Assignment 10
due on Wednesday, July 5, 2017

Name:

Exercise 1 (10 points)**.**
Let $\mathbb{A} := \mathbb{C}^{2\times 2}$ denote the vector space of 2×2 matrices. Let $Y \subseteq \mathbb{A}$ denote the subset of rank 1 matrices. For $y \in \mathbb{A}$ let $c_1(y) \in \mathbb{C}^2$ denote the first column of y and let $c_2(y) \in \mathbb{C}^2$ denote the second column of y. If $y \in Y$, then $c_1(y) = \alpha_y c_2(y)$ for some $\alpha_y \in \mathbb{C}$ or $c_2(y) = 0$.

Let $X \subseteq Y$ denote the subset for which $c_2(y) \neq 0$. We define the function $f : X \to \mathbb{C}$ via $f(y) = \alpha_y$ for every $y \in X$.

Show that $X \subseteq \mathbb{A}$ is locally closed and that f is a regular function on X.

Exercise 2 (10 points)**.**
Let $\mathfrak{S}_3$ denote the symmetric group on 3 symbols. Determine the number of isomorphism types of irreducible $\mathfrak{S}_3$-representations.

Hint: You can use the algebraic Peter-Weyl theorem.

Exercise 3 (20 points)**.**
The 1-dimensional vector space $\mathbb{C}$ with the following action is an irreducible representation of the symmetric group $\mathfrak{S}_k$:

$$\pi v = \mathrm{sgn}(\pi).v$$

for all $v \in \mathbb{C}$, $\pi \in \mathfrak{S}_k$. We call this the *alternating representation* and its isomorphism type is called the *alternating type*.

Let $V := \mathbb{C}^n$. The group $\mathfrak{S}_k \times \mathsf{GL}_n$ acts on the tensor power $V^{\otimes k}$ via

$$(\pi, g)(v_1 \otimes \cdots \otimes v_k) = (g v_{\pi^{-1}(1)}) \otimes \cdots \otimes (g v_{\pi^{-1}(k)})$$

and linear continuation.

The *alternating space* $\bigwedge^k V \subseteq V^{\otimes k}$ is defined as the $\mathfrak{S}_k$-isotypic component of alternating type.

Show that $\bigwedge^k V$ is a GL_n-subrepresentation of $V^{\otimes k}$ and determine the multiplicities of irreducible GL_n-representations in $\bigwedge^k V$.