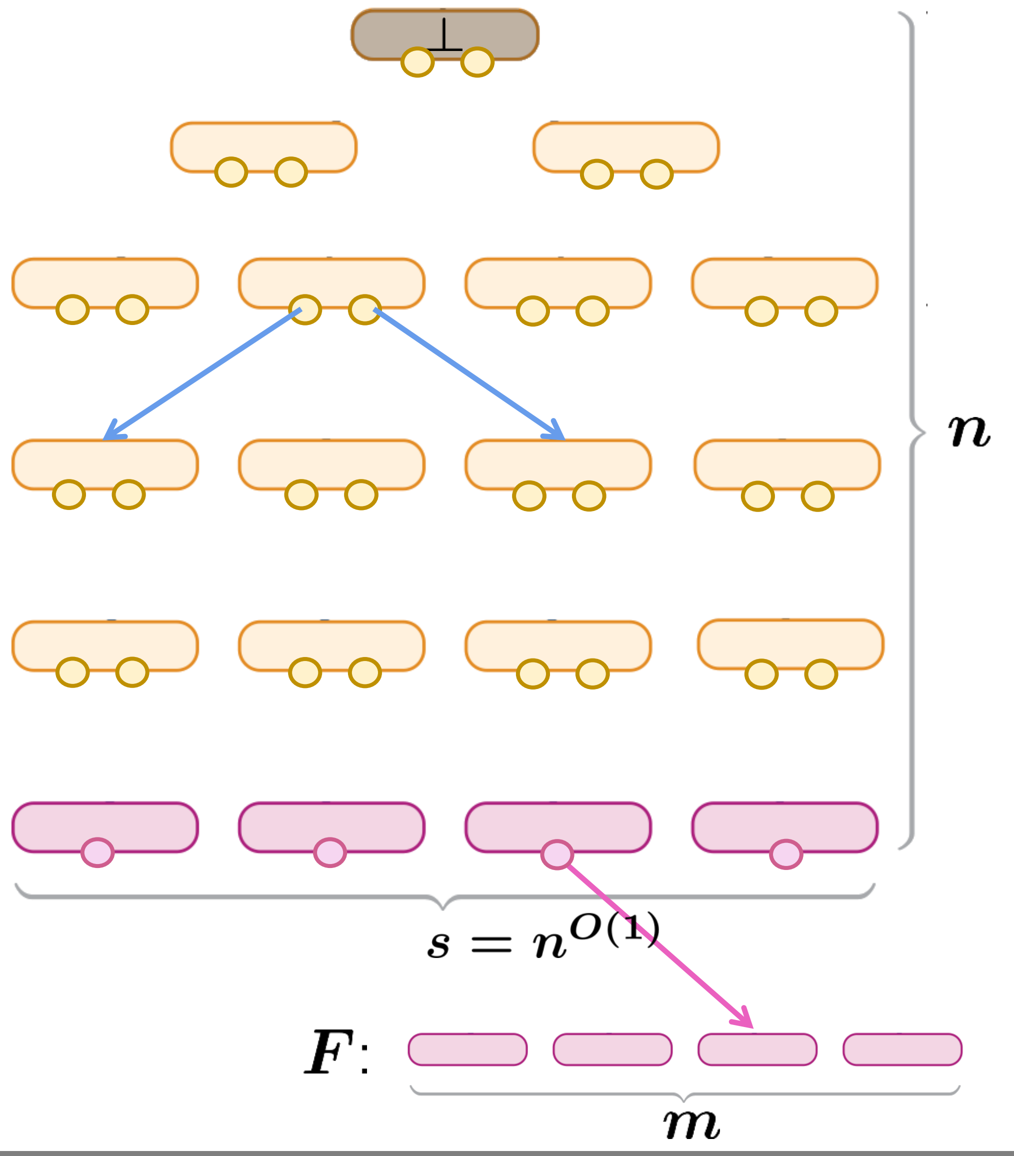

$\mathbf{Ref}(F)$

Encodes "F has short $\underbrace{\text{resolution}}$ refutation"
structured

Variables for each block B:

- $2n$ variables (1 per literal): indicates clause
- $2(\log s)$ variables: 2 pointers to children "derived from B_i and B_j"
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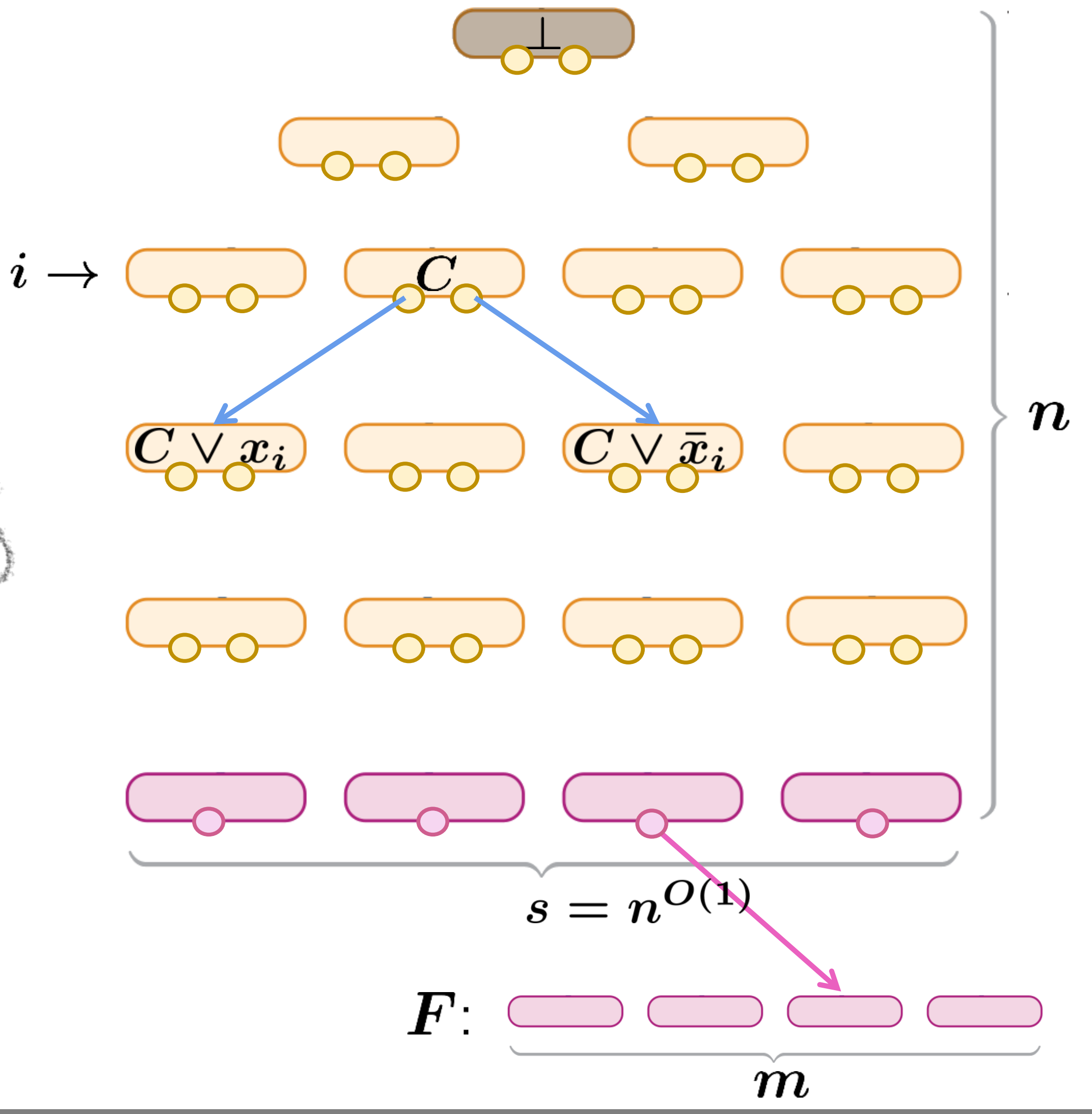

$\mathbf{Ref}(F)$

Encodes "F has short structured resolution refutation"

Variables for each block B:

- $2n$ variables (1 per literal): indicates clause
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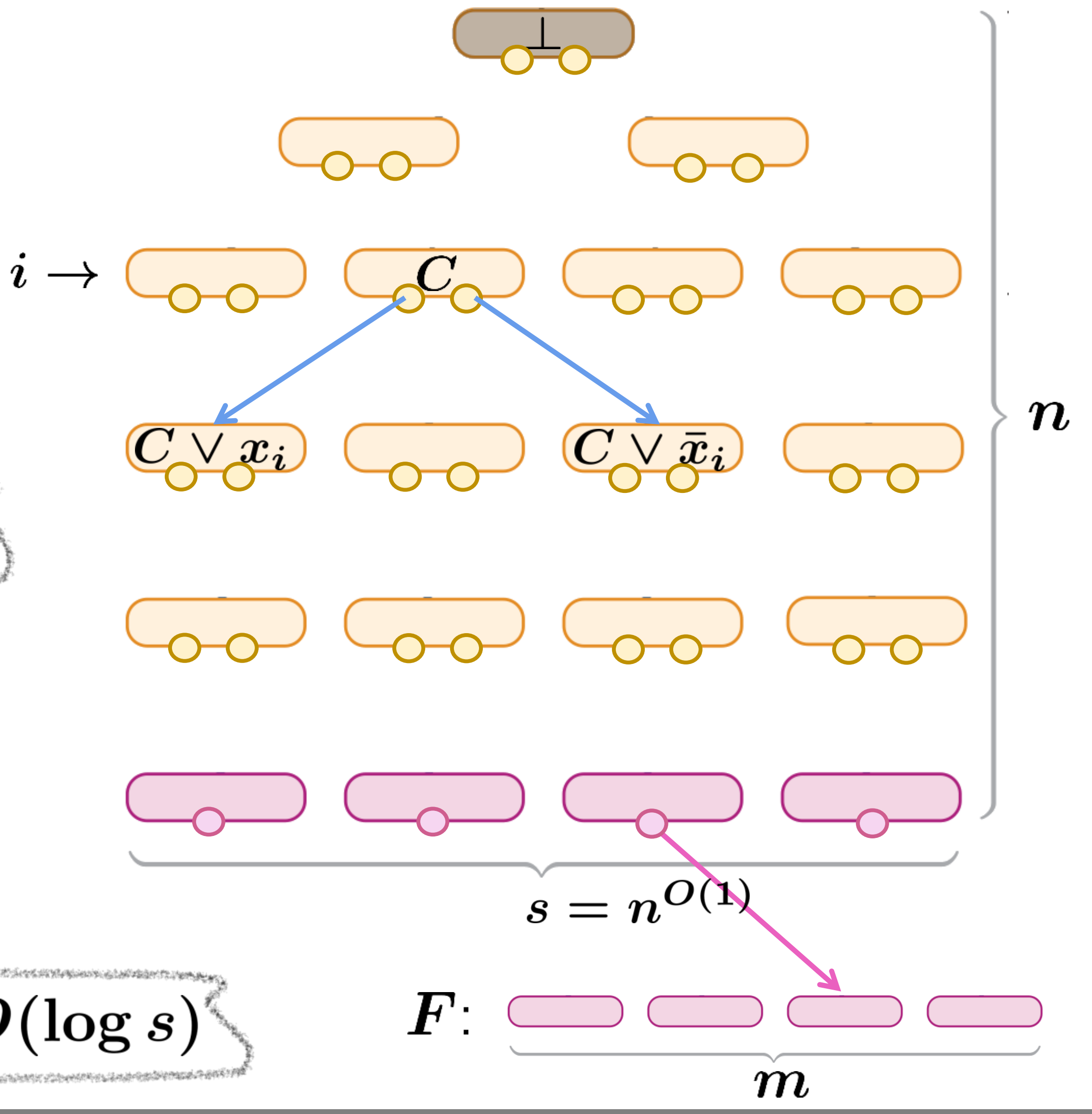
$\boxed{O(n^2 s) \text{ variables}}$



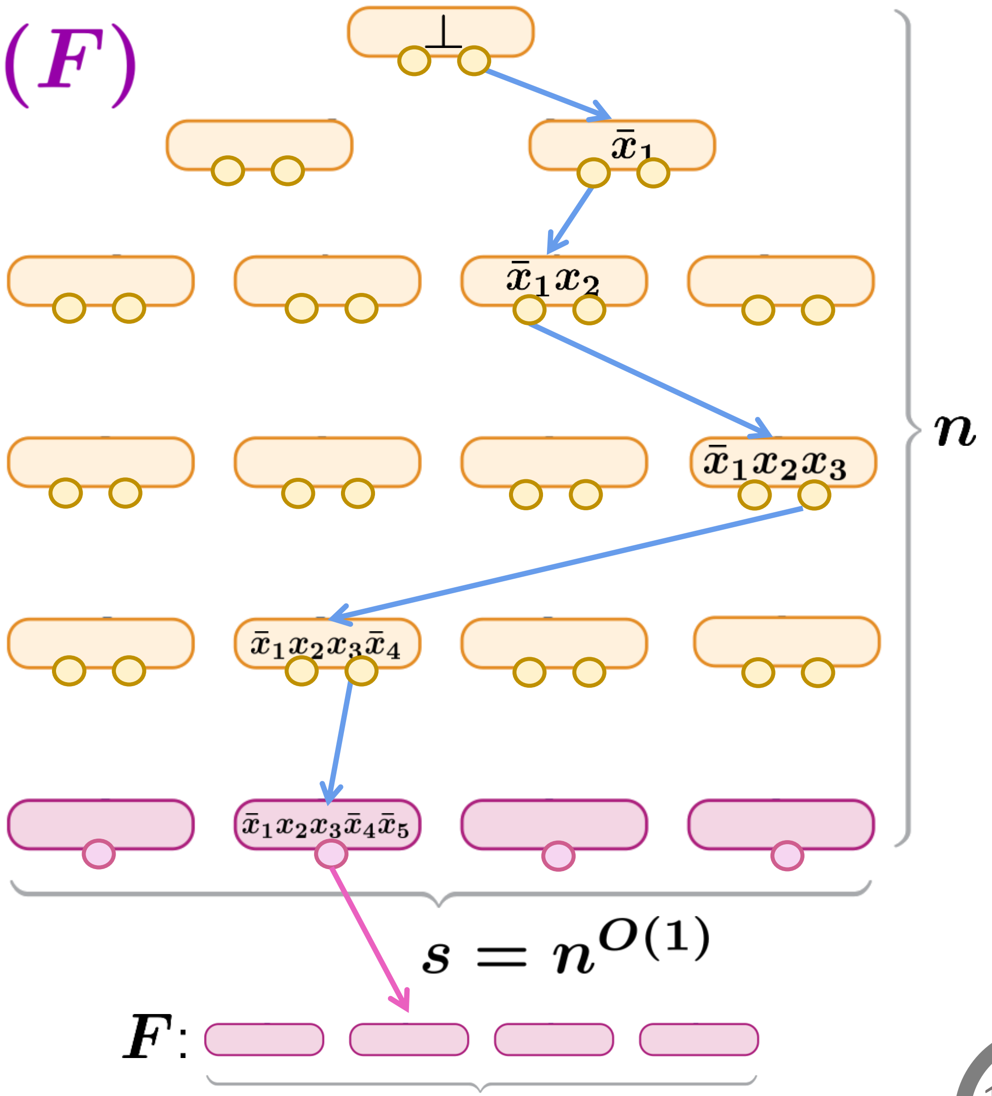
Axioms of $\mathbf{Ref}(F)$:

