Computing linear sections of varieties: quantum entanglement, tensor decompositions and beyond

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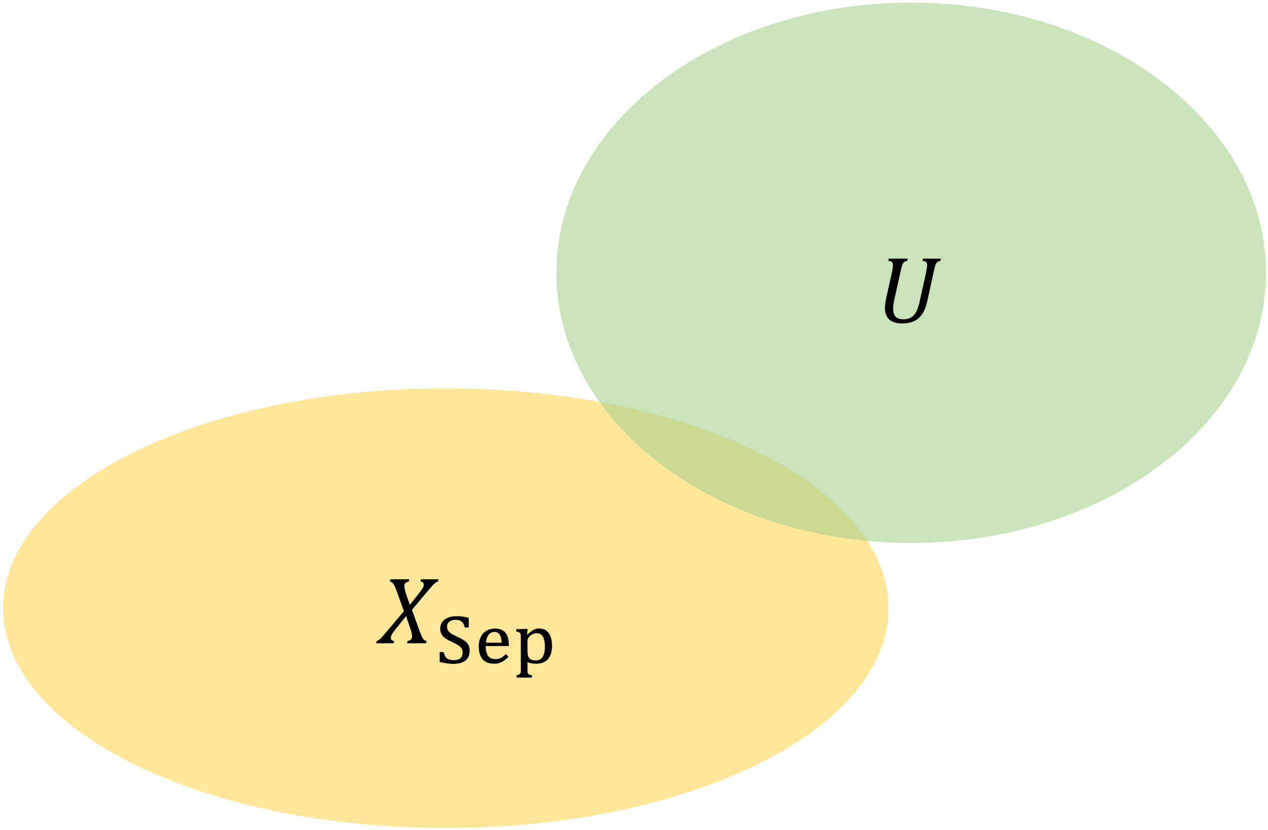
Product tensors: $X_{\mathrm{Sep}} = \{u \otimes v : \quad u, v \in \mathbb{C}^n\} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, determine if U is entangled, i.e. if $U \cap X_{\mathrm{Sep}} = \{0\}$.



Applications: Quantum Information

- Range criterion: For a density operator $\rho \in D(\mathbb{C}^n \otimes \mathbb{C}^n)$,
  $$\mathrm{Im}(\rho) \text{ entangled } \Rightarrow \rho \text{ entangled}$$
- Entangled subspaces can be used to construct entanglement witnesses and quantum error-correcting codes

