A Quadratic Lower Bound against Homogeneous Non-Commutative Circuits

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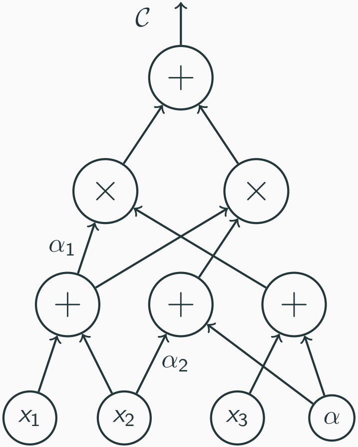
March, 31, 2023

Algebraic Circuits



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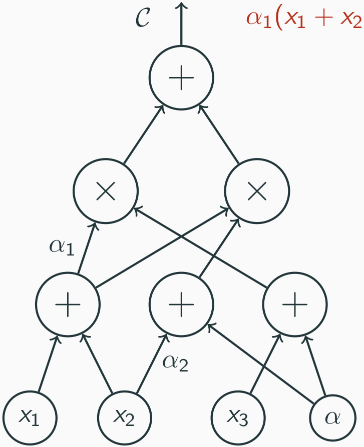

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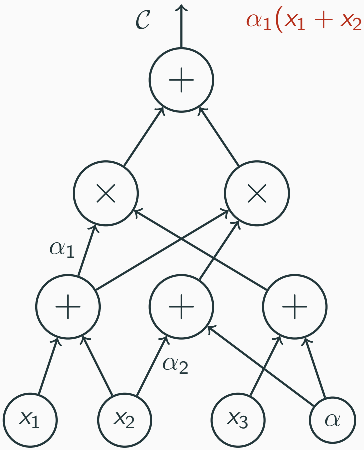

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Polynomials over n variables of degree d.

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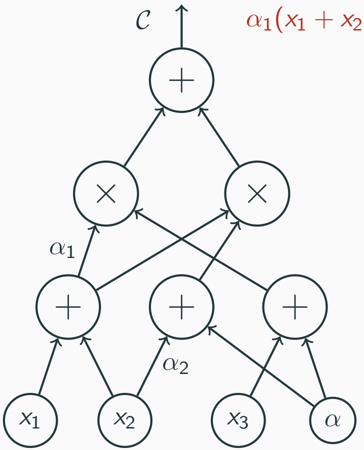

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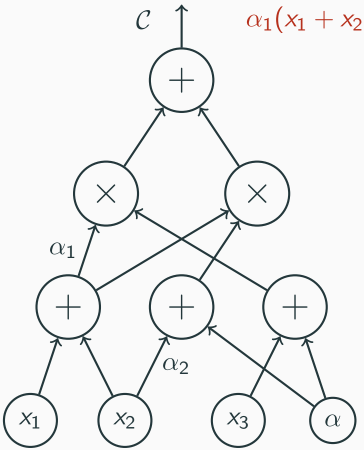

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[Baur-Strassen]: Any algebraic circuit computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(n \log d)$ wires.

The Non-Commutative Setting

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[Carmossino-Impagliazzo-Lovett-Mihajlin]

$\Omega(n^{\omega+\varepsilon})$ for $f_{n,c} \implies \Omega(2^n)$ for $f'_{n,n}$.

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Theorem: Any homogeneous non-commutative circuit computing

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Further, there is a non-commutative circuit of size $O(n \log^2 n)$ that computes $\mathrm{OSym}_{n,n/2}(\mathbf{x})$.

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Main Observation: For any f that is computable by a homogeneous non-commutative circuit of size s,

$$\mu(f) \leq s + 1.$$

A simple proof of an obvious fact

$\mathcal{C}$: Homogeneous non-commutative circuit.