- Root: $\perp$ clause
- Derived: valid resolution step
- Axiom: weakening of axiom

$\boxed{\mathrm{poly}(n) \text{ clauses of width } O(\log s)}$

(1) $\boldsymbol{F}$ is **SAT** $\Rightarrow \mathbf{Ref}(\boldsymbol{F})$ has size-$\boldsymbol{n}^{\boldsymbol{O}(\mathbf{1})}$ resolution refutation

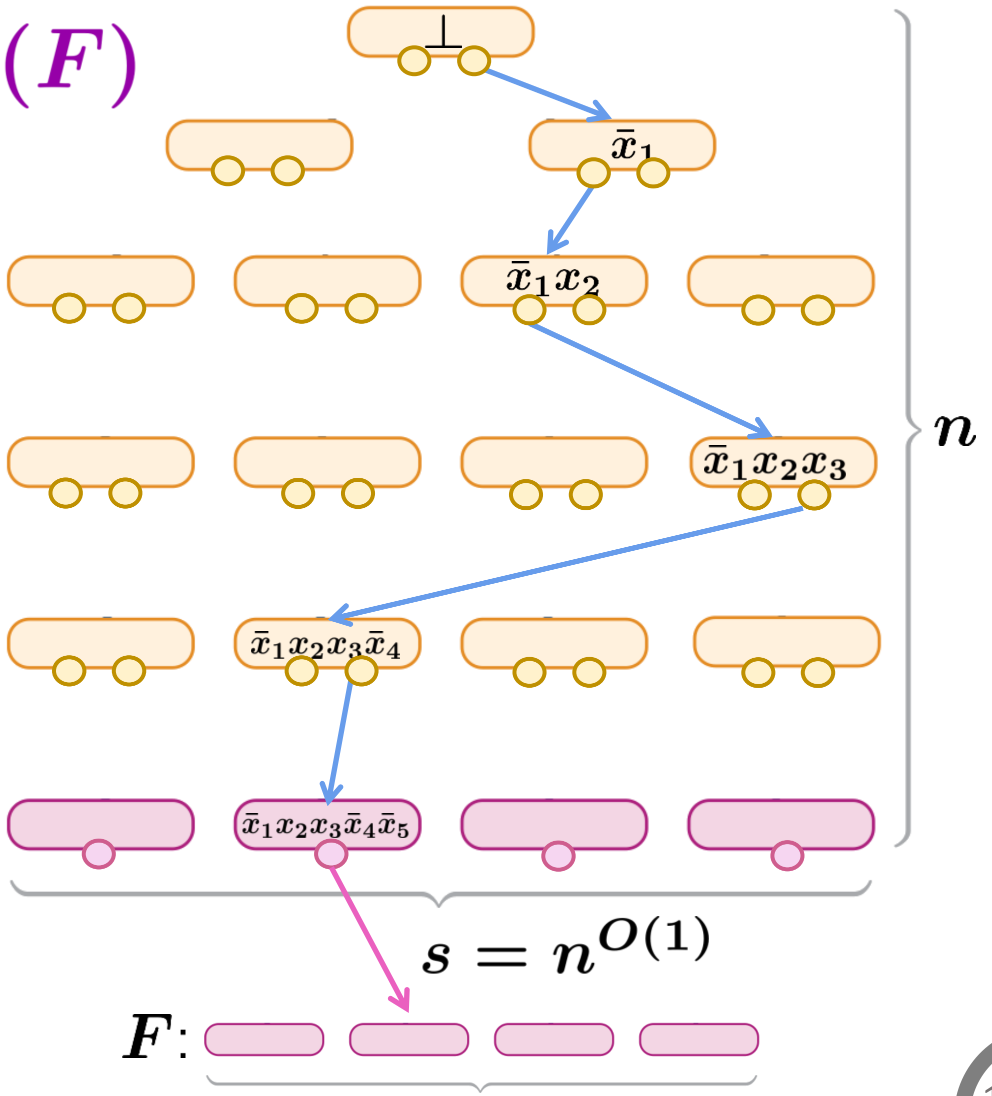
Read-once branching program for $\mathbf{Ref}(\boldsymbol{F})$



(1) $\boldsymbol{F}$ is **SAT** $\Rightarrow$ $\mathbf{Ref}(\boldsymbol{F})$ has size-$\boldsymbol{n}^{\boldsymbol{O}(1)}$ resolution refutation

Read-once branching program for $\mathbf{Ref}(\boldsymbol{F})$

$\boldsymbol{x}^*$ satisfying assignment for $\boldsymbol{F}$



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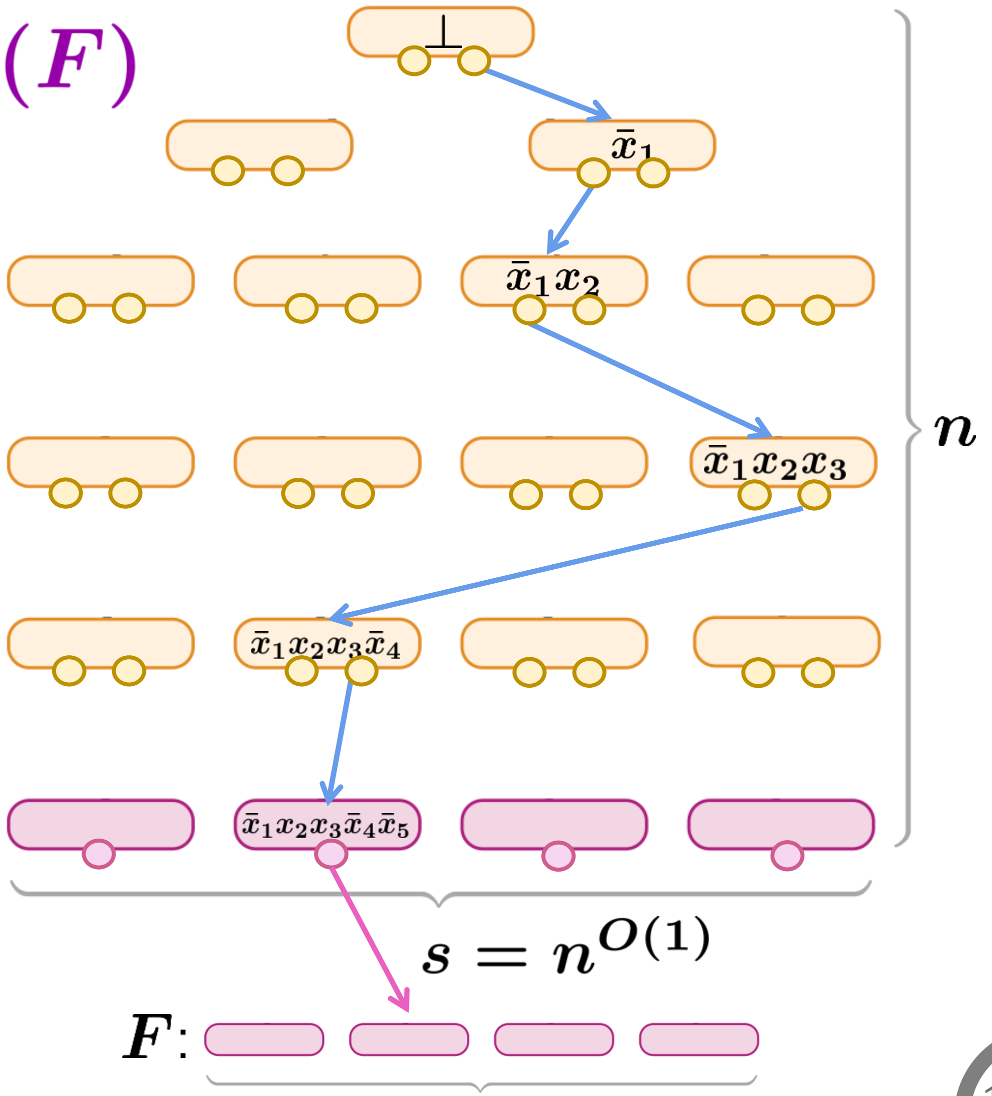


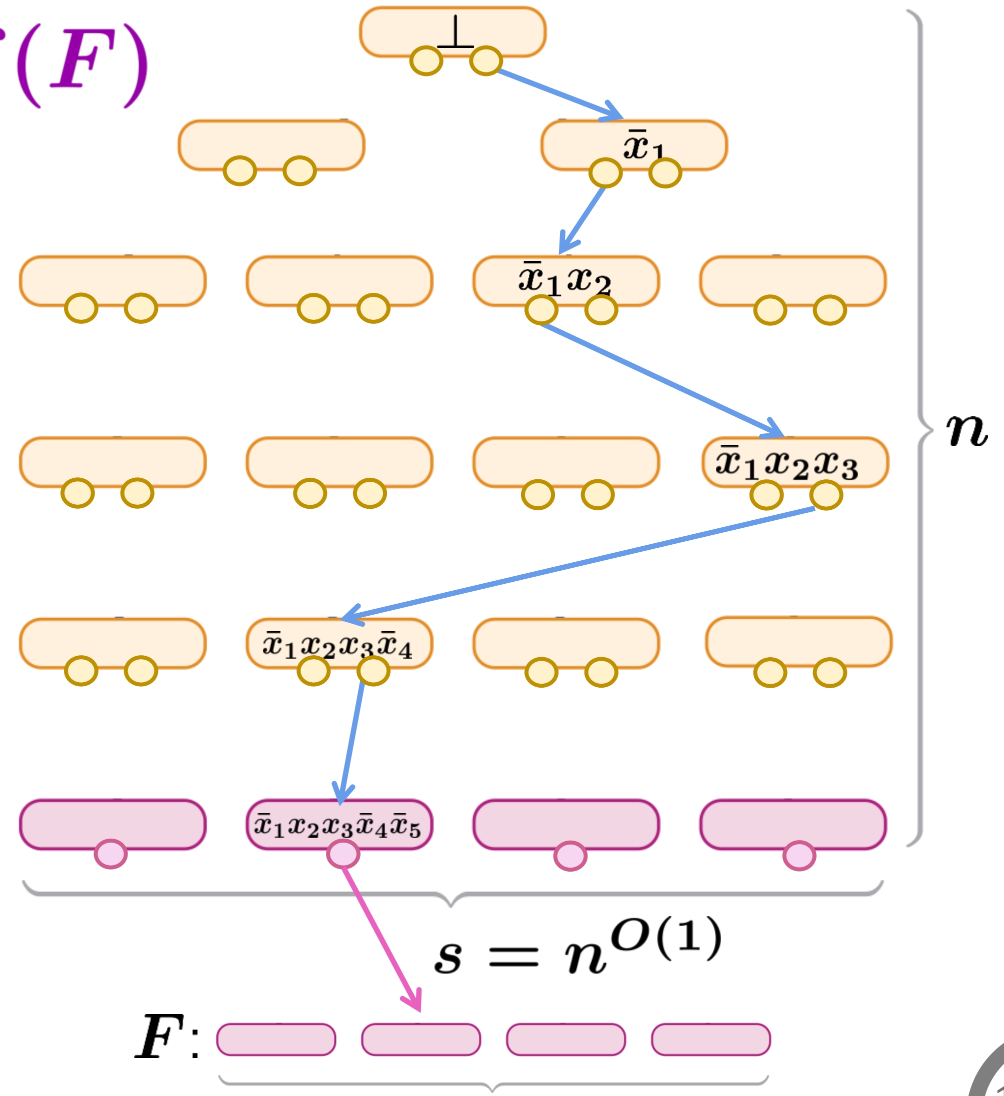
(1) F is **SAT** $\Rightarrow$ $\mathrm{Ref}(F)$ has size-$n^{O(1)}$ resolution refutation

Read-once branching program for $\mathrm{Ref}(F)$

x^* satisfying assignment for F

invariant: x^* falsifies clause in current block B

Start at root and keep invariant
until detect non-valid derivation step or
until reach leaf (cannot be weakening of axiom)



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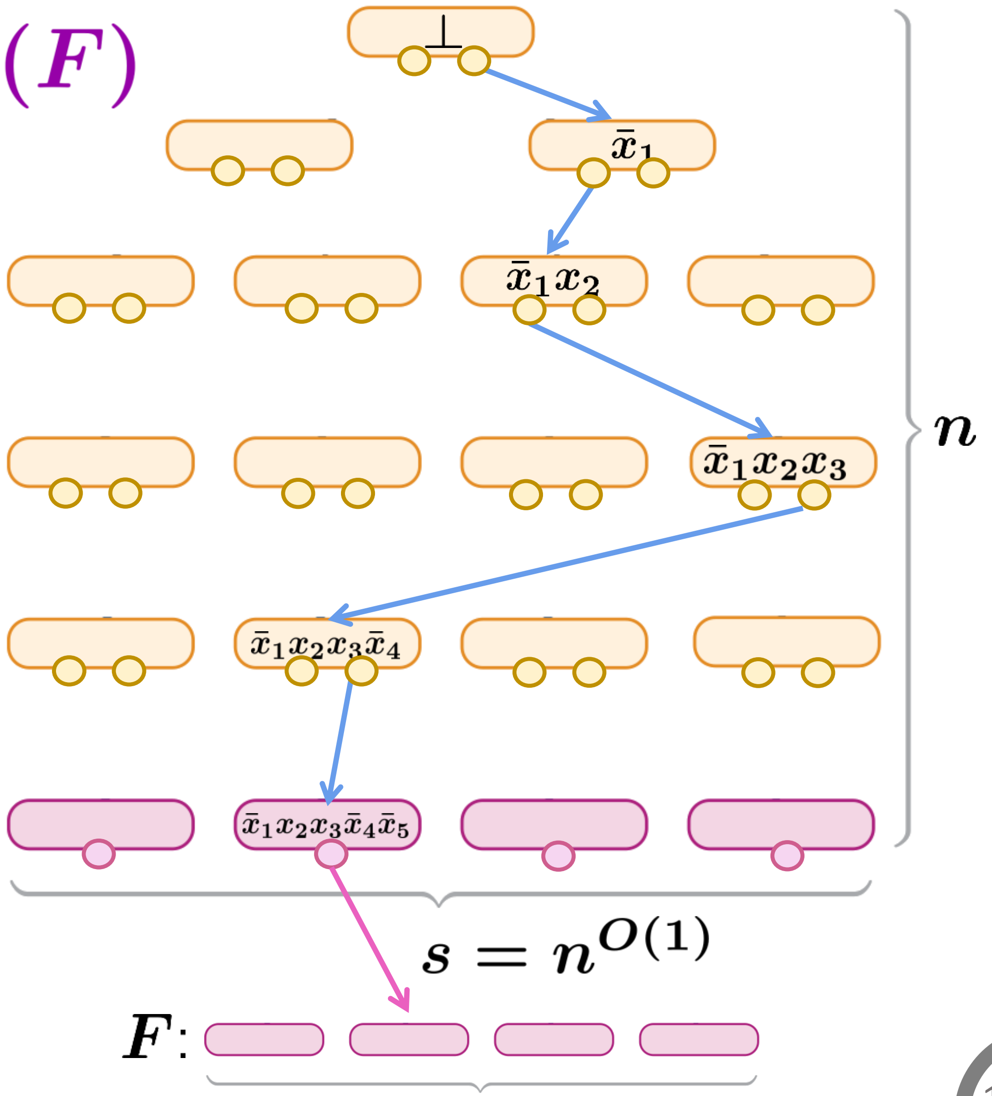
Read-once branching program for $\mathrm{Ref}(F)$

x^* satisfying assignment for F

invariant: x^* falsifies clause in current block B

Start at root and keep invariant
until detect non-valid derivation step or
until reach leaf (cannot be weakening of axiom)

refutation size: $\approx (\#\text{blocks})^2 = n^{O(1)}$



(2) $\boldsymbol{F}$ is **UNSAT** $\Rightarrow$ $\mathbf{Ref}(\boldsymbol{F})$ requires size $\mathbf{2}^{\boldsymbol{\Omega}(\boldsymbol{n})}$ resolution refutation

