On the Extension Complexity of Polytopes Associated with Permutation Groups

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PART I

POLYTOPES vs GROUPS

POPULAR OPTIMIZATION PROBLEM

$\max \sum_i c_i x_i$

VARIABLES OVER A DISCRETE DOMAIN Σ

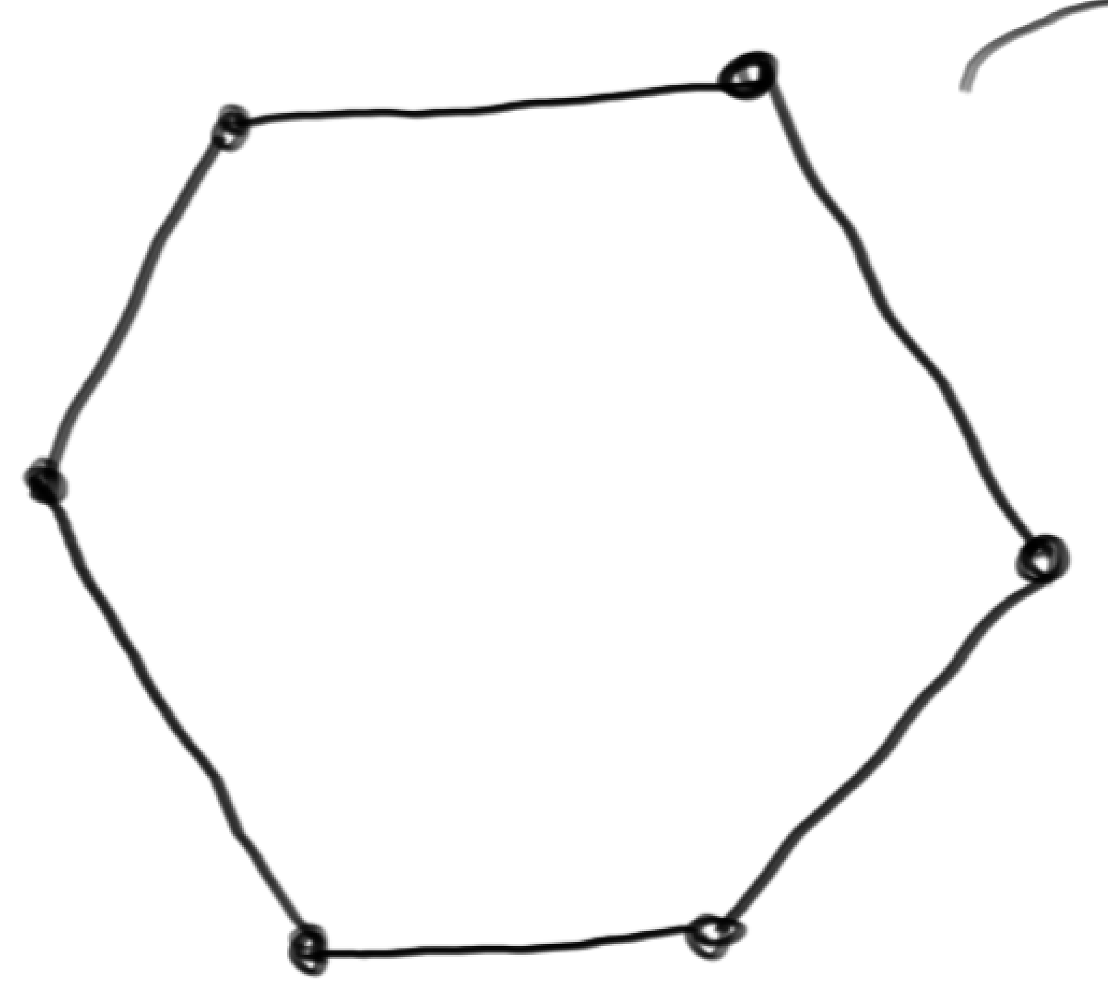
REAL CONSTANTS

$\vec{x} \in \mathcal{F} \subseteq \Sigma^N$

SET OF FEASIBLE SOLUTIONS

POPULAR APPROACH

IDENTIFY POINTS IN $F \subseteq \Sigma^N$
WITH A SET $\hat{F}$ OF VECTORS IN $\mathbb{R}^N$