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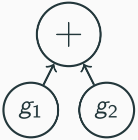

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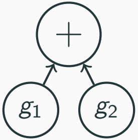

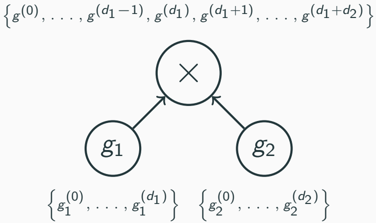

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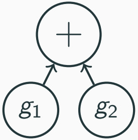
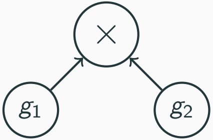


$$\left\{g^{(0)}, \ldots, g^{(d_1-1)}, g^{(d_1)}, g^{(d_1+1)}, \ldots, g^{(d_1+d_2)}\right\}$$



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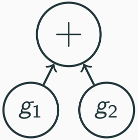

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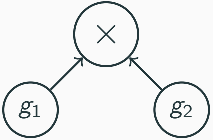

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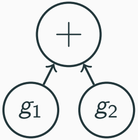

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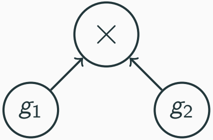

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$$\mu(f) = n + 1.$$

Therefore, $\mu(\mathcal{C}_f) \ge n$.

Using it to prove a “not so obvious” fact

Theorem: There exists an explicit monomial over $\{x_0, x_1\}$ of degree d such that any homogeneous non-commutative circuit computing it must have size $\Omega\left(\frac{d}{\log d}\right)$.

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In this case, if $\mathcal{C}$ is a homogeneous non-commutative circuit of size s, then $\mu_\ell(\mathcal{C}) \leq O(s \log d)$.

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Therefore all we need is a monomial, f, over $\{x_0, x_1\}$ of degree d such that $\mu_\ell(f) \geq \Omega(d)$.

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de Bruijn Sequence (of order $\log d$**)**: It is a cyclic sequence in the alphabet $\{0, 1\}$ in which every string of length $\log d$, occurs exactly once as a substring.

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Question: Can we prove the same lower bound against general non-commutative circuits?

Getting back to the main result

[Baur-Strassen]: If there is a circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

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Note: $f = x_1 B_d(x_0^{(1)}, x_1^{(1)}) + \cdots + x_n B_d(x_0^{(n)}, x_1^{(n)})$ already (almost) has the required property.

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Note: f has a non-homogeneous non-commutative circuit of size $O(n \log^2 d)$.

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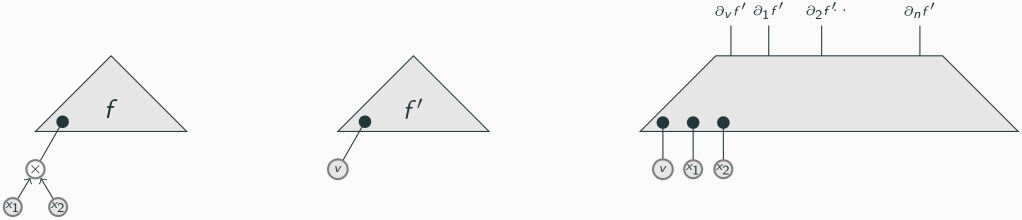

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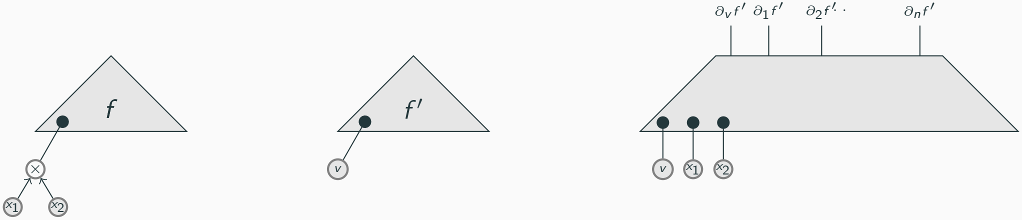

Step 2: Write each of $\{\partial_i f\}_i$ using $\partial_v f'$ and $\{\partial_i f'\}_i$.

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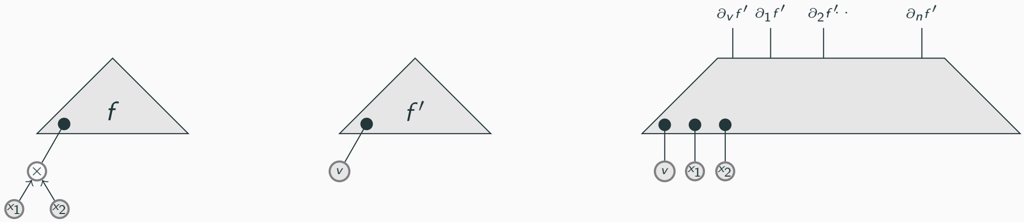

Step 2: Write each of $\{\partial_i f\}_i$ using $\partial_v f'$ and $\{\partial_i f'\}_i$. Add (the ≤ 10 extra) edges accordingly.