$\boxed{\text{(2) } F \text{ is UNSAT} \Rightarrow \mathrm{Ref}(F) \text{ requires size } 2^{\Omega(n)} \text{ resolution refutation}}$

- $w(\mathrm{Ref}(F) \vdash \bot) \geq \tilde{\Omega}(w(\mathrm{PHP}_s^{2s} \vdash \bot)/n)$

  [dRGNPRS'21]

$$\text{(2) } \boldsymbol{F} \text{ is } \textbf{UNSAT} \Rightarrow \mathbf{Ref}(\boldsymbol{F}) \text{ requires size } \mathbf{2}^{\boldsymbol{\Omega(n)}} \text{ resolution refutation}$$

- $$\boldsymbol{w}(\mathbf{Ref}(\boldsymbol{F}) \vdash \perp) \geq \widetilde{\boldsymbol{\Omega}}(\boldsymbol{w}(\mathbf{PHP}^{\boldsymbol{2s}}_{\boldsymbol{s}} \vdash \perp)/\boldsymbol{n}) \geq \widetilde{\boldsymbol{\Omega}}(\boldsymbol{s}/\boldsymbol{n})$$

  [dRGNPRS'21]

(2) $\boldsymbol{F}$ is **UNSAT** $\Rightarrow$ $\mathbf{Ref}(\boldsymbol{F})$ requires size $\mathbf{2}^{\boldsymbol{\Omega(n)}}$ resolution refutation

- $\boldsymbol{w(\mathrm{Ref}(F) \vdash \bot) \geq \widetilde{\Omega}(w(\mathrm{PHP}_s^{2s} \vdash \bot)/n) \geq \widetilde{\Omega}(s/n)}$
  [dRGNPRS'21]
- [BW'01] size of resolution refutation of $\varphi \geq \exp\left(\boldsymbol{\Omega}\left(\frac{(w(\varphi \vdash \bot) - w(\varphi))^2}{\#\mathrm{var}}\right)\right)$

(2) F is **UNSAT** $\Rightarrow$ $\mathbf{Ref}(F)$ requires size $2^{\Omega(n)}$ resolution refutation

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- Recall: $\mathbf{Ref}(F)$ has $O(n^2 s)$ variables and width $O(\log s)$

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$$\mathrm{Ref}(F) \text{ requires size } \exp\left(\tilde{\Omega}\left(\frac{(s/n)^2}{n^2 s}\right)\right) \geq \exp\left(\tilde{\Omega}\left(\frac{s}{n^4}\right)\right)$$

(2) F is **UNSAT** $\Rightarrow$ $\mathrm{Ref}(F)$ requires size $2^{\Omega(n)}$ resolution refutation

- $w(\mathrm{Ref}(F) \vdash \bot) \geq \widetilde{\Omega}(w(\mathrm{PHP}_s^{2s} \vdash \bot)/n) \geq \widetilde{\Omega}(s/n)$
  [dRGNPRS'21]
- [BW'01] size of resolution refutation of $\varphi \geq \exp\left(\Omega\left(\frac{(w(\varphi\vdash\bot)-w(\varphi))^2}{\#\text{var}}\right)\right)$
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choose $s \geq n^5$

$\boxed{\text{(2) } F \text{ is } \textbf{UNSAT} \Rightarrow \mathrm{Ref}(F) \text{ requires size } 2^{\Omega(n)} \text{ resolution refutation}}$

- $w(\mathrm{Ref}(F) \vdash \bot) \geq \tilde{\Omega}(w(\mathrm{PHP}_s^{2s} \vdash \bot)/n) \geq \tilde{\Omega}(s/n)$

  [dRGNPRS'21]

- [BW'01] size of resolution refutation of $\varphi \geq \exp\left(\Omega\left(\frac{(w(\varphi \vdash \bot) - w(\varphi))^2}{\#\text{var}}\right)\right)$

- Recall: $\mathrm{Ref}(F)$ has $O(n^2 s)$ variables and width $O(\log s)$

$$\mathrm{Ref}(F) \text{ requires size } \exp\left(\tilde{\Omega}\left(\frac{(s/n)^2}{n^2 s}\right)\right) \geq \exp\left(\tilde{\Omega}\left(\frac{s}{n^4}\right)\right)$$

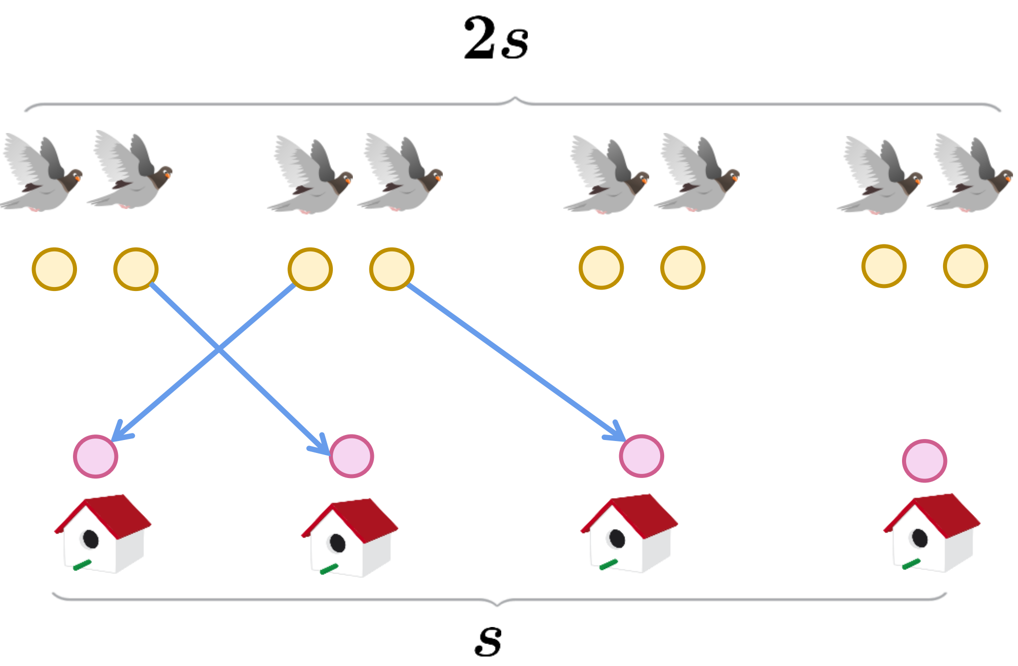
$\boxed{\text{choose } s \geq n^5}$

OBS. this same lower bound proof works for PC, SA:

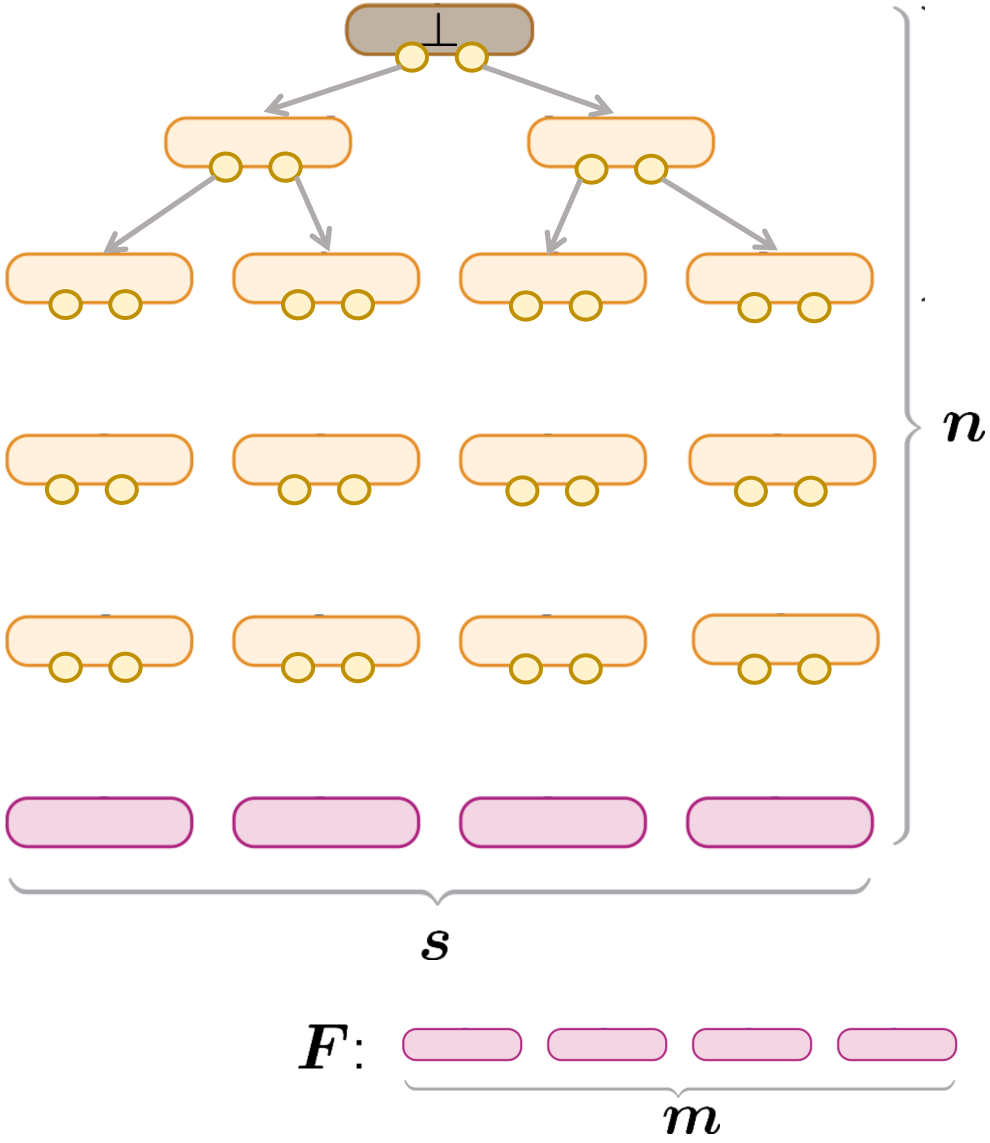
- degree lower bound for PHP_s^{2s}
- similar size-degree relation

$$\mathbf{PHP}_{\boldsymbol{s}}^{\boldsymbol{2s}} \leq_{\widetilde{\boldsymbol{O}}(\boldsymbol{n})} \mathbf{Ref}(\boldsymbol{F})$$

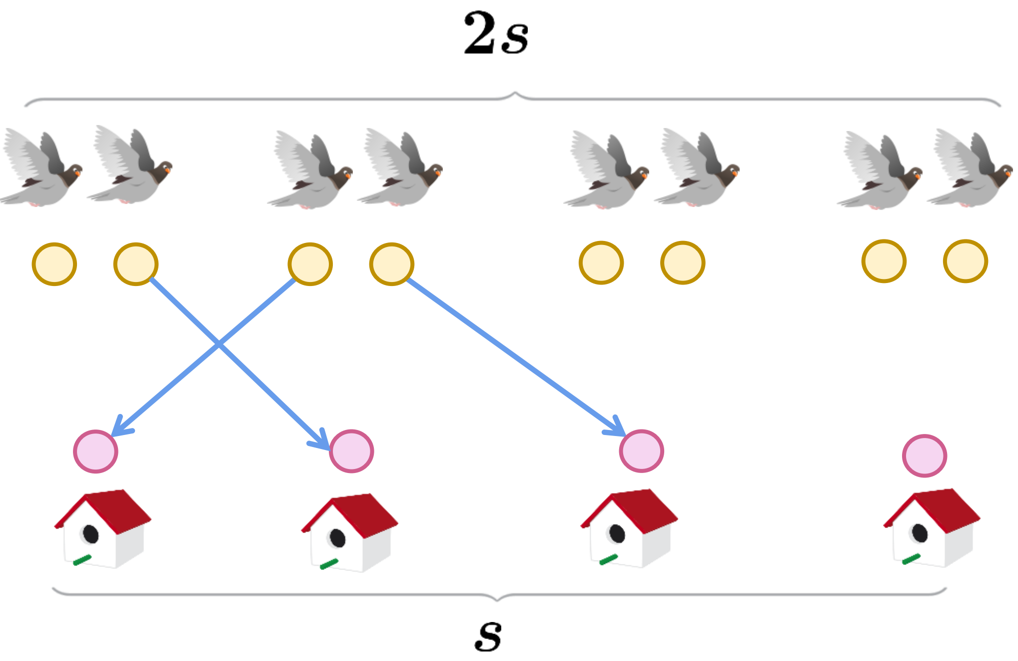
$\mathrm{PHP}_s^{2s} \leq_{\widetilde{O}(n)} \mathrm{Ref}(F)$