$P(F) = \mathrm{CONV}(\hat{F})$

OPTIMIZE $\sum_i c_i x_i$

OVER $P(F)$

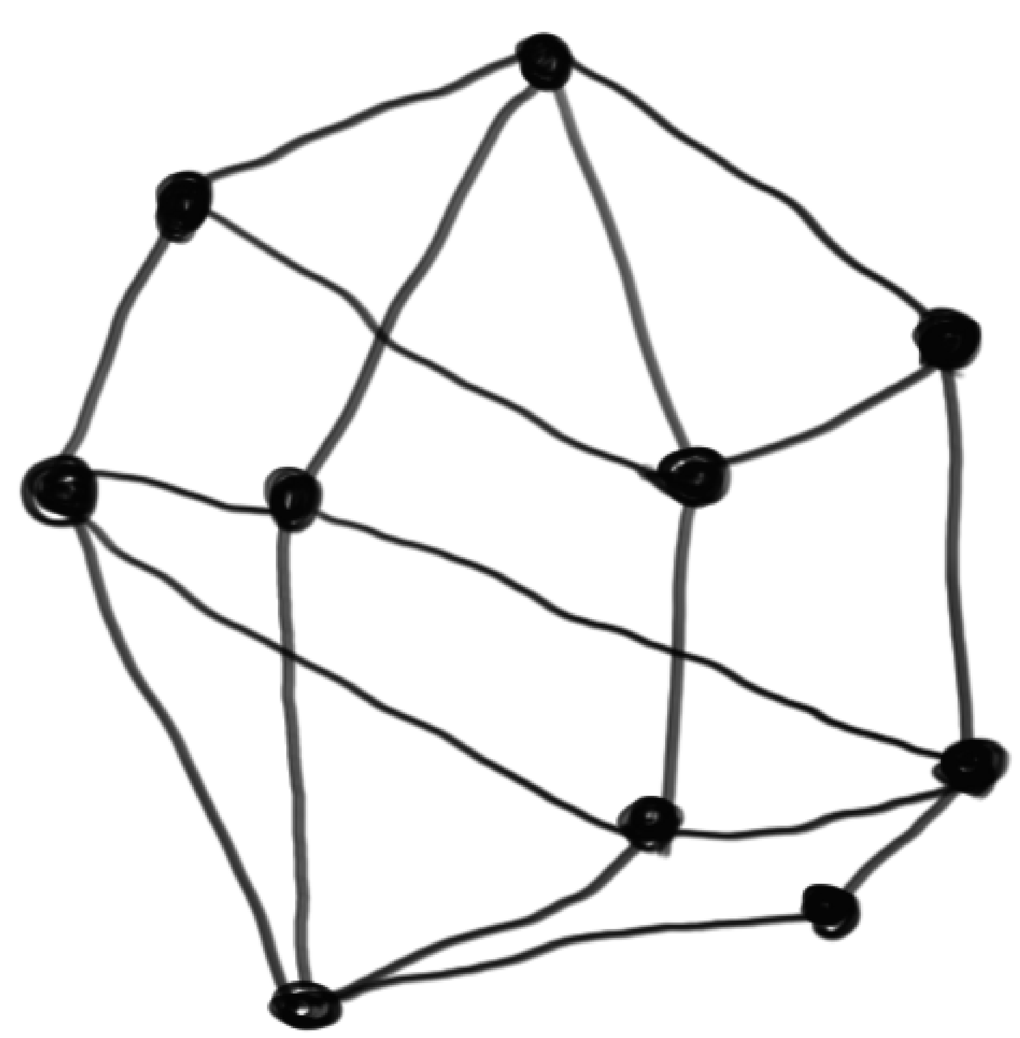
ISSUE: THE POLYTOPE P(F)

MAY HAVE SUPERPOLYNOMIALLY MANY FACETS.