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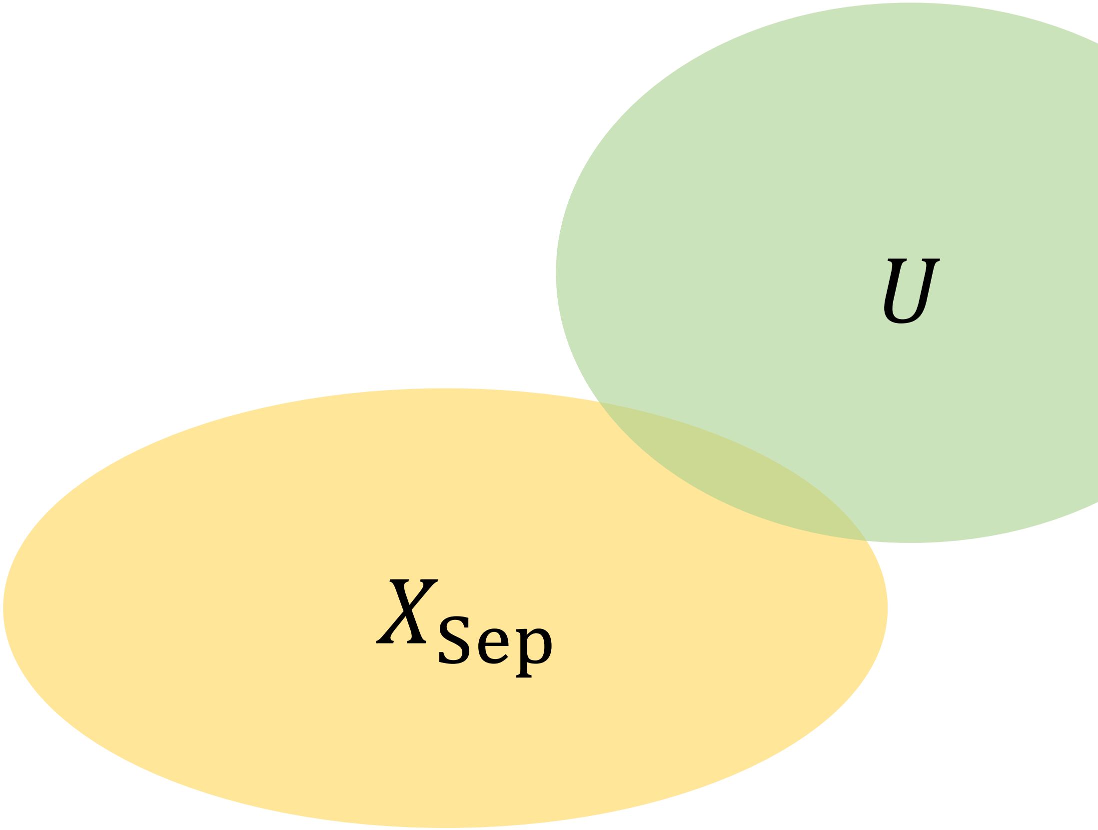

Outline:

1. Algorithm (Nullstellensatz Certificate)
2. Algorithm to recover elements of $U \cap X_{\mathrm{Sep}}$, with applications to tensor decompositions
3. Generalization to arbitrary conic variety X
4. Robust generalization of Hilbert's Nullstellensatz for this problem

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Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$,
determine if U is entangled, i.e. if $U \cap X_{\mathrm{Sep}} = \{0\}$.

[Buss et al 1999]: This is NP-Hard in the worst case.

[Barak et al 2019]: Best known algorithm takes $2^{\tilde{O}(\sqrt{n})}$ time.

[Classical AG, Parthasarathy 01]: $\dim(U) > (n-1)^2 \quad \Rightarrow \quad U$ is not entangled

U generic and $\dim(U) \leq (n-1)^2 \quad \Rightarrow \quad U$ is entangled

Algorithm (deg. 2 N.C.): Takes poly(n)-time and outputs either:

"Hay in a haystack problem"

1. Fail, or
2. A certificate that U is entangled

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[Buss et al 1999]: This is NP-Hard in the worst case.

Works-Extremely-Well Theorem [JLV 22]:

U generic and $\dim(U) \leq \frac{1}{4}(n-1)^2 \Rightarrow$ Algorithm outputs a certificate that U is entangled

U generic and $\dim(U) \leq (n-1)^2 \quad \Rightarrow \quad U$ is entangled

Algorithm (deg. 2 N.C.): Takes poly(n)-time and outputs either: "Hay in a haystack problem"

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The Algorithm
(Nullstellensatz Certificate)

Product tensors: $X_{\mathrm{Sep}} = \{u \otimes v: \quad u, v \in \mathbb{C}^n\} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, determine if U is entangled, i.e. if $U \cap X_{\mathrm{Sep}} = \{0\}$.

Idea: Problem is difficult because it's non-linear

($X_{\mathrm{Sep}} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$ isn't a linear subspace).

Make it linear: Instead check if $U \cap \mathrm{Span}(X_{\mathrm{Sep}}) = \{0\}$.

Doesn't work: $\mathrm{Span}(X_{\mathrm{Sep}}) = \mathbb{C}^n \otimes \mathbb{C}^n$.

Works extremely well already for $d = 2$!