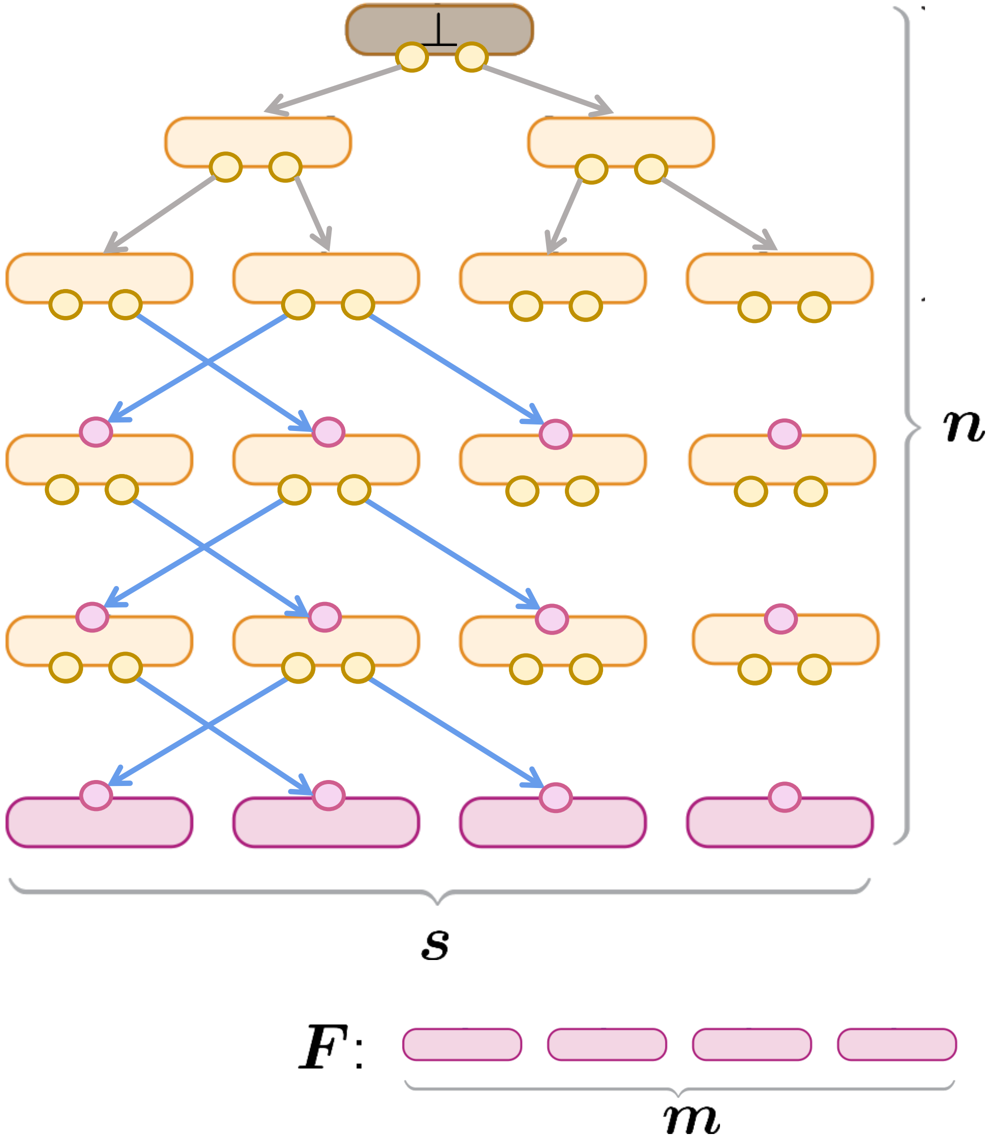
$\leq_{\widetilde{O}(n)}$



$\mathrm{PHP}_s^{2s} \leq_{\widetilde{O}(n)} \mathrm{Ref}(F)$



$\leq_{\widetilde{O}(n)}$

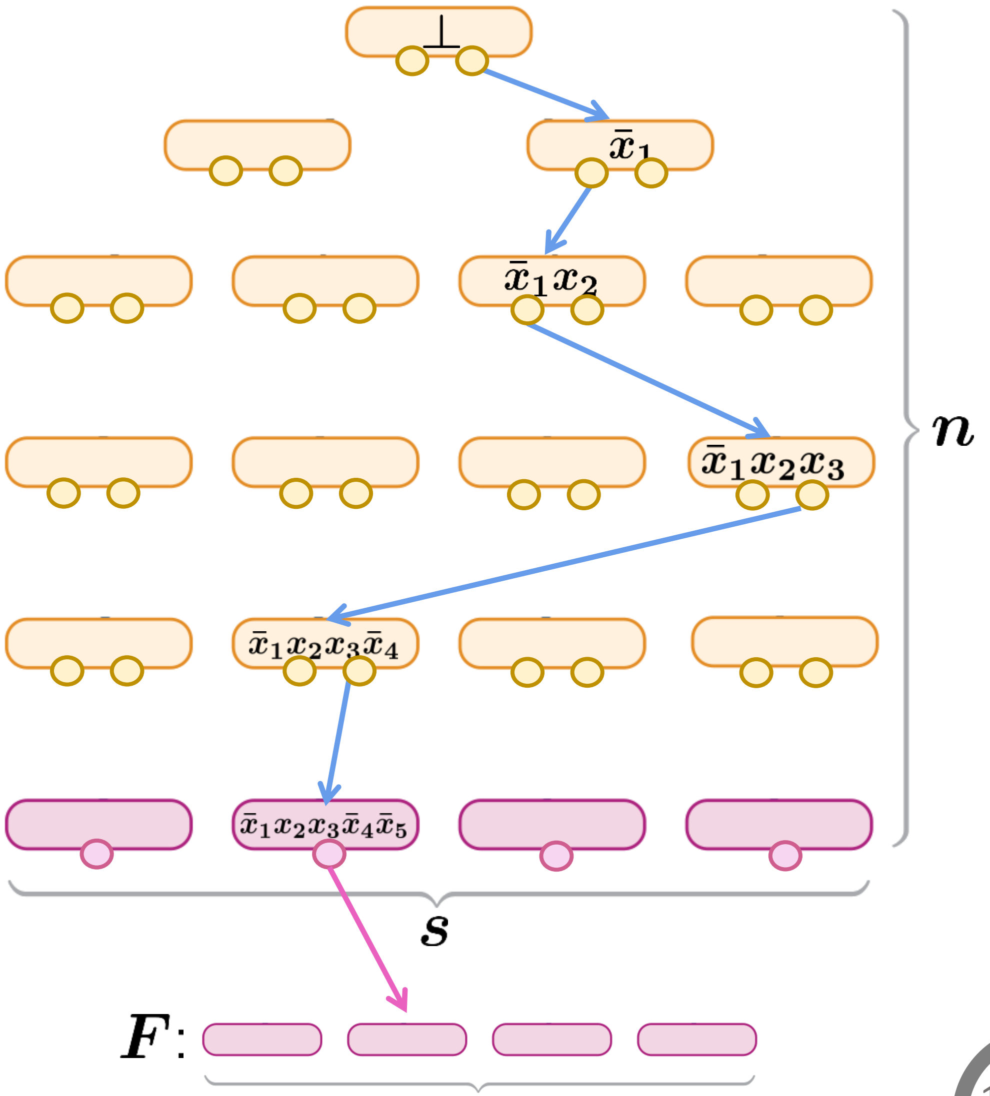


Tree-Like Resolution

(1) $\boldsymbol{F}$ is **SAT** $\Rightarrow$ $\mathbf{Ref}(\boldsymbol{F})$ has size-$\boldsymbol{n}^{\boldsymbol{O(1)}}$ resolution refutation

Read-once branching program for $\mathbf{Ref}(\boldsymbol{F})$

size: $\mathbf{poly}(\boldsymbol{n})$

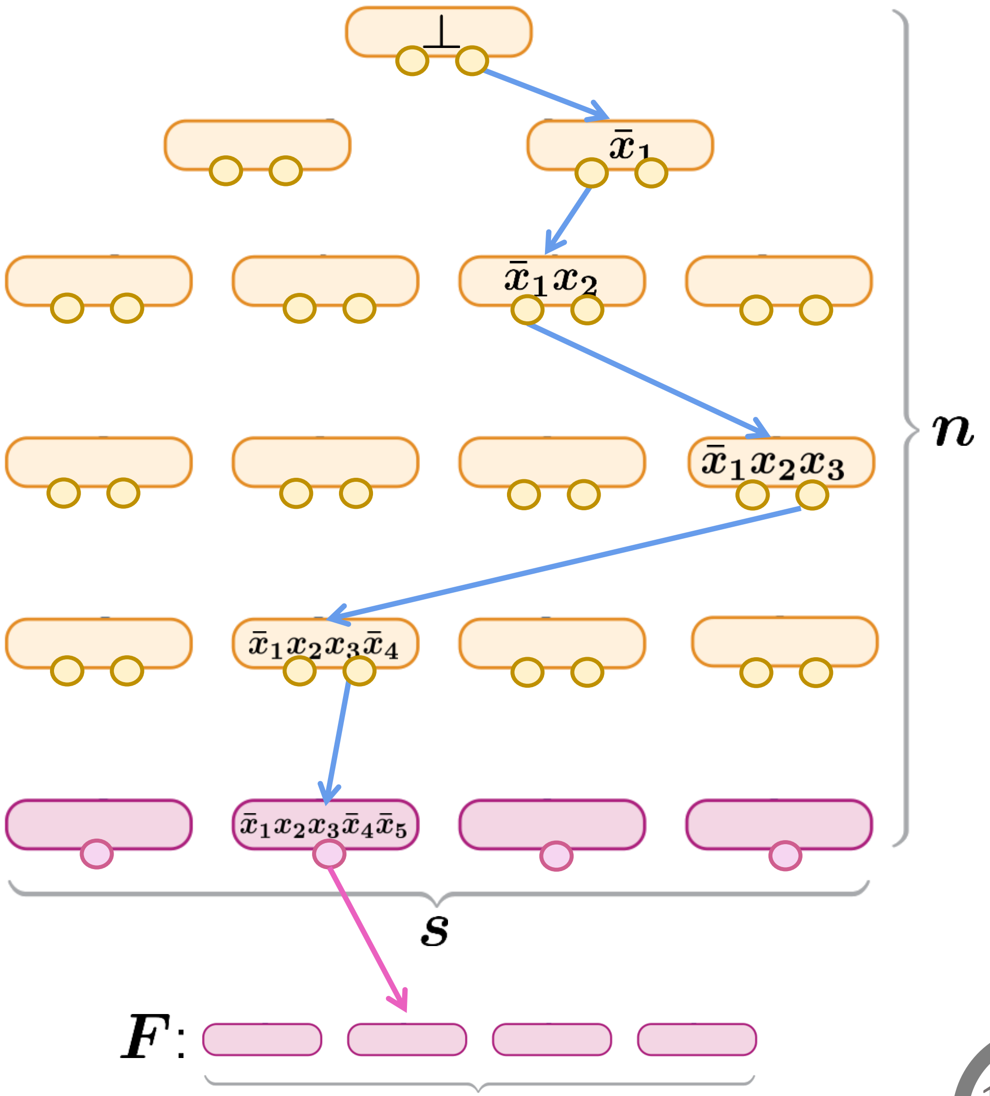


(1) F is **SAT** $\Rightarrow$ $\mathrm{Ref}(F)$ has size-$n^{O(1)}$ resolution refutation

Read-once branching program for $\mathrm{Ref}(F)$

size: $\mathbf{poly}(n)$

Decision tree for $\mathrm{Ref}(F)$



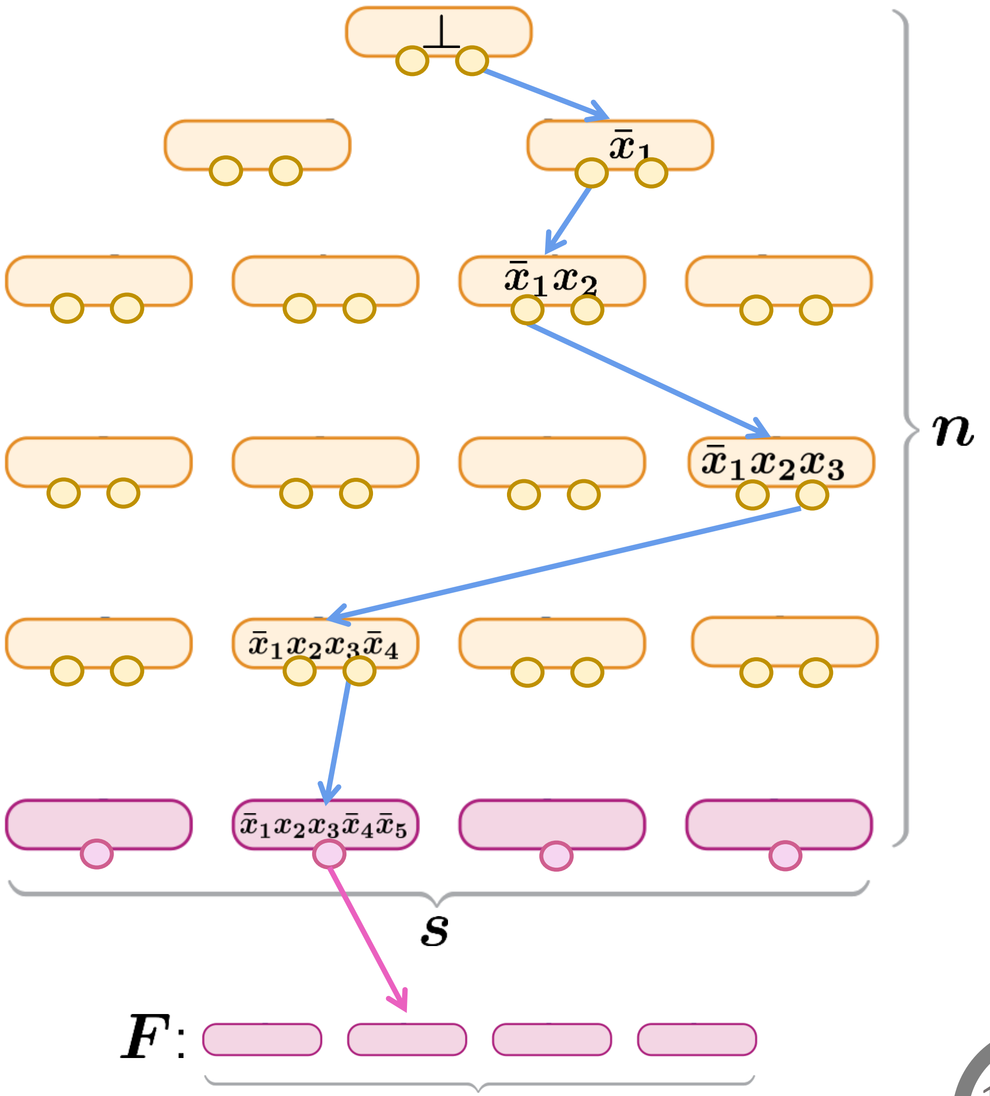
(1) F is **SAT** $\Rightarrow$ $\mathrm{Ref}(F)$ has size-$n^{O(1)}$ resolution refutation