DESCRIPTION OF P(F) MAY REQUIRE SUPERPOLYNOMIALLY MANY INEQUALITIES

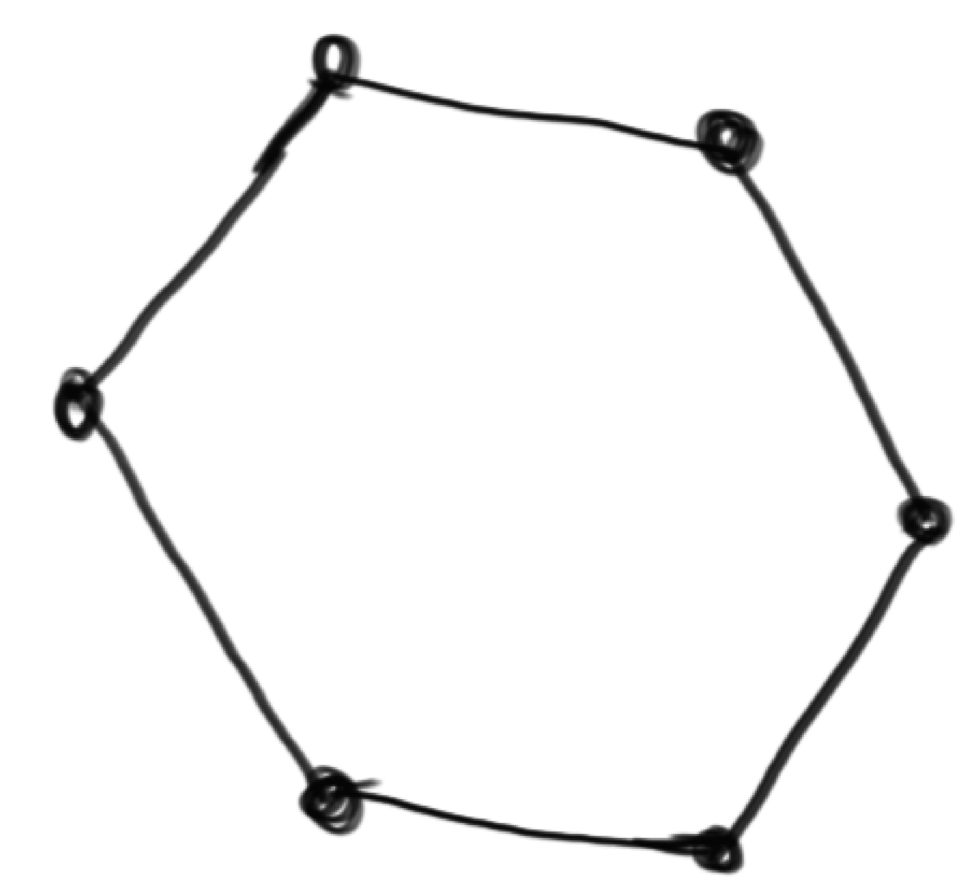
WORK AROUND: EXTENDED FORMULATIONS

$Q \subseteq \mathbb{R}^{N+N'}$ $\quad N' = \mathrm{POLY}(N)$



LINEAR PROJECTION π

$\longrightarrow$



MAXIMIZE $\sum c_i x_i$ HERE

GET A SOLUTION HERE

EXAMPLE: PERMUTAHEDRON

$$\mathcal{P}_N = \{ a_1 a_2 \ldots a_m \in [N]^N : a_i \neq a_j \text{ FOR } i \neq j \}$$

