Consistency of circuit lower bounds with bounded theories

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Talk based on joint work with **Jan Bydžovský** (Vienna) and **Jan Krajíček** (Prague).

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Status of circuit lower bounds

- Interested in **unrestricted** (non-uniform) Boolean circuits.

- Proving a lower bound such as NP $\not\subseteq$ SIZE$[n^2]$ seems out of reach.

Frontiers

$\mathsf{ZPP}^{\mathsf{NP}} \not\subseteq \mathsf{SIZE}[n^k]$ [Kobler-Watanabe'90s]

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- **Frontier 1**: Lower bounds for **deterministic** class P^{NP}?

While we have lower bounds for larger classes, **there is an important issue**:

- **Frontier 2**: Known results only hold on **infinitely many input lengths.**

a.e. versus i.o. results in algorithms and complexity

- **Mystery:** Existence of mathematical objects of certain sizes making computations easier only around corresponding input lengths.

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- Issue **not** restricted to complexity lower bounds:

Sub-exponential time generation of canonical prime numbers [Oliveira-Santhamam'17].

The logical approach

- We discussed two frontiers in complexity theory:

1. Understand relation between P^{NP} and say $\mathsf{SIZE}[n^2]$.
2. Establish almost-everywhere circuit lower bounds.

- This work investigates these challenges from the **perspective of mathematical logic**.

Investigating complexity through logic

- Theories in the standard framework of first-order logic.

- Investigation of complexity results that can be established under certain axioms.

Example: Does theory T prove that SAT can be solved in polynomial time?

- **Complexity Theory** that considers **efficiency** and **difficulty of proving correctness**.

Bounded Arithmetics

- Fragments of Peano Arithmetic (PA).
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Example: **Theory $\boldsymbol{I\Delta_0}$** [Parikh'71].

$I\Delta_0$ employs the language $\mathcal{L}_{PA} = \{0, 1, +, \cdot, <\}$.

14 axioms governing these symbols, such as:

1. $\forall x \;\; x + 0 = x$
2. $\forall x \forall y \;\; x + y = y + x$
3. $\forall x \;\; x = 0 \lor 0 < x$

$\ldots$

Bounded formulas and bounded induction

Induction Axioms. $I\Delta_0$ also contains the induction principle

$$\psi(0) \land \forall x\,(\psi(x) \to \psi(x+1)) \to \forall x\,\psi(x)$$

for each **bounded formula** $\psi(x)$ (additional free variables are allowed in ψ).

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- **Roughly, this shifts the perspective from computability to complexity theory.**

Theories PV, S_2^1, and T_2^1

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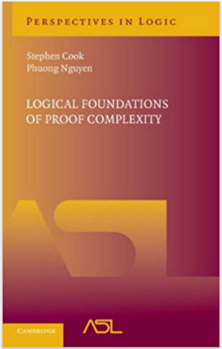
$$\text{PV} \approx \text{T}_2^0 \subseteq S_2^1 \subseteq T_2^1 \subseteq S_2^2 \subseteq T_2^2 \subseteq \ldots \subseteq \bigcup_i T_2^i \approx I\Delta_0 + \Omega_1$$

Resources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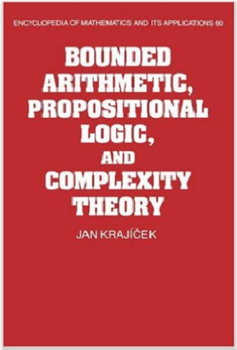
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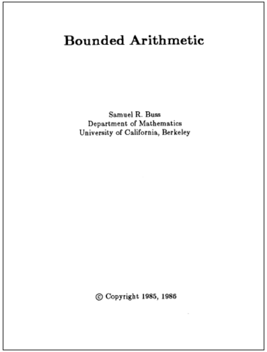
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Samuel R. Buss
Department of Mathematics
University of California, Berkeley
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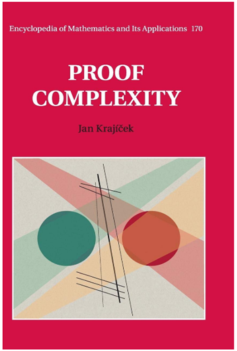
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COMPLEXITY
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Formalizations in Bounded Arithmetic

- Many complexity results have been formalized in such theories.