Making [Baur-Strassen] work in the homogeneous setting

Target: If there is a homogeneous circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a homogeneous circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

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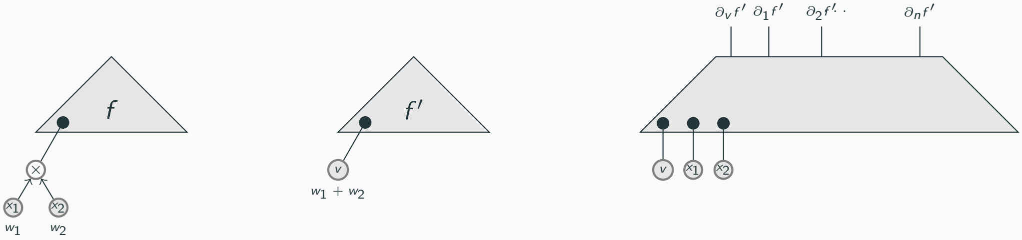

Making [Baur-Strassen] work in the non-commutative setting

Formal derivatives (with respect to the first position)

Given a polynomial f and a variable x, f can be uniquely written as

$$f = x \cdot f_0 + f_1$$

where no monomial in f_1 contains x in the first position.

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Chain rules can be defined formally as well.

Lemma: If there is a homogeneous NC circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a homogeneous NC circuit of size at most $5s$ that simultaneously compute $\{\partial_{1,x_1} f, \ldots, \partial_{1,x_n} f\}$.

Polynomial with a large measure

$$\boxed{f = \mathrm{OSym}_{n,\frac{n}{2}+1}(\mathbf{x}) = \sum_{1 \le i_1 < \cdots < i_{\frac{n}{2}+1} \le n} \left(\prod_{j=1}^{\frac{n}{2}+1} x_{i_j} \right)}$$

Polynomial with a large measure

$x_{\frac{n}{2}+1}x_{\frac{n}{2}+2} \quad \cdots \quad x_2x_{\frac{n}{2}+2} \quad \cdots \quad \cdots \quad x_{n-2}x_{n-1} \quad \cdots \quad x_{\frac{n}{2}-1}x_{n-1} \quad x_{n-1}x_n \quad \cdots \quad x_{\frac{n}{2}}x_n$

$(1, \frac{n}{2})$

$\vdots$

$(1, 1)$

$\vdots$

$\vdots$

$(\frac{n}{2}-2, \frac{n}{2})$

$\vdots$

$(\frac{n}{2}-2, 1)$

$(\frac{n}{2}-1, \frac{n}{2})$

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$\vdots$

$(\frac{n}{2} - 2, 1)$

$(\frac{n}{2} - 1, \frac{n}{2})$

$\vdots$

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Defining the matrix $\mathcal{M}(f)$

$x_k x_l$

(j, i) $\quad \boxed{\mathrm{coeff}_{x_k x_l}(\partial_i f^{(j)})}$

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$$\mu(\mathbb{D}(f)) \ge \mathsf{rank}(\mathcal{M}(f)) = \Omega(n^2).$$

The lower bound is tight

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Use the following fact recursively.

$$\mathrm{OSym}_{n,d}(x_1,\ldots,x_n) = \mathrm{OSym}_{n-1,d-1}(x_1,\ldots,x_{n-1}) \cdot x_n + \mathrm{OSym}_{n-1,d}(x_1,\ldots,x_{n-1}).$$

Better Upper bound in the non-homogeneous setting

There is a non-commutative circuit of size $O(n \log^2 n)$ that computes all the elementary symmetric polynomials simultaneously.

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Think of $\quad f = \prod_{i=1}^{\frac{n}{2}}(1 + tx_i), g = \prod_{i=\frac{n}{2}+1}^{n}(1 + tx_i) \in \mathbb{F}\langle \mathbf{x} \rangle [t].$

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Do polynomial multiplication recursively $\log n$ times. Note that polynomial multiplication can be done in time $O(n \log n)$ using FFT.

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If true, then the answer to the second question is "yes".

Thank you!