Read-once branching program for $\mathrm{Ref}(F)$

size: $\mathbf{poly}(n)$

Decision tree for $\mathrm{Ref}(F)$

size $\approx$ # root-to-leaf paths $\approx s^n$



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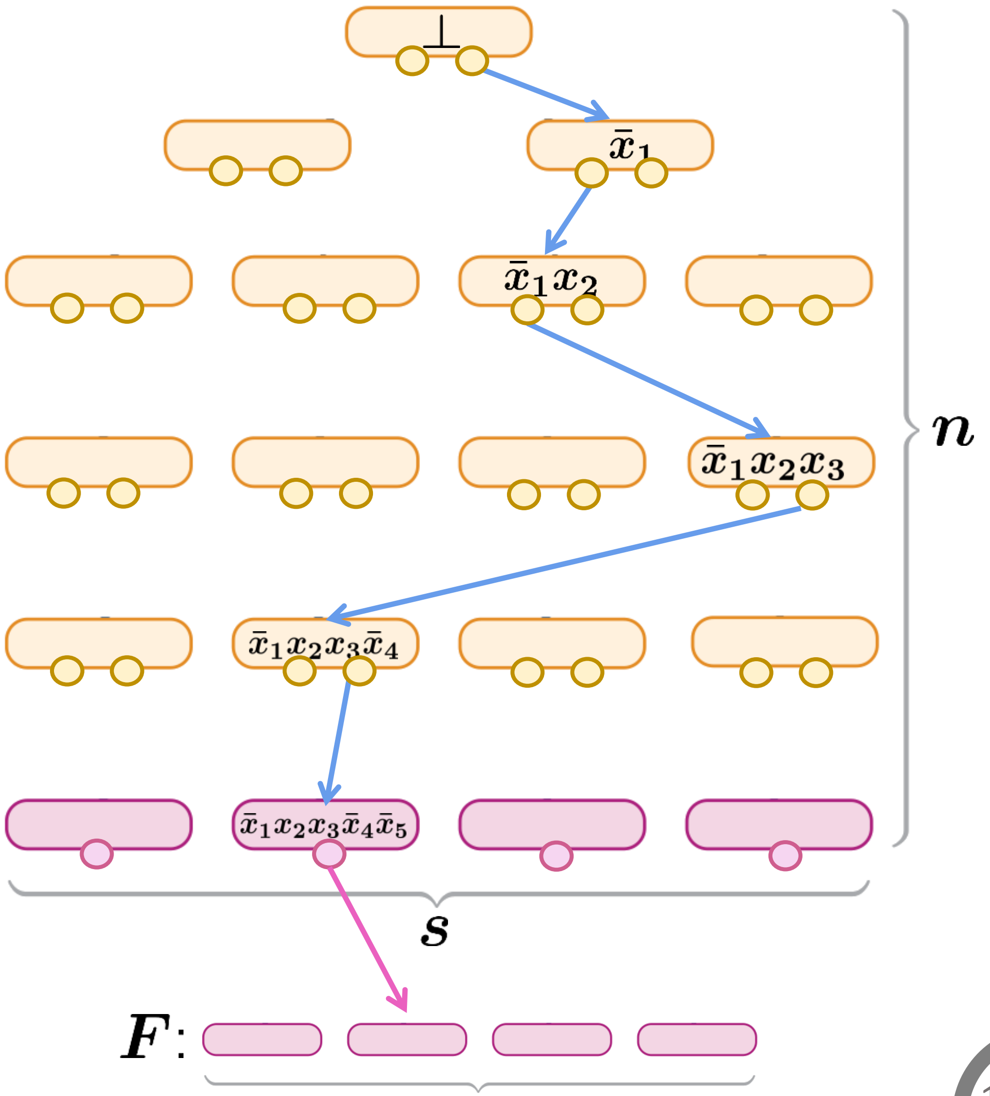
Read-once branching program for $\mathbf{Ref}(\boldsymbol{F})$

size: $\mathbf{poly}(\boldsymbol{n})$

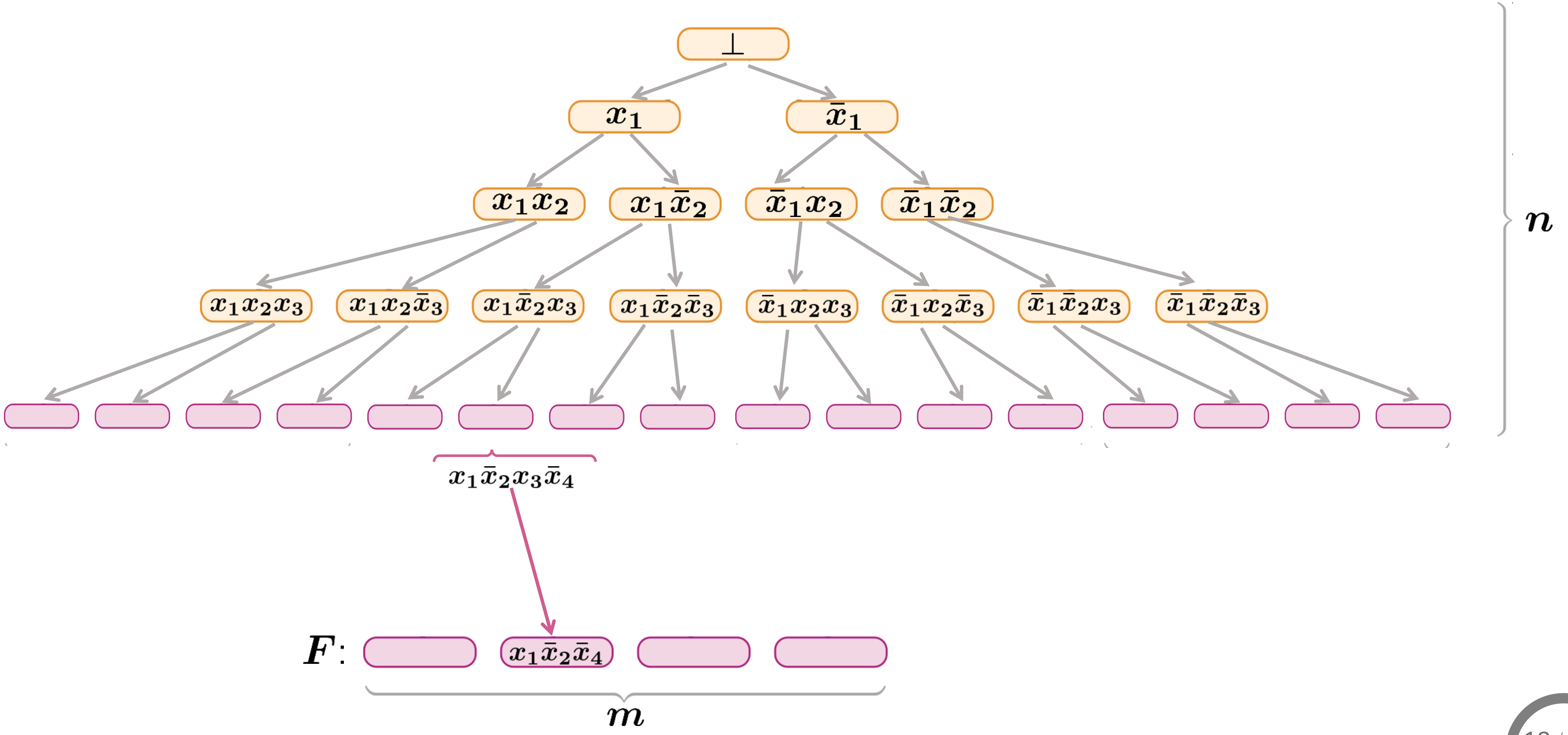
Decision tree for $\mathbf{Ref}(\boldsymbol{F})$

size $\approx$ # root-to-leaf paths $\approx \boldsymbol{s}^{\boldsymbol{n}}$

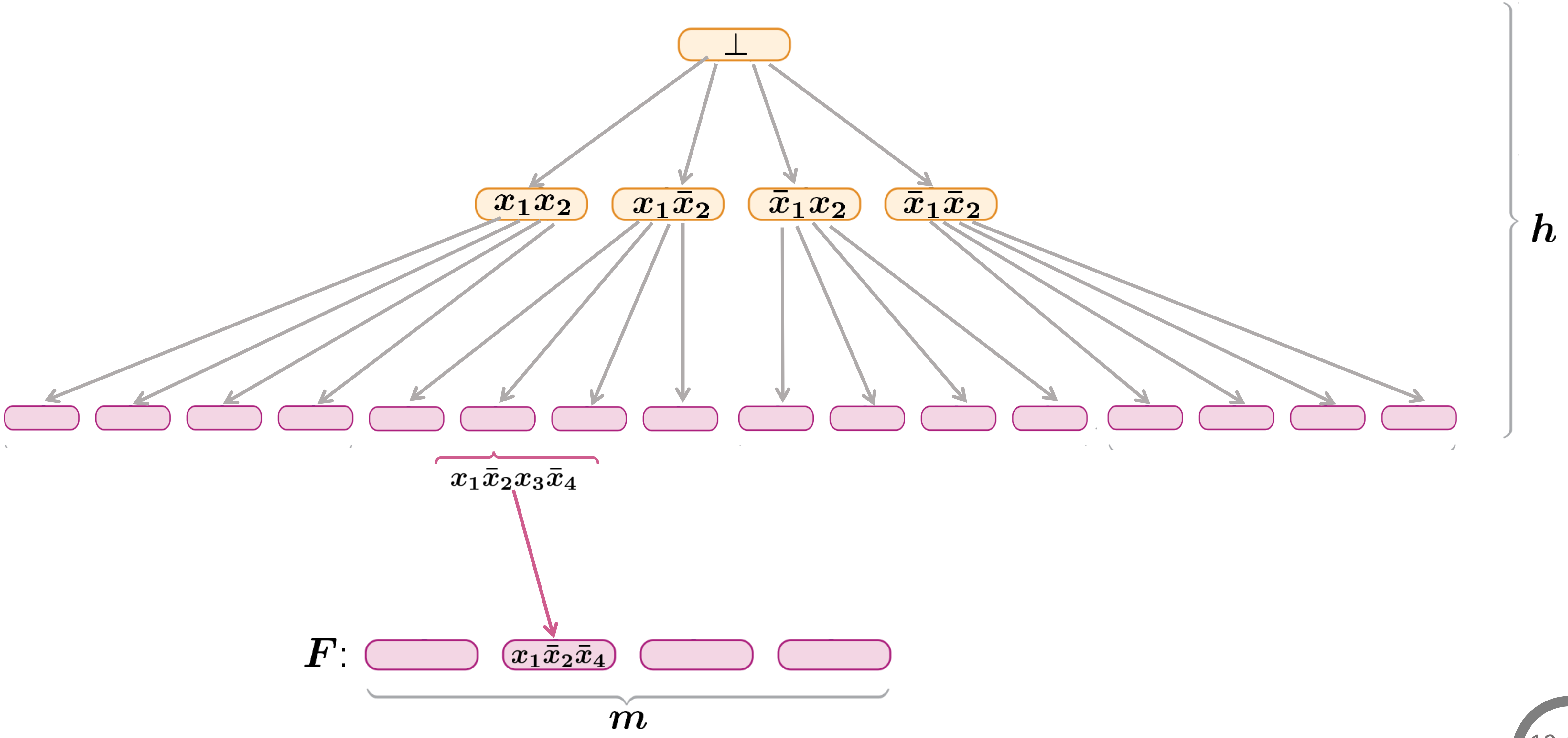
(don't expect upper bound to hold: would imply NP $\subseteq$ QP)



“Universal refutation” (complete Binary tree: depth n)



“Universal refutation” (complete tree: depth $h \ll n$)



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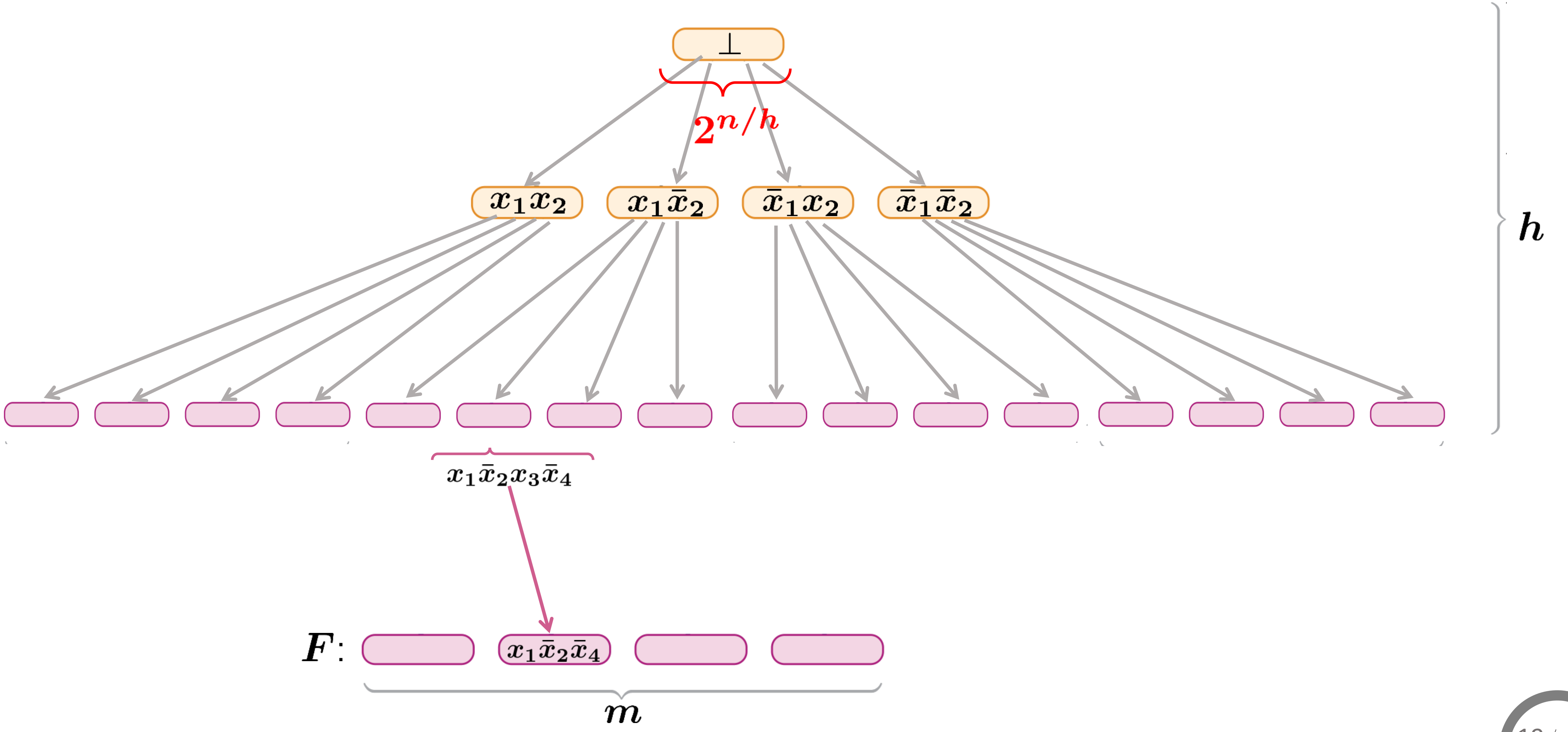