$$P(\mathcal{P}_4) =$$

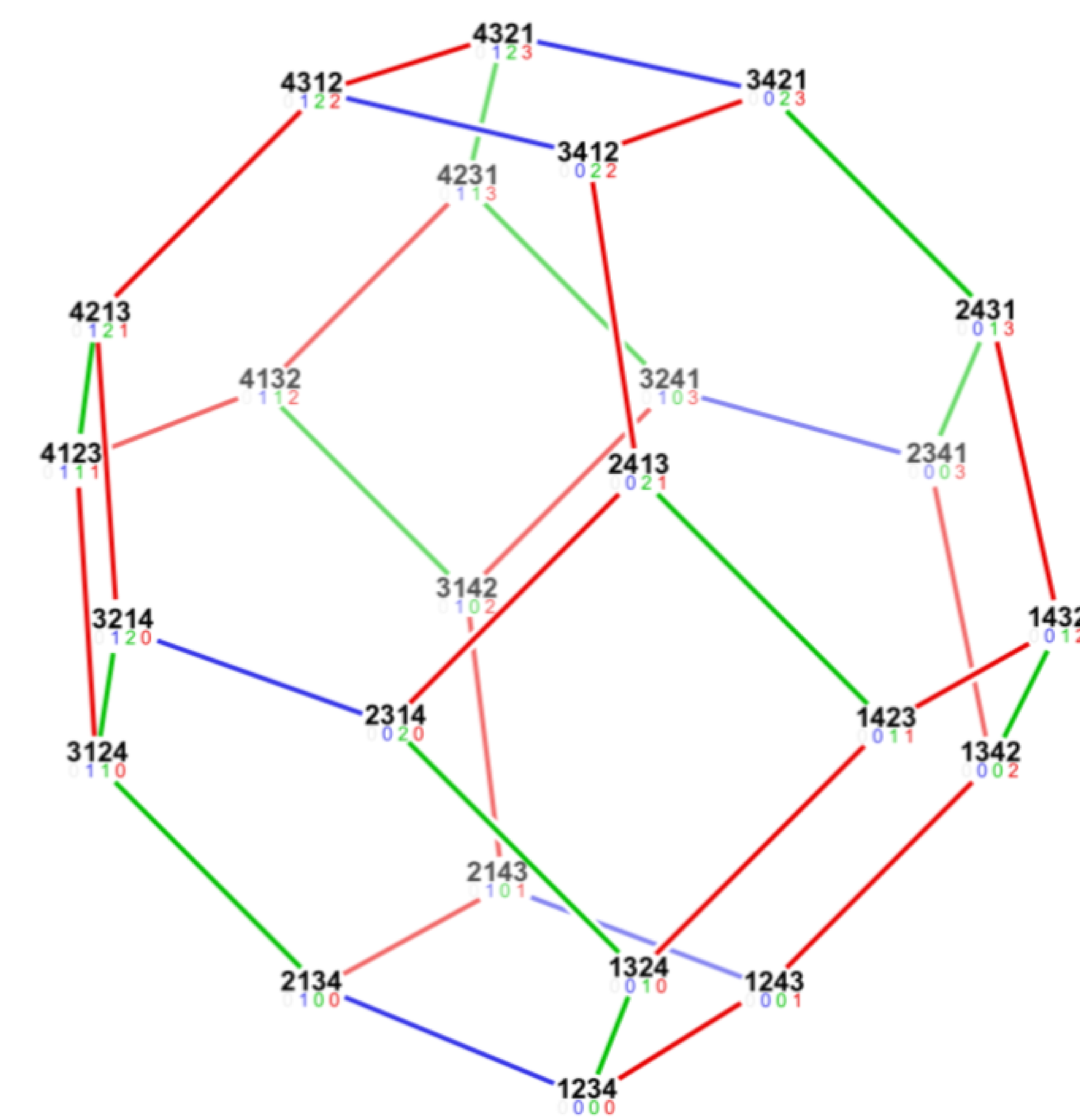


IMAGE FROM WIKIPEDIA

$P(\mathcal{P}_N)$ HAS $2^{\Omega(N)}$ FACETS

BUT $\Theta(N \log N)$ EXT. FORM.

WHICH GROUPS HAVE POLYNOMIAL EXTENDED FORMULATIONS

$G \leq SYM([N])$ $\quad\quad$ $\sigma \in G \Rightarrow$ REPRESENT σ BY

$a_1 a_2 \dots a_N \in [N]^N$

$$\hat{G} = \{ (a_1, a_2, \dots, a_N) \in \mathbb{R}^N : a_1 a_2 \dots a_N \in G \}$$

G-HEDRON: $P(G) = CONV(\hat{G})$

EX. $P(ALT_N)$ IS KNOWN AS ALTERNAHEDRON

ALSO HAS $\Theta(N \log N)$-SIZE EXT. FORM. (WELTGE-2012)

WHICH GROUPS HAVE POLYNOMIAL EXTENDED FORMULATIONS

SYM_N , ALT_N (BIRKHOFF GOEMANS, WELTGE)

REFLECTION GROUPS (KAIBEL, PASIKOVICH, HUMPHREYS)

SPECIAL CASE OF MAIN THEOREM OF THIS WORK.

THM: X IS A GRAPH OF TREEWIDTH t AND MAX DEGREE Δ $\Rightarrow$

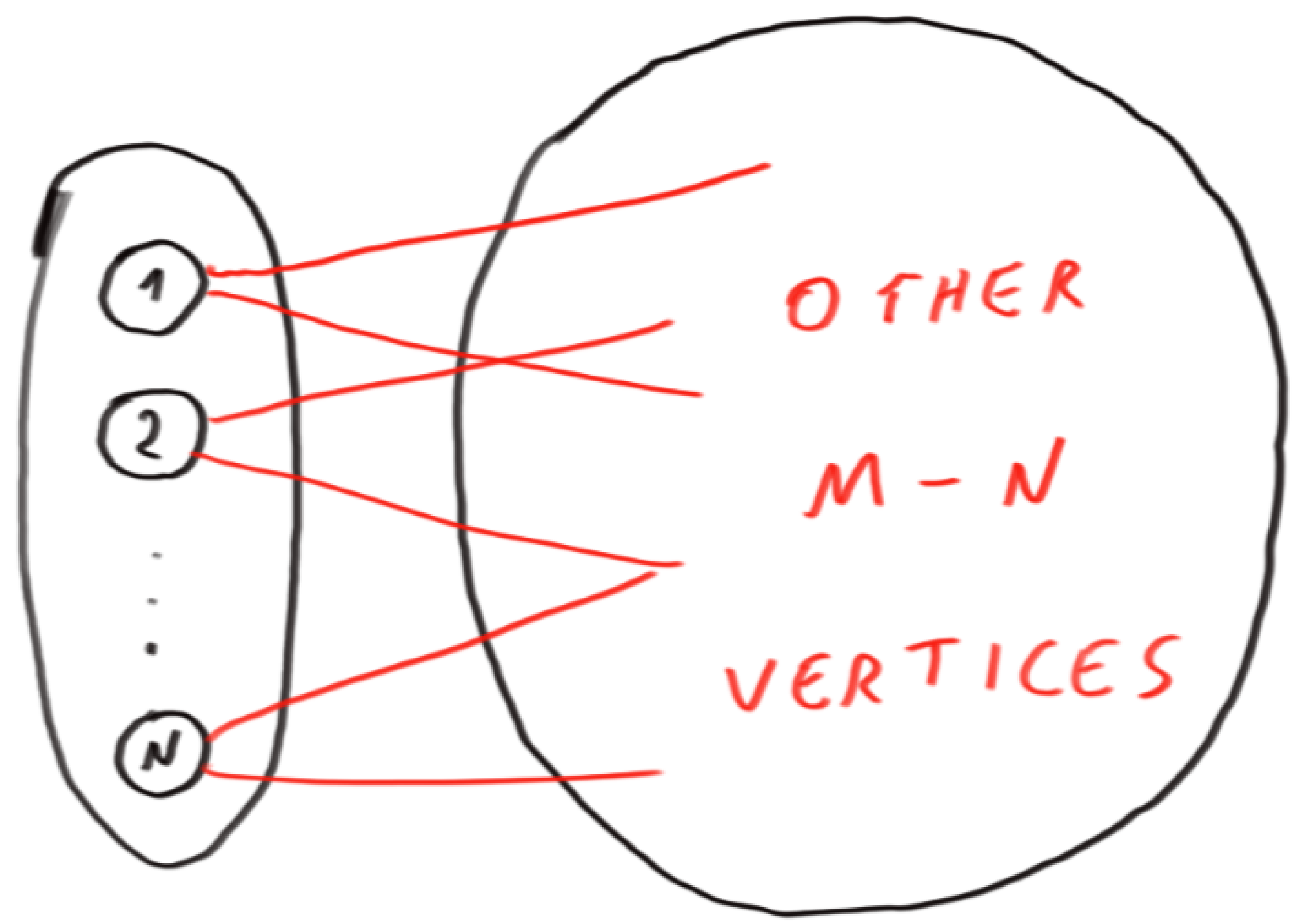
$$\text{x.c.}(\mathrm{AUT}(X)) = 2^{O(t\Delta \log \Delta)} \cdot |X|^{O(t)}$$

