Border rank and homogeneous complexity classes

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(based on joint work with P. Dutta, C. Ikenmeyer, G. Jindal, V. Lysikov)
We study complexity measures on complex homogeneous polynomials

\[ f \in S^d \mathbb{C}^N = \mathbb{C}[x_1, \ldots, x_N]_d. \]

**Plan:**

- Waring rank and border Waring rank
- Kumar's *product plus constant* model
- Generalization to other complexity classes
The Waring rank of $f$ is

$$\text{WR}(f) = \min \left\{ r : f = \ell_1^d + \cdots + \ell_r^d \text{ for some } \ell_j \in S_1^1 \mathbb{C}^n \right\};$$

the border Waring rank of $f$ is

$$\overline{\text{WR}}(f) = \min \left\{ r : f = \lim_{\varepsilon \to 0} f_\varepsilon \text{ for a sequence } f_\varepsilon \text{ with } \text{WR}(f_\varepsilon) \leq r \right\}.$$

Clearly $\overline{\text{WR}}(f) \leq \text{WR}(f)$.

There are examples where the inequality is strict:

$$\text{WR}(x^{d-1}y) = d$$
$$\overline{\text{WR}}(x^{d-1}y) = 2.$$
A debordering result for $\text{WR}$ is an inequality of the form

$$(\text{some complexity measure of } f) \leq (\text{some function of } \text{WR}(f)).$$

**Theorem.** [Bläser-Dörfler-Ikenmeyer]

$$\text{abpw}(f) \leq \text{WR}(f).$$

**Bold Conjecture.**

For $f \in S^d \mathbb{C}^N$, then $\text{WR}(f) \leq O(d) \cdot \text{WR}(f)$.

- True for small $\text{WR}(f)$:  
  [Sylvester, Segre, Buczynski-Landsberg, Ballico-Bernardi, Chiantini];

- True when $\text{WR}(f)$ nearly maximal: 
  [Blekherman-Teitler].
Debordering border Waring rank - cont’d

**Theorem.** [DGIJL] For \( f \in S^d \mathbb{C}^N \), if \( \text{WR}(f) = r \), then

\[
\text{WR}(f) \leq d \cdot \binom{2r - 2}{r - 1}.
\]

**Idea of the proof:**

Three ingredients:

(i) We may assume \( f \) can be written in \( r \) variables.

(ii) We may assume \( \deg(f) \geq r \).

(iii) Generalized additive decompositions allow one to give bounds in this range.

Previously only general bounds were of the form \( \text{WR}(f) \leq O(d^r) \) or \( \text{WR}(f) \leq O(r^d) \) which is almost trivial using just “ingredient (i)”.
Kumar’s product plus constant model

Let $f \in \mathbb{C}[x_1, \ldots, x_N]$. The Kumar’s complexity of $f$ is

$$K_c(f) = \min \left\{ r : f = \alpha \left( \prod_{j=1}^{r} (1 + \ell_j) - 1 \right) \text{ for some } \ell_j \in S^1 \mathbb{C}^N, \alpha \in \mathbb{C} \right\}$$

Example. Set $\omega = \exp(2\pi i / d)$.

$$\ell^d = (1 + \omega^0 \ell) \cdots (1 + \omega^{d-1} \ell) - 1 \quad \text{so} \quad K_c(\ell^d) = d.$$  

However $K_c(f)$ is not always finite. In fact, if $f$ is homogeneous, then $K_c(f)$ is finite if and only if $f = \ell^d$.

The border Kumar’s complexity of $f$ is

$$\overline{K_c}(f) = \min \left\{ r : f = \lim_{\varepsilon \to 0} f_\varepsilon \text{ for a sequence } f_\varepsilon \text{ with } K_c(f_\varepsilon) \leq r \right\}$$

$K_c(f)$ is finite for every polynomial $f$.

Theorem. [Kumar]
For $f \in S^d \mathbb{C}^N$, one has $\overline{K_c}(f) \leq \deg(f) \cdot WR(f)$. 
A converse of Kumar’s result

How good is the bound $K_c(f) \leq \deg(f) \cdot \text{WR}(f)$?

Example.

$$x_1 \cdots x_n = \lim_{\varepsilon \to 0} \varepsilon^n (\prod_{j=1}^n (1 + \frac{1}{\varepsilon} x_j) - 1)$$

One has

$$K_c(x_1 \cdots x_n) = n \quad \text{WR}(x_1 \cdots x_n) = 2^{n-1}.$$ 

Except for this case, $K_c$ is roughly equivalent to $\text{WR}$.

**Theorem.** [DGIJL]  
For $f \in S^d \mathbb{C}^N$, either $f$ is a product of linear forms or

$$\text{WR}(f) \leq K_c(f) \leq \deg(f) \cdot \text{WR}(f).$$
Generalizing the *product plus constant* model

For $i = 1, \ldots, r$, let $X_i$ be an $m \times m$ matrix of linear forms. Then

$$A = (\text{id}_m + X_1) \cdots (\text{id}_m + X_r) - \text{id}_m$$

is a matrix whose entries are (non-homogeneous) polynomials of degree $d$ without constant term.

Idea: Fix $m$ and define a complexity measure for $f$ in terms of the value of $r$ in the expression of $A$.

We recover the completeness of ABPs of width 3 for VF [Ben-Or and Cleve].

**Theorem.** [DGIJL] If $f \in S^d \mathbb{C}^N$ has a formula of depth $\delta$, then $f$ can be expressed as an entry of $A$ for some $r \leq 4^\delta$ and $m = 3$.  

Parity-alternating elementary symmetric functions

The $d$-th homogeneous component of $A$ is

$$\bar{e}_d(X_1, \ldots, X_r),$$

the elementary symmetric polynomial in non-commuting variables.

Fix $m = 2$ and specialize $X_i = (0^t x_i^t)$ if $i$ is odd, $X_i = (x_i^t 0^t)$ if $i$ is even. Let $C = \bar{e}_d(X_1, \ldots, X_r)$. One of the entries of $C$ is

$$c_{r,d} = \sum_{(i_1, \ldots, i_d)} x_{i_1} \cdots x_{i_d}$$

where the sum is over parity-alternating increasing sequences.

For $f \in S^d \mathbb{C}^N$, define

$$r_c(f) = \min \{ r : f = c_{r,d}(\ell_1, \ldots, \ell_r) \text{ for some } \ell_i \in S^1 \mathbb{C}^N \}$$

and let $r_c$ be the corresponding border complexity.

**Theorem.** [DGIJL]

$\text{VNP} \not\subseteq \text{VQP}$ if and only if $r_c(\text{perm}_m)$ grows super-quasipolynomially.
What next?

• Debordering Waring rank:
  • study the geometry of approximating curves;
  • explore other models equivalent to Waring rank.

• Homogeneous polynomials defining complexity classes:
  • GCT and obstructions;
  • geometric methods for orbit-closures.