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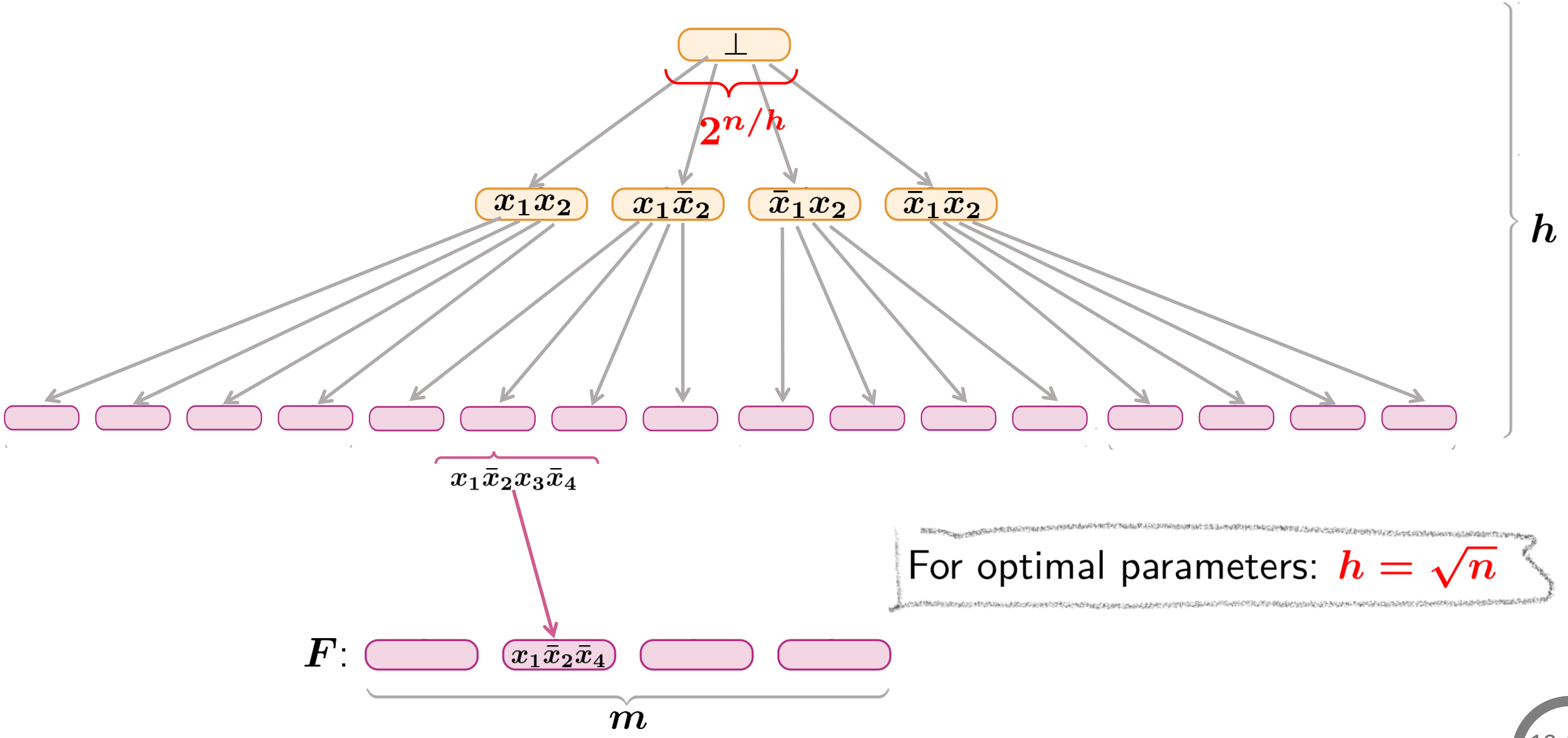


For optimal parameters: $h = \sqrt{n}$

ShallowRef(F)

Variables for each block B:

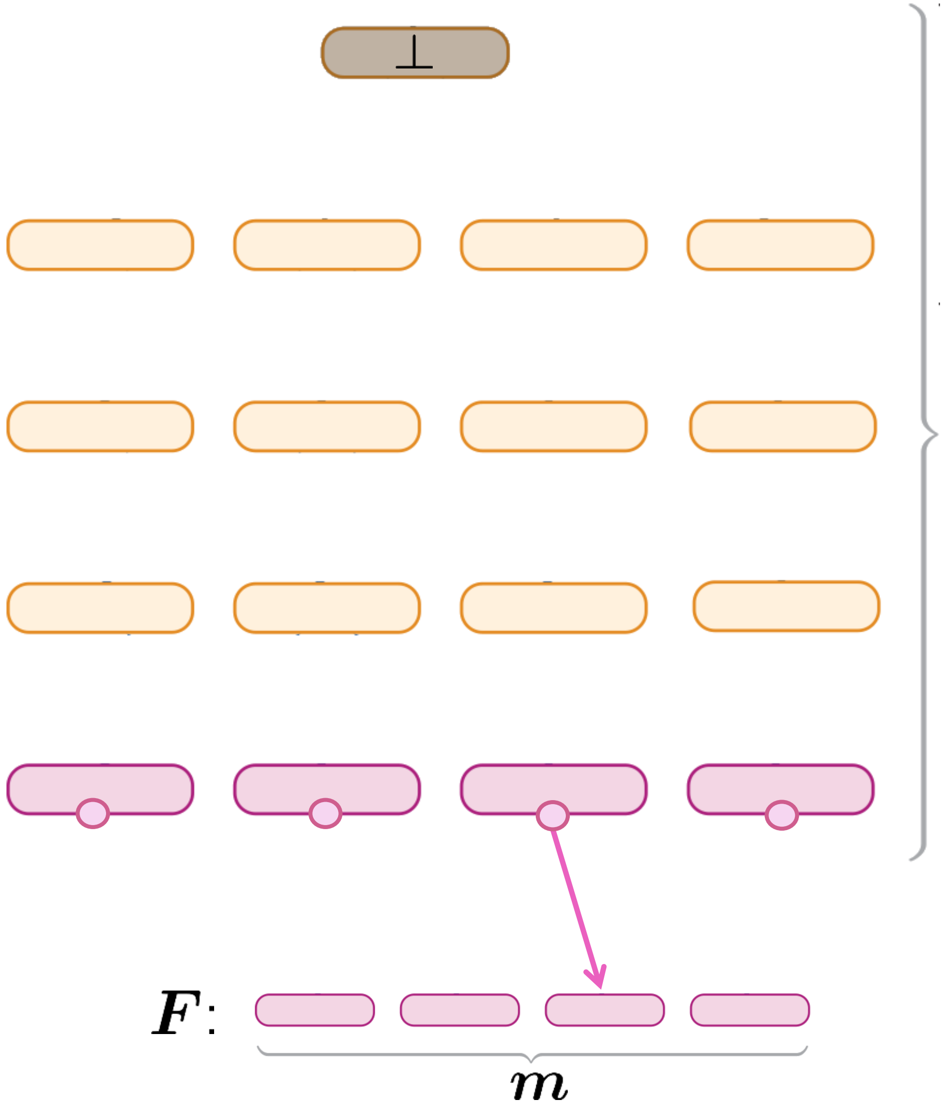
- $2n$ variables (1 per literal): indicates clause
- $2(\log s)$ variables: 2 pointers to children "derived from B_i and B_j"
- $\log m$ variables: axiom-index $j \in [m]$

$O(n^2 s)$ variables

Axioms:

- Root: $\perp$ clause
- Derived: valid resolution step
- Axiom: weakening of axiom

clauses of width $O(\log s)$



$\mathbf{ShallowRef}(F)$

Variables for each block B:

- $2n$ variables (1 per literal): indicates clause
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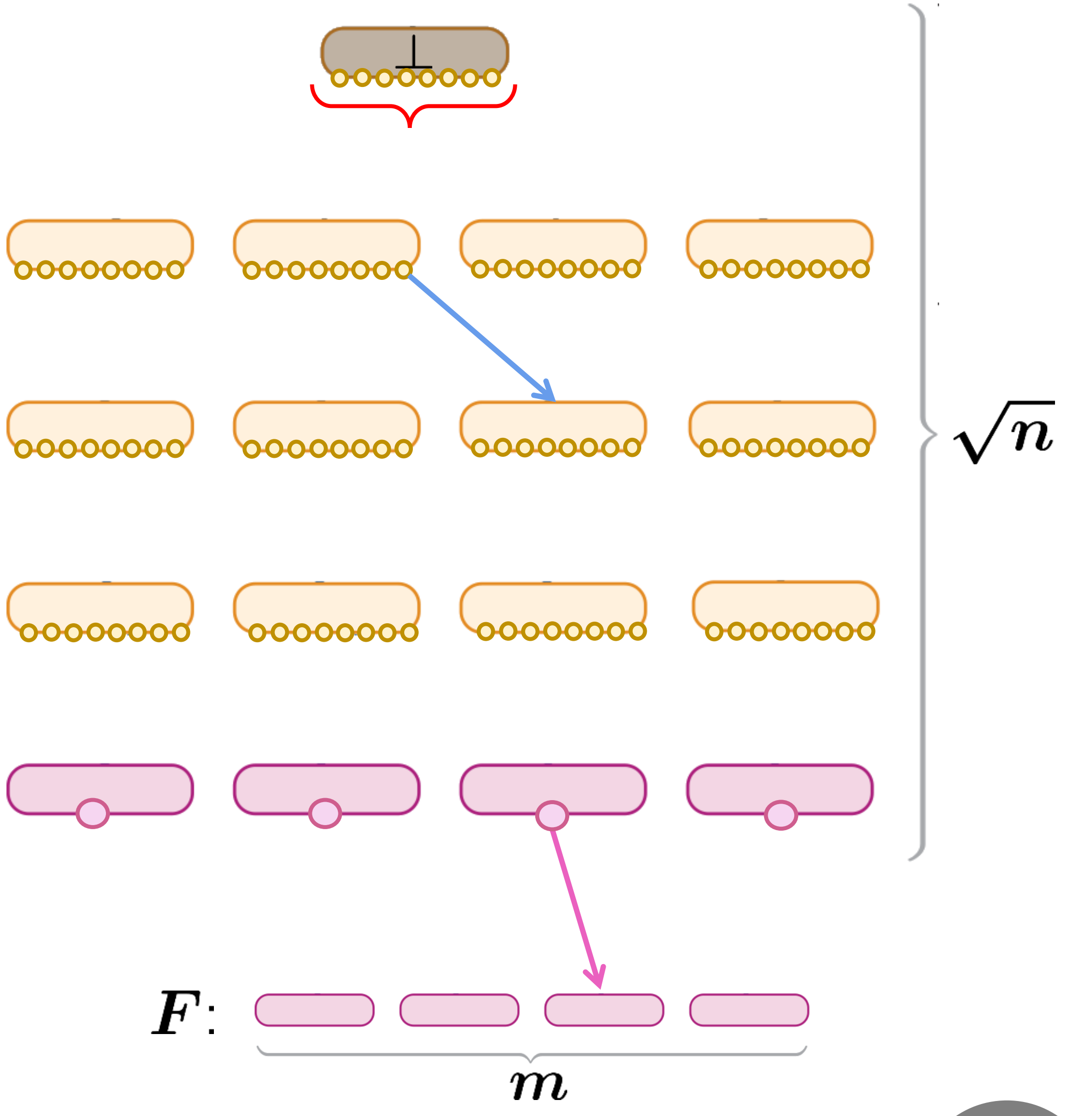


ShallowRef(F)

Variables for each block B:

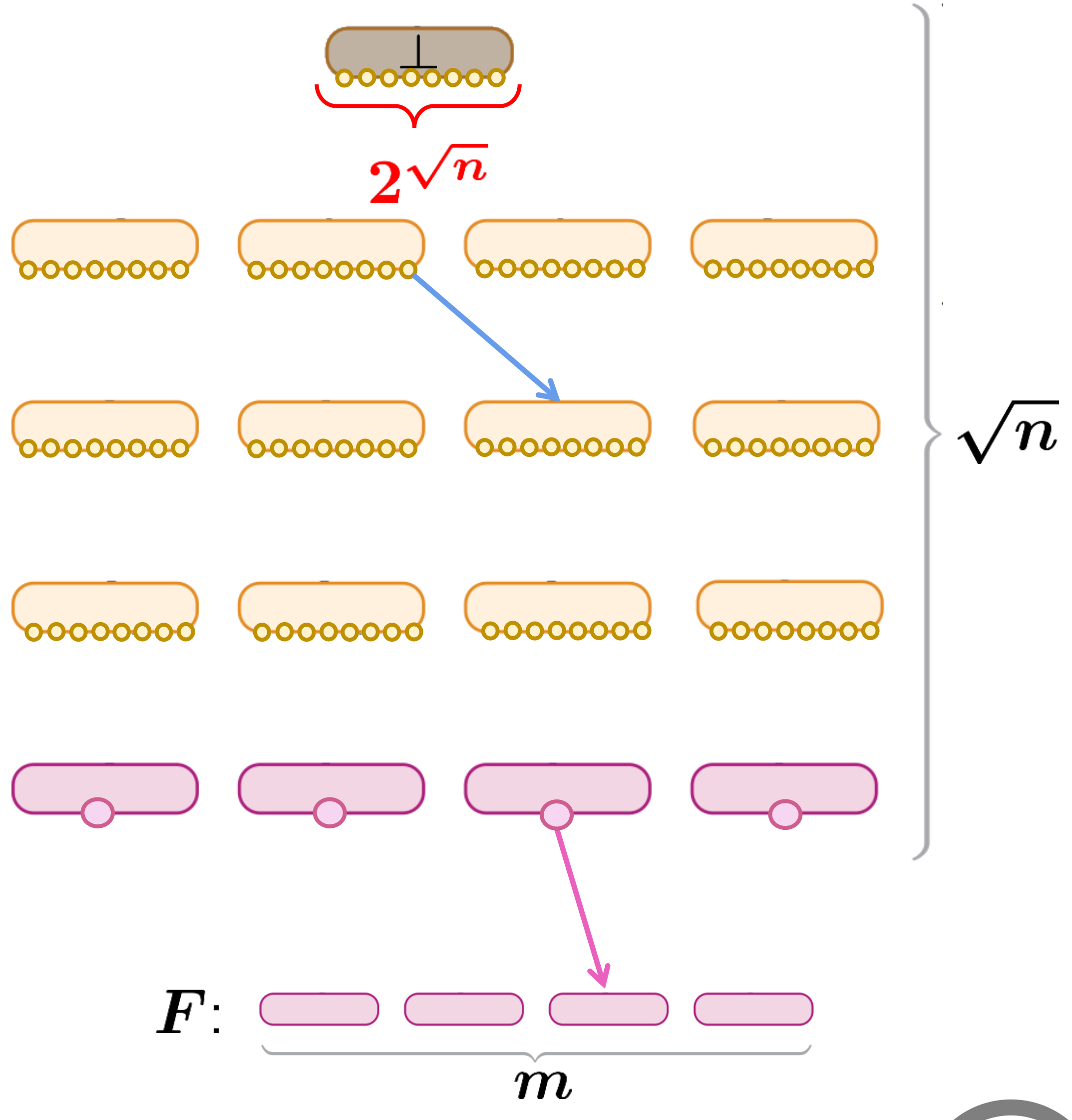
- $2n$ variables (1 per literal): indicates clause
- $O(2^{\sqrt{n}})$ variables: $2^{\sqrt{n}}$ pointers to children
  "derived from all children"
- $\log m$ variables: axiom-index $j \in [m]$

$\boxed{2^{O(\sqrt{n})} \text{ variables}}$

Axioms:

- Root: $\perp$ clause
- Derived: valid resolution step
- Axiom: weakening of axiom