Cook-Levin Theorem in PV [folklore].

PCP Theorem in PV [Pich'15].

Parity $\notin$ AC^0, k-Clique $\notin$ mSIZE$[n^{\sqrt{k}/1000}]$ in APC$^1 \subseteq T^2_2$ [Muller-Pich'19].

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Rest of the talk: **Independence of complexity results** from bounded arithmetic.

Unprovability and circuit complexity

- Using $\mathcal{L}_{PV}$, we can refer to circuit complexity:

$$\exists y\ (\mathsf{Ckt}(y) \wedge \mathsf{Vars}(y) = n \wedge \mathsf{Size}(y) \leq 5n \wedge \forall x\ (|x| = n \rightarrow (\mathsf{Eval}(y, x) = 1 \leftrightarrow \mathsf{Parity}(x) = 1)))$$

n is the "feasibility" parameter (formally, the length of another variable N).

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Two directions: unprovability of **LOWER** bounds and unprovability of **UPPER** bounds.

Unprovability of circuit **LOWER** bounds

- Initiated by Razborov in the nineties under a different formalization.

Motivation: Why are complexity lower bounds so difficult to prove?

Also: potential source of hard tautologies; self-referential arguments and implications.

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Example: Is it the case that $T^2_2 \nvdash k$-Clique $\notin$ SIZE$[n^{\sqrt{k}/100}]$?

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Example: Is it the case that $T^2_2 \nvdash k$-Clique $\notin$ SIZE$[n^{\sqrt{k}/100}]$?

- Extremely interesting, but not much is known in terms of **unconditional** unprovability results for strong theories such as PV.

Unprovability of circuit **UPPER** bounds

- We currently cannot rule out that SAT $\in$ SIZE$[10n]$. Can we at least show that easiness of SAT doesn't follow from certain axioms?

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   – By Godel's completeness theorem, there is a model M of T where SAT is hard.
   – M satisfies many known results in algorithms and complexity theory.

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2. **Formal evidence** that SAT is computationally hard:

   – By Godel's completeness theorem, there is a model M of T where SAT is hard.
   – M satisfies many known results in algorithms and complexity theory.

3. **Consistency of lower bounds:** Adding to T axiom stating that SAT is hard will never lead to a contradiction. We can develop a theory where circuit lower bounds exist.

Some works on unprovability of circuit upper bounds

- Cook-Krajicek, 2007: "*Consequences of the provability of* $\mathsf{NP} \subseteq \mathsf{P}/\mathsf{poly}$".

Initiated a systematic investigation. Conditional unprovability results.

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- Bydzovsky-Muller, 2018: "*Polynomial time ultrapowers and the consistency of circuit lower bounds.*".

Model-theoretic proof of a slightly stronger statement.

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PV might still show that every $L \in \mathsf{P}$ is infinitely often in $\mathsf{SIZE}[n^k]$.

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PV might still show that every $L \in$ P is infinitely often in SIZE$[n^k]$.

- Recall issue mentioned earlier in the talk:

We lack techniques to show hardness with respect to every large enough input length.

This work

- T^1_2 and weaker theories cannot establish circuit upper bounds of the form n^k for classes contained in P^{NP}.

- Unprovability of infinitely often upper bounds, i.e., models where hardness holds almost everywhere.

- All results are **unconditional**.