PART II

GROUPS vs GRAPHS

LET $[N] = \{1, \dots, N\}$.

A GROUP $G \subseteq SYM([N])$ IS EMBEDDABLE IN A CONNECTED GRAPH X WITH $M \geq N$ VERTICES IF...



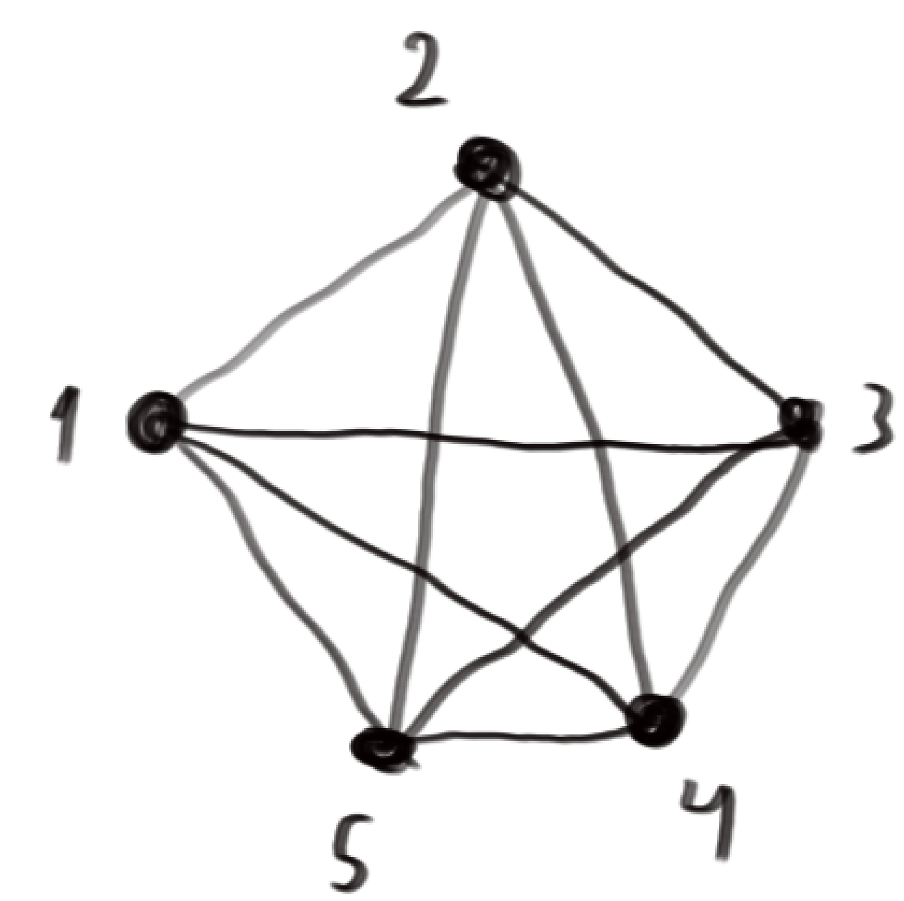
AUT(X) STABILIZES $[N] = \{1, \dots, N\}$