$\boxed{\text{clauses of width } O(\log s)}$



$\mathbf{ShallowRef}(F)$

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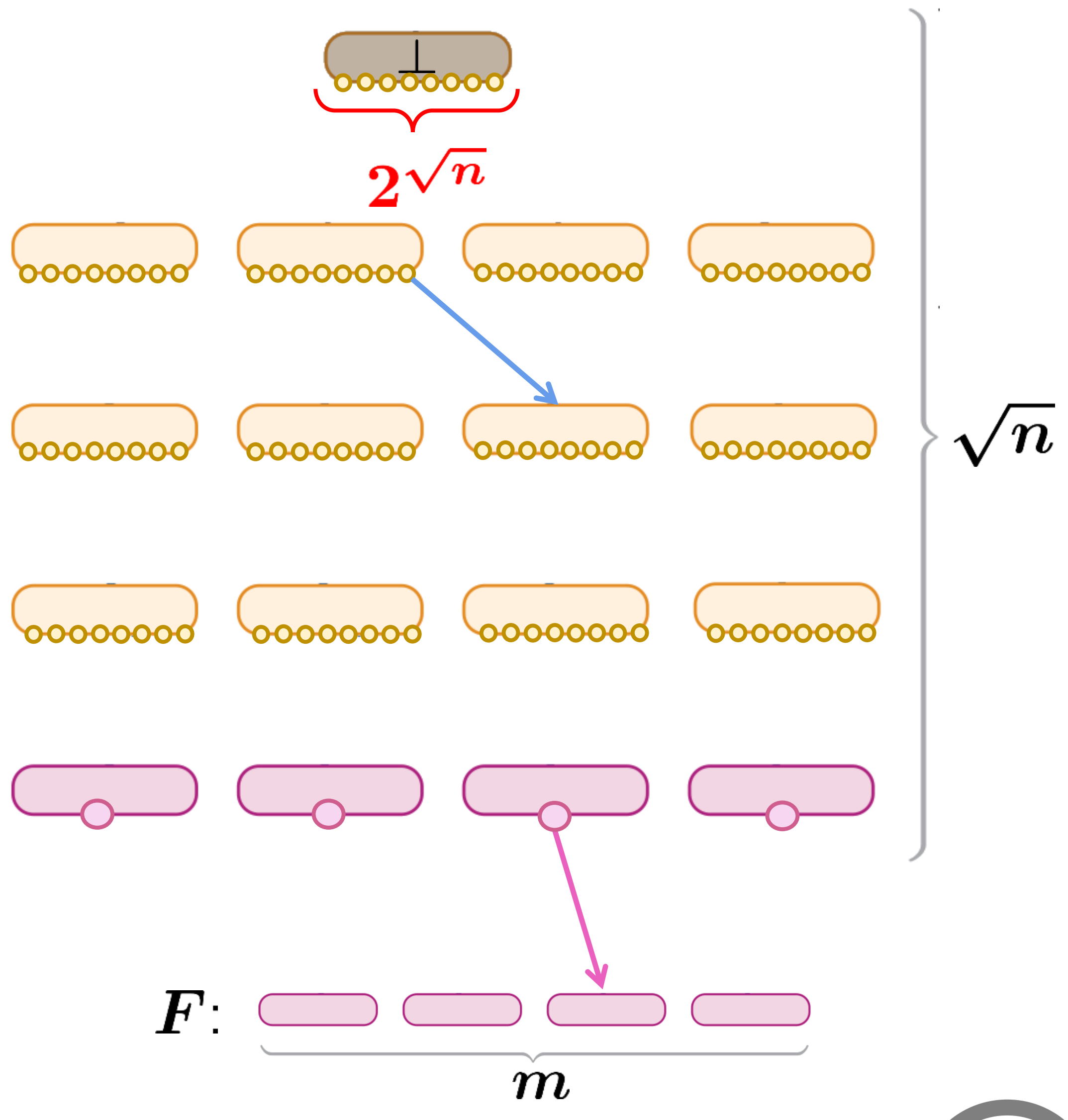
- $O(2^{\sqrt{n}})$ variables: $2^{\sqrt{n}}$ pointers to children
  "derived from all children"
- $\log m$ variables: axiom-index $j \in [m]$

$2^{O(\sqrt{n})}$ variables

OBS. Bounded degree expander between layers (requires $2^{\Omega(\sqrt{n})}$ blocks/layer)

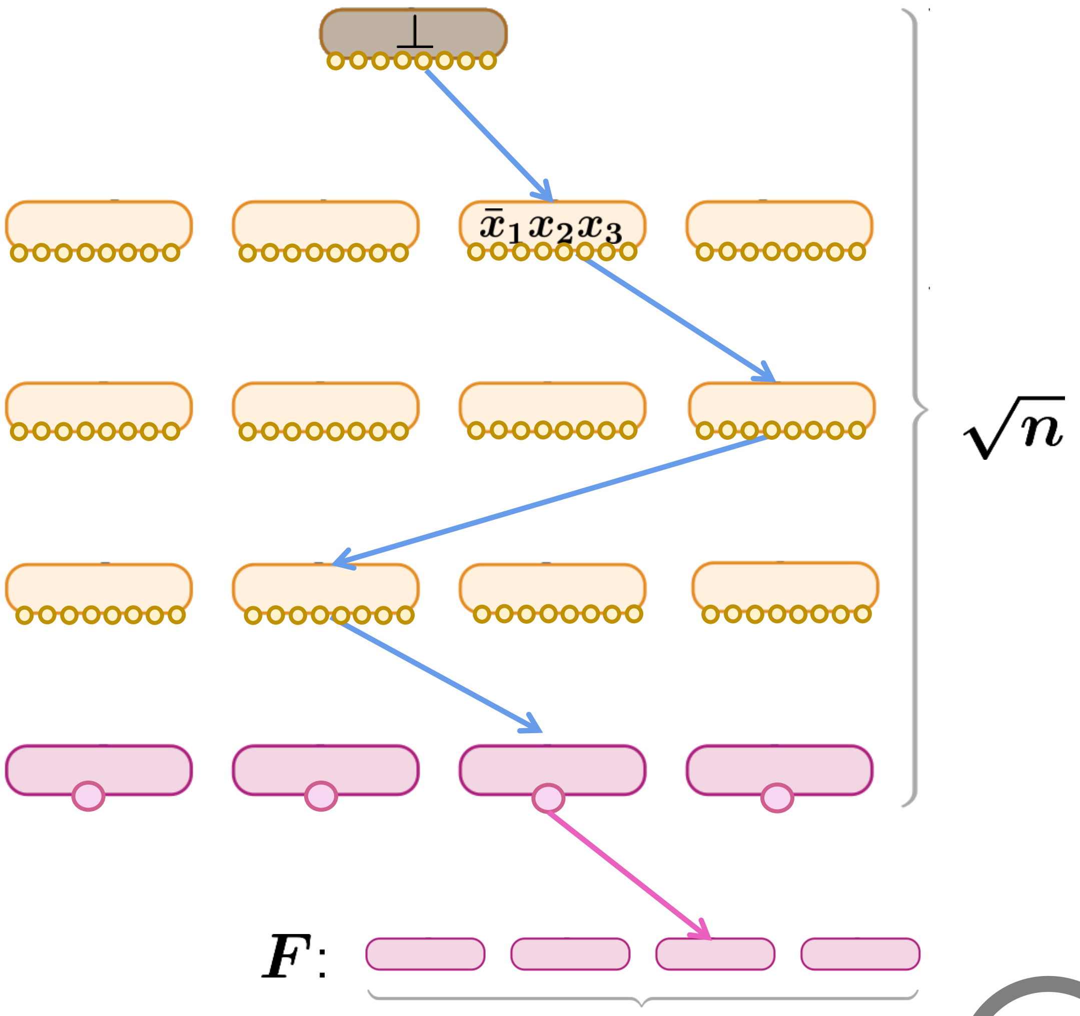
- Root: $\perp$ clause
- Derived: valid resolution step
- Axiom: weakening of axiom

clauses of width $O(\log s)$



(1) $\boldsymbol{F}$ is **SAT** $\Rightarrow$ $\mathbf{ShallowRef}(\boldsymbol{F})$ has size-$\mathbf{2}^{O(\sqrt{n})}$ tree-like resolution refutation

Decision tree for $\mathbf{ShallowRef}(\boldsymbol{F})$



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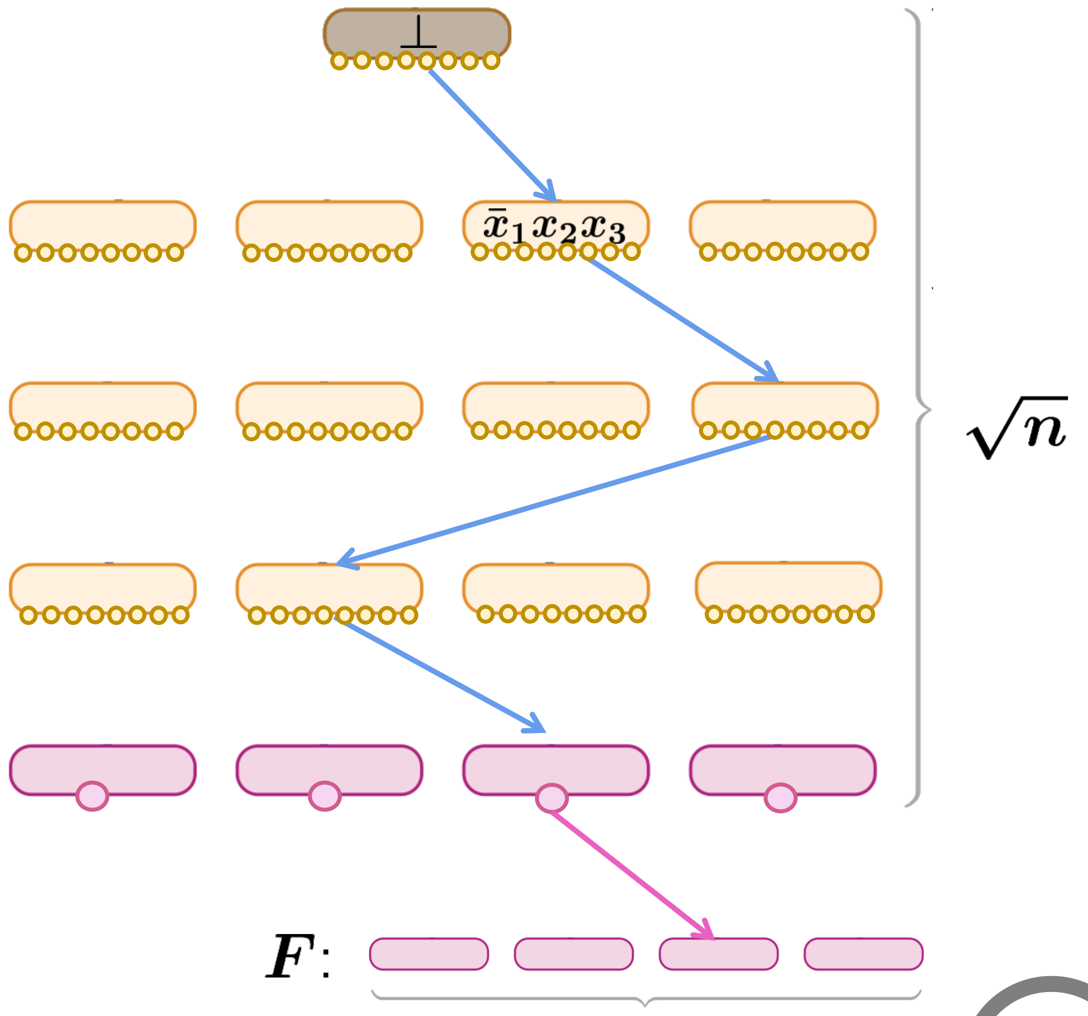
Decision tree for $\mathbf{ShallowRef}(\boldsymbol{F})$

$\boldsymbol{x}^*$ satisfying assignment for $\boldsymbol{F}$

invariant: $\boldsymbol{x}^*$ falsifies clause in block $\boldsymbol{B}$

Start at root and keep invariant
until detect non-valid derivation step or
until reach leaf (cannot be weakening of axiom)



(1) F is **SAT** $\Rightarrow$ $\mathbf{ShallowRef}(F)$ has size-$2^{O(\sqrt{n})}$ tree-like resolution refutation

Decision tree for $\mathbf{ShallowRef}(F)$

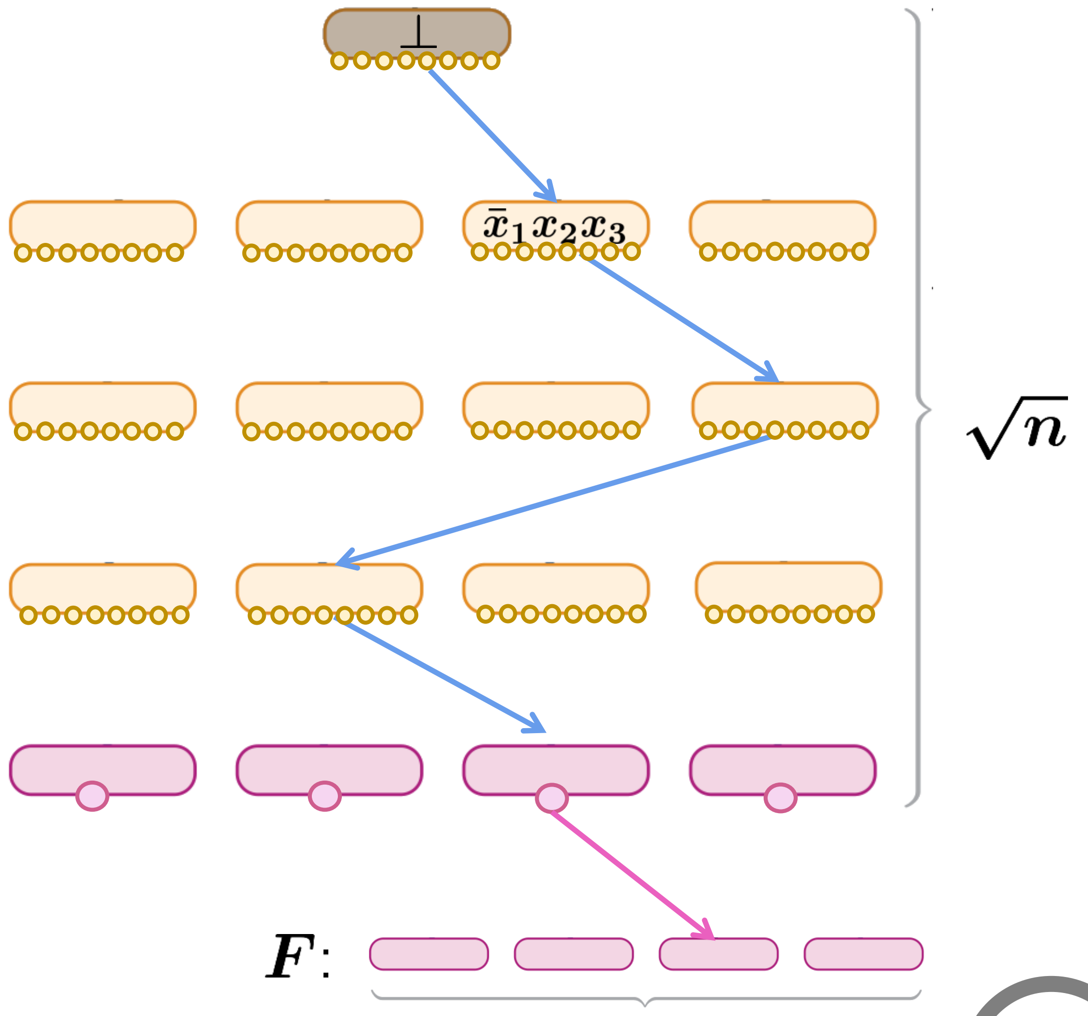
x^* satisfying assignment for F

invariant: x^* falsifies clause in block B

Start at root and keep invariant
until detect non-valid derivation step or
until reach leaf (cannot be weakening of axiom)