Our main result

Theorem 1 (Informal): For each $k \geq 1$,

$$T_2^1 \nvdash \mathsf{P}^{\mathsf{NP}} \subseteq \text{i.o.SIZE}[n^k]$$

$$S_2^1 \nvdash \mathsf{NP} \subseteq \text{i.o.SIZE}[n^k]$$

$$\mathsf{PV} \nvdash \mathsf{P} \subseteq \text{i.o.SIZE}[n^k]$$

Our main result

Theorem 1 (Informal): For each $k \geq 1$,

$$T^1_2 \nvdash \mathsf{P}^{\mathsf{NP}} \subseteq \text{i.o.SIZE}[n^k]$$

$$S^1_2 \nvdash \mathsf{NP} \subseteq \text{i.o.SIZE}[n^k]$$

$$\mathsf{PV} \nvdash \mathsf{P} \subseteq \text{i.o.SIZE}[n^k]$$

Extensions. $\mathsf{True}_1 \stackrel{\text{def}}{=} \forall\Sigma^b_1(\mathcal{L}_{PV})$-sentences true in $\mathbb{N}$ can be included in first item.

Example: $\forall x\, (\exists y\, (1 < y < x \wedge y \,|\, x) \leftrightarrow f_{\mathsf{AKS}}(x) = 0)$

$T^1_2 \cup \mathsf{True}_1$ proves that Primes $\in \text{SIZE}[n^c]$ for some $c \in \mathbb{N}$, but not that $\mathsf{P}^{\mathsf{NP}} \subseteq \text{i.o.SIZE}[n^k]$.

A more precise statement

- $\mathcal{L}_{PV}$-formulas $\varphi(x)$ interpreted over $\mathbb{N}$ can define languages in P, NP, etc.

- The sentence $\mathsf{UB}_k^{\text{i.o.}}(\varphi)$ expresses that the corresponding n-bit boolean functions are computed infinitely often by circuits of size n^k:

$$\forall 1^{(\ell)} \, \exists 1^{(n)} (n \geq \ell) \, \exists C_n(|C_n| \leq n^k) \, \forall x(|x| = n), \; \varphi(x) \equiv (C_n(x) = 1)$$

Theorem

For any of the following pairs of an $\mathcal{L}_{PV}$-theory T and a uniform complexity class $\mathcal{C}$:

(a) $T = T_2^1$ *and* $\mathcal{C} = \mathsf{P}^{\mathsf{NP}}$,

(b) $T = S_2^1$ *and* $\mathcal{C} = \mathsf{NP}$,

(c) $T = \mathsf{PV}$ *and* $\mathcal{C} = \mathsf{P}$,

there is an $\mathcal{L}_{PV}$-formula $\varphi(x)$ defining a language $L \in \mathcal{C}$ such that T does not prove the sentence $\mathsf{UB}_k^{\text{i.o.}}(\varphi)$.

High-level ideas

- Two approaches (forget the "i.o." condition for now):

$$T_2^1 \nvdash \mathsf{P}^{\mathsf{NP}} \subseteq \text{i.o.SIZE}[n^k]$$
$$S_2^1 \nvdash \mathsf{NP} \subseteq \text{i.o.SIZE}[n^k]$$

Main ingredient is the use of **"logical" Karp-Lipton theorems**.

$$\mathsf{PV} \nvdash \mathsf{P} \subseteq \text{i.o.SIZE}[n^k]$$

Extract from (non-uniform) circuit upper bound proofs a **"uniform construction"**.

Bounded theories and a.e. vs i.o. circuit bounds

Parikh's Theorem. Let $A(\vec{x}, y)$ be a bounded formula.

If $I\Delta_0 \vdash \forall \vec{x}\, \exists \boldsymbol{y}\, A(\vec{x}, y)$ then $I\Delta_0 \vdash \forall \vec{x}\, \exists \boldsymbol{y} \leq \boldsymbol{t}(\vec{\boldsymbol{x}})\, A(\vec{x}, y)$.

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- We use similar results to "tame" i.o. upper bounds in bounded arithmetic.

Example: If $T^1_2 \vdash \text{SAT} \in \text{i.o.SIZE}[n^k]$ then $T^1_2 \vdash \text{SAT} \in \text{SIZE}[n^{k'}]$.