$$G = \{ \varphi|_{[N]} : \varphi \in AUT(X) \}$$

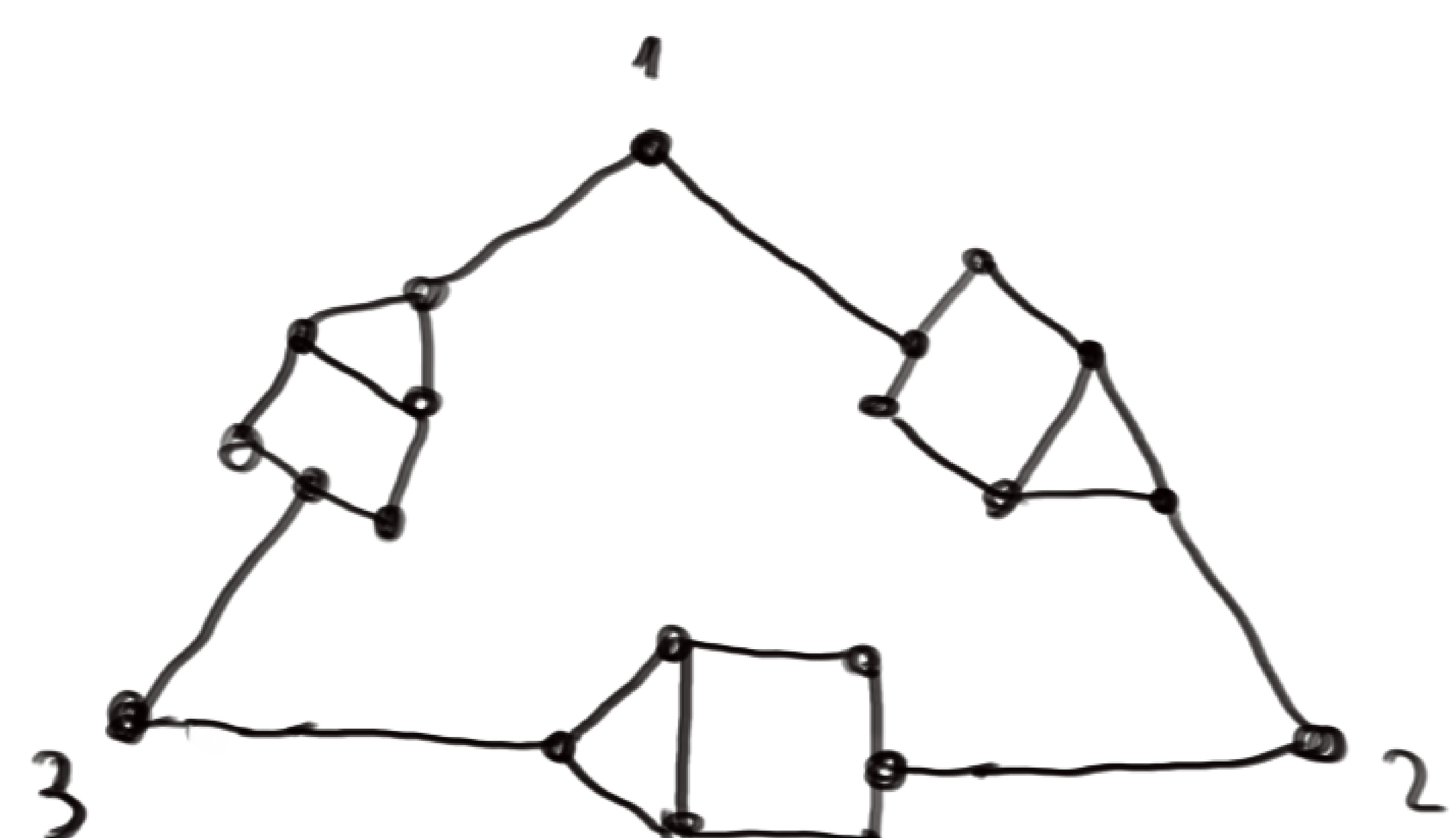
GRAPH EMBEDDABILITY COMPLEXITY : g.e.c(G)

MINIMUM M S.T. $\exists X$ ON M VERTICES S.T $G \hookrightarrow X$

EX: SYMMETRIC GROUP SYM([N])



CYCLIC GROUP C_N



BABAI, BOWER 1969:

ANY SUBGROUP $G \subseteq SYM([N])$ CAN BE EMBEDDED IN A GRAPH WITH $O(N+|G|)$ VERTICES.