OBS. Bounded degree Δ expander

tree-like refutation size $\approx \Delta^{\sqrt{n}} = 2^{O(\sqrt{n})}$



(2) $\boldsymbol{F}$ is **UNSAT** $\Rightarrow$ $\mathbf{ShallowRef}(\boldsymbol{F})$ requires tree-like res refutations size $\mathbf{2}^{\boldsymbol{\Omega(n)}}$

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- $\boldsymbol{w}(\mathbf{ShallowRef}(\boldsymbol{F}) \vdash \bot) \geq \boldsymbol{\Omega}(\boldsymbol{w}(\mathbf{GPHP}^{\mathbf{2}^{(\boldsymbol{c}+\mathbf{1})\sqrt{\boldsymbol{n}}}}_{\mathbf{2}^{\boldsymbol{c}\sqrt{\boldsymbol{n}}}} \vdash \bot)/\boldsymbol{n})$

(2) $\boldsymbol{F}$ is **UNSAT** $\Rightarrow$ $\mathbf{ShallowRef}(\boldsymbol{F})$ requires tree-like res refutations size $\mathbf{2^{\Omega(n)}}$

- $w(\mathbf{ShallowRef}(F) \vdash \bot) \geq \Omega(w(\mathbf{GPHP}_{2^{c\sqrt{n}}}^{2^{(c+1)\sqrt{n}}} \vdash \bot)/n) \geq \Omega(2^{c\sqrt{n}}/n)$

(2) $\boldsymbol{F}$ is **UNSAT** $\Rightarrow$ $\mathbf{ShallowRef}(\boldsymbol{F})$ requires tree-like res refutations size $\mathbf{2}^{\boldsymbol{\Omega(n)}}$

- $\boldsymbol{w}(\mathbf{ShallowRef}(\boldsymbol{F}) \vdash \perp) \geq \boldsymbol{\Omega}(\boldsymbol{w}(\mathbf{GPHP}^{\mathbf{2}^{(\boldsymbol{c}+\mathbf{1})\sqrt{\boldsymbol{n}}}}_{\mathbf{2}^{\boldsymbol{c}\sqrt{\boldsymbol{n}}}} \vdash \perp)/\boldsymbol{n}) \geq \boldsymbol{\Omega}(\mathbf{2}^{\boldsymbol{c}\sqrt{\boldsymbol{n}}}/\boldsymbol{n})$
- [BW'01] size of tree-like resolution refutation of $\boldsymbol{\varphi} \geq \mathbf{2}^{\boldsymbol{w}(\boldsymbol{\varphi} \vdash \perp) - \boldsymbol{w}(\boldsymbol{\varphi})}$

(2) $\boldsymbol{F}$ is **UNSAT** $\Rightarrow$ $\mathbf{ShallowRef}(\boldsymbol{F})$ requires tree-like res refutations size $\mathbf{2}^{\boldsymbol{\Omega(n)}}$

- $w(\mathbf{ShallowRef}(F) \vdash \bot) \geq \Omega(w(\mathbf{GPHP}^{2^{(c+1)\sqrt{n}}}_{2^{c\sqrt{n}}} \vdash \bot)/n) \geq \Omega(2^{c\sqrt{n}}/n)$
- [BW'01] size of tree-like resolution refutation of $\varphi \geq 2^{w(\varphi \vdash \bot) - w(\varphi)}$
- $\mathbf{ShallowRef}(F)$ has width $O(1)$ and $\#$ variables: $\mathbf{poly}(n) \cdot 2^{(c+1)\sqrt{n}}$

(2) F is **UNSAT** $\Rightarrow$ $\mathbf{ShallowRef}(F)$ requires tree-like res refutations size $2^{\Omega(n)}$

- $w(\mathbf{ShallowRef}(F) \vdash \bot) \geq \Omega(w(\mathbf{GPHP}^{2^{(c+1)\sqrt{n}}}_{2^{c\sqrt{n}}} \vdash \bot)/n) \geq \Omega(2^{c\sqrt{n}}/n)$
- [BW'01] size of tree-like resolution refutation of $\varphi \geq 2^{w(\varphi \vdash \bot) - w(\varphi)}$
- $\mathbf{ShallowRef}(F)$ has width $O(1)$ and # variables: $\mathbf{poly}(n) \cdot 2^{(c+1)\sqrt{n}}$

$\mathbf{ShallowRef}(F)$ requires tree-like res refutation size $2^{\Omega(2^{c\sqrt{n}}/n)} \geq 2^{\Omega(n)}$

- $w(\mathrm{ShallowRef}(F) \vdash \bot) \geq \Omega(w(\mathrm{GPHP}^{2^{(c+1)\sqrt{n}}}_{2^{c\sqrt{n}}} \vdash \bot)/n) \geq \Omega(2^{c\sqrt{n}}/n)$
- [BW'01] size of tree-like resolution refutation of $\varphi \geq 2^{w(\varphi \vdash \bot) - w(\varphi)}$
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$\mathrm{ShallowRef}(F)$ requires tree-like res refutation size $2^{\Omega(2^{c\sqrt{n}}/n)} \geq 2^{\Omega(n)}$

(2) F is **UNSAT** $\Rightarrow$ $\mathbf{ShallowRef}(F)$ requires dag-like res refutations size $\mathbf{2^{\Omega(n)}}$

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- [BW'01] size of tree-like resolution refutation of $\varphi \geq 2^{w(\varphi \vdash \bot) - w(\varphi)}$
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OBS. this same lower bound holds for PC:

- degree lower bound for $\mathbf{GPHP}^{\mathbf{poly}(s)}_{s}$
- similar size-degree relation

Generalization

If tree-like resolution is automatable:

1. in time $\boldsymbol{n^{o(\log n)}}$ then ETH is false
2. in time $\boldsymbol{n^{O(\log^{1-\epsilon} n)}}$ then $\mathsf{NP} \subseteq \mathsf{DTIME}(\boldsymbol{2^{O(n^{1-\epsilon/2})}})$
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Classical result in parameterized complexity [ADF'95]

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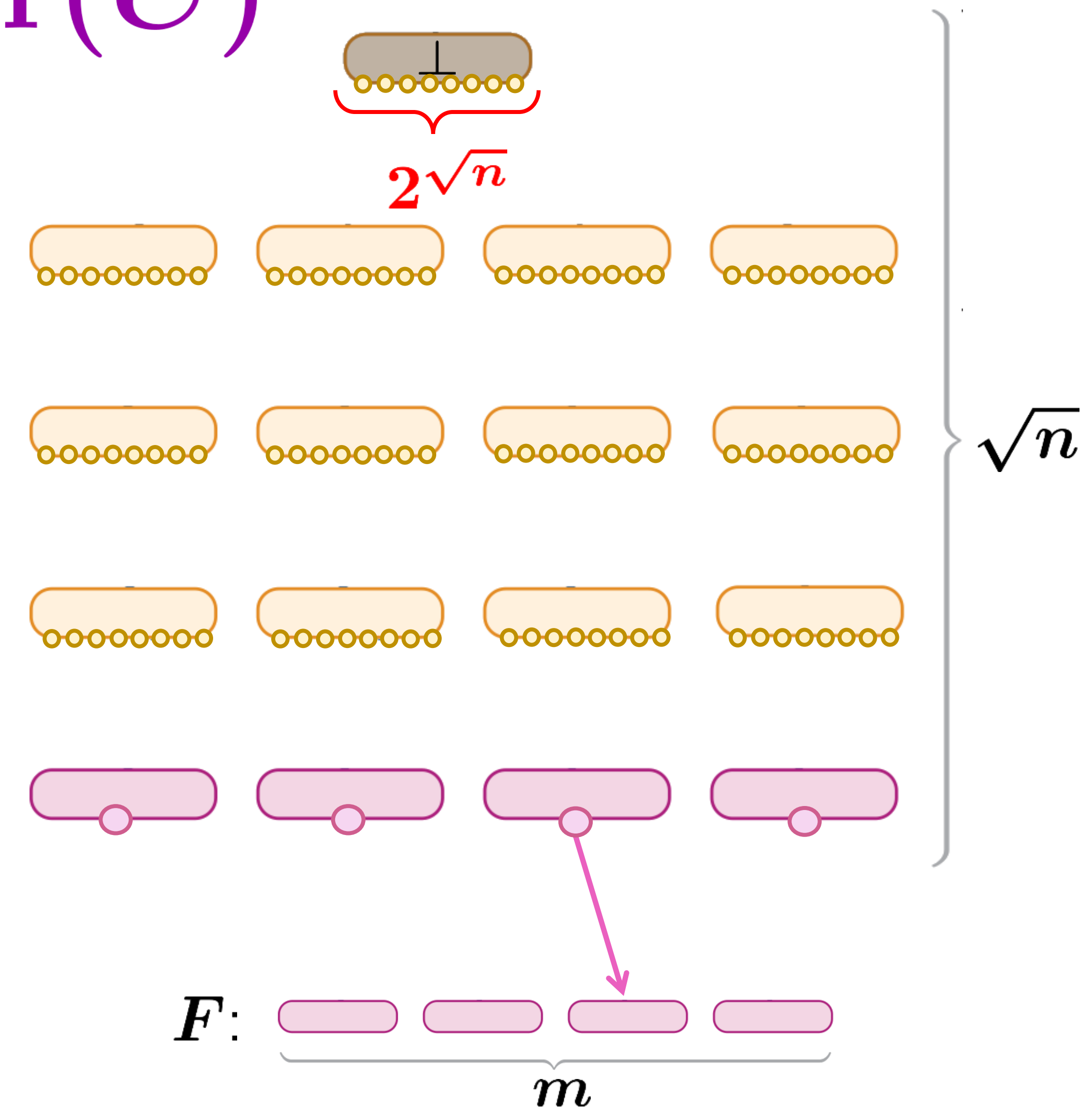
Main Theorem

$\exists$ algorithm that given n-variate circuit C of size m outputs CNF $\mathcal{A}(C)$ in time $\mathbf{poly}(m) \cdot \mathbf{2}^{O(\sqrt{n})}$ s.t.

1. C is SAT $\Rightarrow R^*(\mathcal{A}(C)) \leq \mathbf{poly}(m) \cdot \mathbf{2}^{O(\sqrt{n})}$
2. C is UNSAT $\Rightarrow PC(\mathcal{A}(C)) \geq \mathbf{2}^{\Omega(n)}$

$\mathbf{ShallowRef}(C)$



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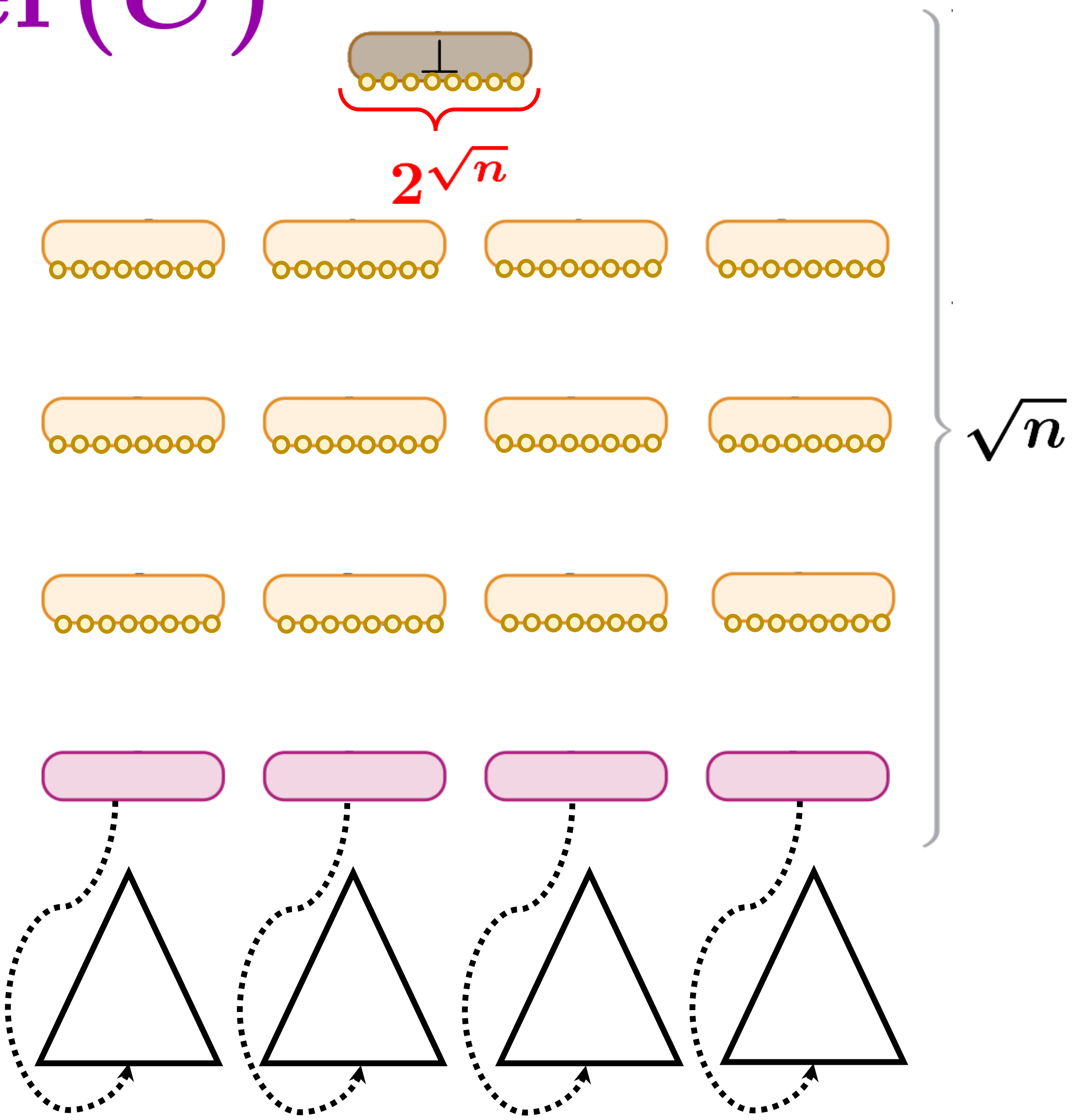

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Thanks!