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- Not every language is paddable, and more delicate arguments are needed.

Concluding Remarks: Logic and P vs NP

- A major question is to establish the unprovability of P = NP:

For a function symbol $f \in \mathcal{L}_{PV}$, consider the universal sentence

$$\varphi_{\mathsf{P}=\mathsf{NP}}(f) \stackrel{\text{def}}{=} \forall x \, \forall y \; \psi_{\mathsf{SAT}}(x, y) \rightarrow \psi_{\mathsf{SAT}}(x, f(x))$$

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- Reduces to the study of unprovability of circuit **lower** bounds (Theorem 2 in our work).
- Motivates **both** research directions (**unprovability of upper and lower bounds**).

Thank you

Approach 1: "Logical" Karp-Lipton theorems

- A few unconditional circuit lower bounds in complexity theory use KL theorems. For instance, $\mathsf{ZPP}^{\mathsf{NP}} \not\subseteq \mathsf{SIZE}[n^k]$ can be derived from:

[Kobler-Watanabe'98] If $\mathsf{NP} \subseteq \mathsf{SIZE}[\mathrm{poly}]$ then $\mathsf{PH} \subseteq \mathsf{ZPP}^{\mathsf{NP}}$.

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[Cook-Krajicek'07] If $\mathsf{NP} \subseteq \mathsf{SIZE}[\mathsf{poly}]$ and **this is provable in a theory** $T \in \{\mathsf{PV}, S^1_2, T^1_2\}$, then PH collapses to a complexity class $\mathcal{C}_T \subseteq \mathsf{P}^{\mathsf{NP}}$.

Approach 2: A "bridge" between uniform and non-uniform circuits

If PV $\vdash$ P $\subseteq$ SIZE$[n^k]$, try to extract from PV-proof a "uniform" circuit family for each $L \in$ P.

This would contradict known separation P $\not\subseteq$ P-unifom-SIZE$[n^k]$ [Santhanam-Williams'13].

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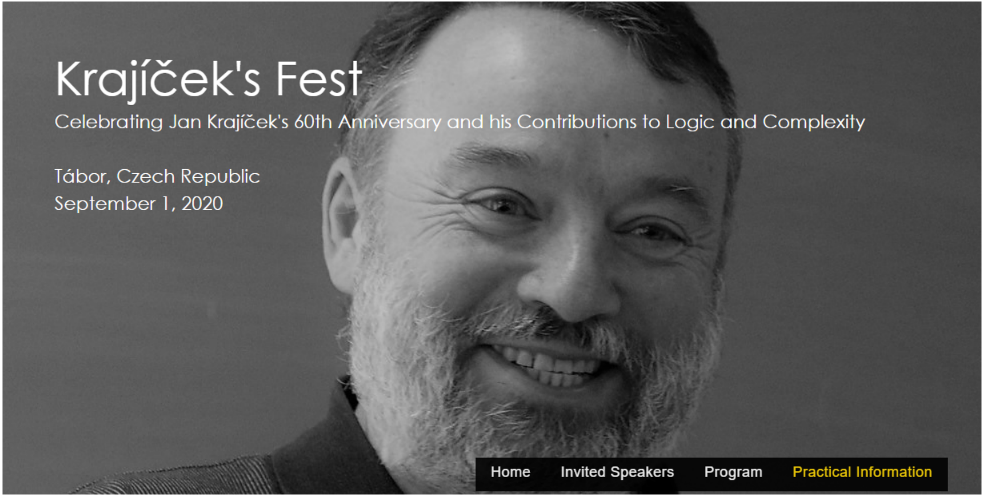
- Theorem 1 (c) strengthens Krajicek-Oliveira to rule out PV $\vdash$ P $\subseteq$ i.o.SIZE$[n^k]$.

Complications appear because Santhanam-Williams doesn't provide a.e. lower bounds.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