THEREFORE WORST CASE FOR g.r.c OF SUBGROUPS OF $SYM([N])$ IS $O(N!)$

OPEN PROBLEM: GIVE A FAMILY OF GROUPS WITH SUPERPOLYNOMIAL g.r.c

BABAI - 1969: IS THE g.r.c OF ALT_N SUPERPOLYNOMIAL IN N ?

A RELATED PROBLEM:

GIVEN A GROUP G, FIND A GRAPH X SUCH THAT $G \simeq \mathrm{AUT}(X)$.

FRUCHT-1949: ANY GROUP IS ISOMORPHIC TO THE AUTOMORPHISM GROUP OF A 3-REGULAR GRAPH.

CONTRAST WITH: FOR $N \geq 6$, $\mathrm{SYM}([N])$ CANNOT BE EMBEDDED IN CONNECTED GRAPHS OF MAX DEGREE LESS THAN $N-1$.

LIEBECK-1983: $\mathrm{ALT}_N \approx \mathrm{AUT}(X) \Rightarrow |X| \geq 2^{\Omega(N)}$.

CONTRAST WITH: OPEN WHETHER g.l.c OF ALT_N IS SUPERPOLYNOMIAL.

WHICH GROUPS HAVE POLYNOMIAL EXTENDED FORMULATIONS

MAIN THEOREM:

IF $G \subseteq SYM([N])$ IS EMBEDDABLE IN A GRAPH X WITH $M \geq N$ VERTICES TREEWIDTH t AND MAX-DEGREE Δ THEN

$$x.c.(G) = 2^{O(t \Delta \log \Delta)} \cdot M^{O(t)}$$

PART III

FORMAL LANGUAGES VS POLYTOPES AND GROUPS

WHICH FORMAL LANGUAGES HAVE SMALL EXTENSION COMPLEXITY?

BY LANGUAGE WE MEAN

$$L_1, L_2, \ldots, L_N, \ldots \qquad L_N \in (\Sigma_N)^N$$

WE ALLOW THE ALPHABET TO GROW WITH THE SIZE OF STRINGS

WHICH FORMAL LANGUAGES HAVE SMALL EXTENSION COMPLEXITY ?

(EXPLICIT IN TIWARY 2015)

IF L_N IS COMPUTABLE BY NON. UNIF. NONDET. READ-ONCE OBLIVIOUS BRANCHING PROGRAMS OF SIZE $S(N)$ THEN $x.c(L_N) \leq S(N)^{O(1)}$

$$\Downarrow$$

ONLINE NONDET. TURING MACHINE WORKING IN SPACE $R(N) \Rightarrow x.c.(L_N) \leq 2^{O(R(N))} \cdot N$

$$\Downarrow$$

REGULAR LANGUAGES $\Rightarrow x.c.(L_N) \leq O(N)$

WHICH FORMAL LANGUAGES HAVE SMALL EXTENSION COMPLEXITY?

OPEN PROBLEM: SUPPOSE $L_N = \Sigma^N \cap L(G)$ FOR SOME CONTEXT-FREE GRAMMAR G.

$$\text{IS } x.c.(L_N) = N^{f(|G|)} \text{ ?}$$

IN OTHER WORDS DO CONTEXT-FREE LANGUAGES HAVE POLYNOMIAL x.c.?

EX: REGULAR CFG'S ARE 1-HOMOGENEOUS.

(PARSE TREES ARE LINES)

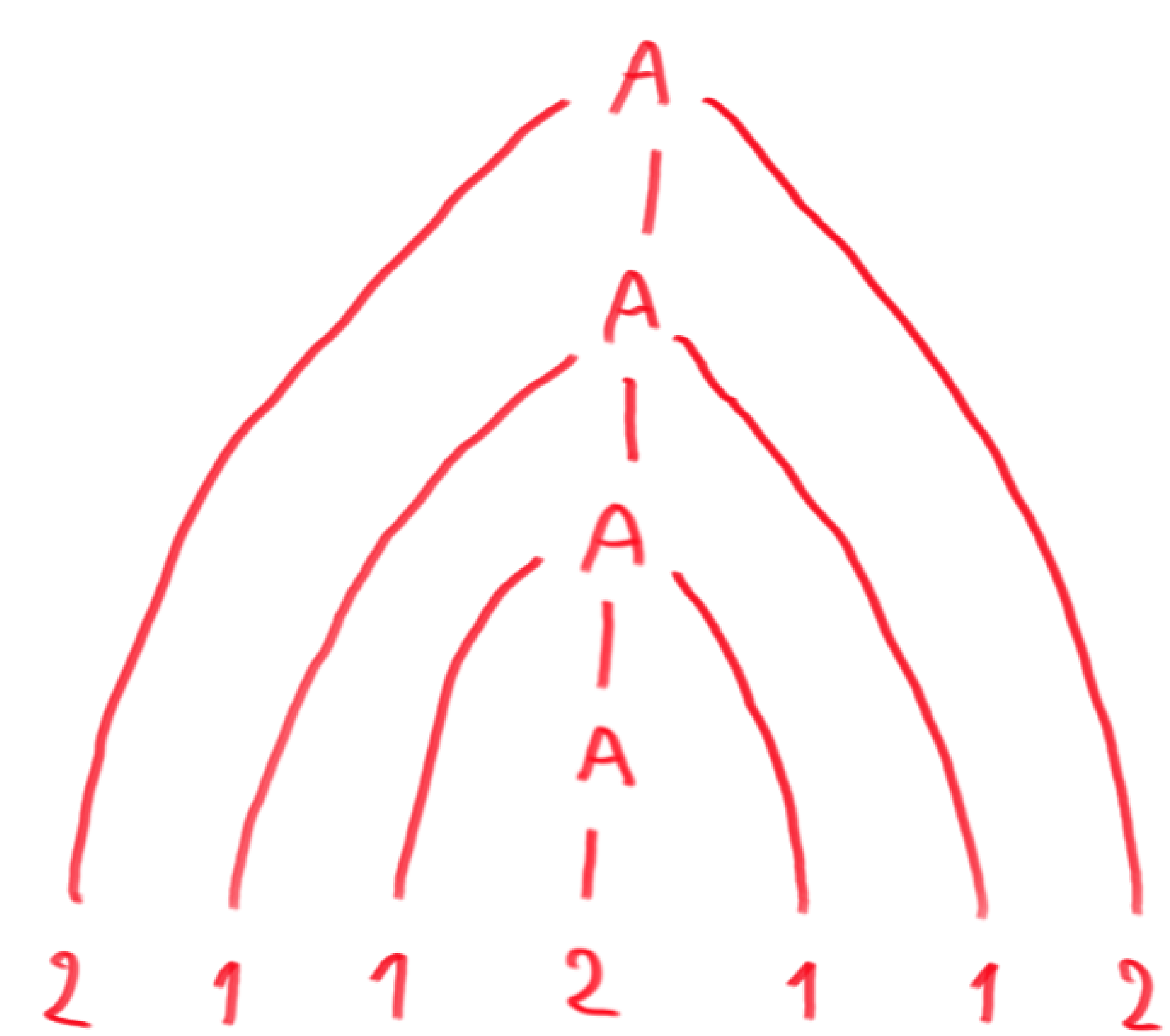
ANY LANGUAGE CAN BE ACCEPTED BY NON UNIF. FAMILIES OF 1-HOMOG CFG'S.

PALINDROME LANGUAGE IS ACCEPTED BY CONST. SIZE 1-HOMOGENEOUS CFG.

G: $A \to 1 \mid 2$

$A \to 1\,A\,1$

$A \to 2\,A\,2$



EVERY WORD OF SIZE N IS PARSED BY TREES OF SAME SHAPE.

h-HOMOGENEOUS CFG'S

SAY THAT A CFG IS h-HOMOGENEOUS IF FOR EACH N, THE STRINGS OF LENGTH N ARE ACCEPTED USING ONE OUT OF $h(N)$ PARSE-TREES.

THEOREM: IF L_N IS ACCEPTED BY A h-HOMOGENEOUS CFG $\mathbb{G}$ THEN

$$\mathrm{x.c}(\mathcal{P}(L_N)) \leq h(N) \cdot |\mathbb{G}|^{O(1)} \cdot N^{O(1)}$$

THEOREM: IF $G \leq \mathrm{SYM}([N])$ CAN BE EMBEDDED ON A GRAPH OF SIZE M, TREEWIDTH τ AND MAX-DEGREE Δ THEN THERE IS A 1-HOMOGENEOUS CFG $\mathbb{G}$ OF SIZE

$$2^{O(\tau \cdot \Delta \log \Delta)} \cdot M^{O(\tau)} \quad \text{S.T.} \quad L(\mathbb{G}) = G.$$

OPEN PROBLEMS

1) IS $g.l.c(ALT_N) \geq N^{\Omega(1)}$?

NOTE THAT $x.c(P(ALT_N)) = \Theta(N \log N)$

2) IS $x.c(\mathcal{L}(\mathbb{G})) \leq N^{f(|\mathbb{G}|)}$ FOR SOME COMPUTABLE FUNCTION f ?

3) GIVE A GROUP WITH SUPERPOLYNOMIAL x.c.

4) GIVE A GROUP WITH SUPERPOLYNOMIAL g.l.c.

THANK YOU!