Lift it up: Let $I(X_{\mathrm{Sep}})_d^{\perp} = \mathrm{Span}\{(u \otimes v)^{\otimes d}: \; u, v \in \mathbb{C}^n\} = S^d(\mathbb{C}^n) \otimes S^d(\mathbb{C}^n)$

Check if $S^d(U) \cap I(X_{\mathrm{Sep}})_d^{\perp} = \{0\}$.

Product tensors: $X_{\text{Sep}} = \{u \otimes v : \quad u, v \in \mathbb{C}^n\} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, determine if U is entangled, i.e. if $U \cap X_{\text{Sep}} = \{0\}$.

Hilbert's Nullstellensatz:

$U \cap X = \{0\} \quad \Longleftrightarrow \quad$ For some $d \in \mathbb{N}$ it holds that

$$I(U)_d + I(X)_d = \mathbb{C}[x_{1,1}, \dots, x_{n,n}]_d$$

$$\Longleftrightarrow$$

$$S^d(U) \cap I(X_{\text{Sep}})_d^{\perp} = \{0\}$$

Works extremely well already for $d = 2$!

$$I(X_{\text{Sep}})_d^{\perp} = \{(u \otimes v)^{\otimes d} : u, v \in \mathbb{C}^n\} = S^d(\mathbb{C}^n) \otimes S^d(\mathbb{C}^n)$$

Product tensors: $X_{\mathrm{Sep}} = \{u \otimes v: \; u, v \in \mathbb{C}^n\} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, determine if U is entangled, i.e. if $U \cap X_{\mathrm{Sep}} = \{0\}$.

$$I(X_{\mathrm{Sep}})_2^{\perp} := \mathrm{Span}\{(u \otimes v)^{\otimes 2}: \; u, v \in \mathbb{C}^n\} = S^2(\mathbb{C}^n) \otimes S^2(\mathbb{C}^n)$$

Takes poly(n) time to check

Algorithm (2nd level of Nullstellensatz certificate):

If $S^2(U) \cap I(X_{\mathrm{Sep}})_2^{\perp} = \{0\}$, output U is entangled

Otherwise, output Fail

Correctness: $u \otimes v \in U \;\Rightarrow (u \otimes v)^{\otimes 2} \in S^2(U) \cap I(X_{\mathrm{Sep}})_2^{\perp}$

$\Rightarrow$ Algorithm outputs Fail.

Product tensors: $X_{\mathrm{Sep}} = \{u \otimes v : \; u, v \in \mathbb{C}^n\} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$,
determine if U is entangled, i.e. if $U \cap X_{\mathrm{Sep}} = \{0\}$

Works-Extremely-Well Theorem [JLV 22]:
U generic and $\dim(U) \leq \frac{1}{4}(n-1)^2 \quad \Rightarrow \quad S^2(U) \cap I(X_{\mathrm{Sep}})_2^{\perp} = \{0\}$.

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$\Rightarrow$ Algorithm outputs Fail.

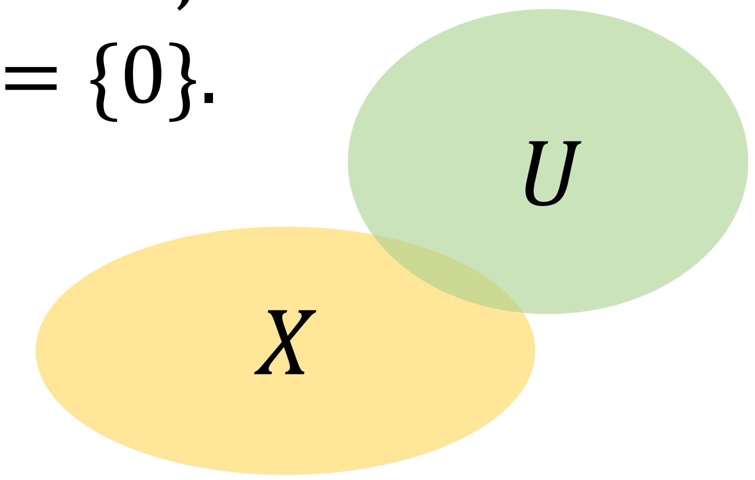
Algorithm runtime to certify $U \cap X_{\mathbf{Sep}} = \{0\}$

d	$\dim(U)$	time
3	3	0.01 s
4	8	0.03 s
5	13	0.08 s
6	20	0.20 s
7	29	0.49 s
8	39	1.06 s
9	50	2.24 s
10	63	5.56 s

Analogous hierarchies for other notions of entanglement (any conic variety)

Let $X \subseteq \mathbb{C}^N$ be any conic variety (for example, $X = X_{\text{Sep}} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$)

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^N$,
determine if U avoids X, i.e. if $U \cap X = \{0\}$.



Let $X \subseteq \mathbb{C}^N$ be any conic variety (for example, $X = X_{\mathrm{Sep}} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$)

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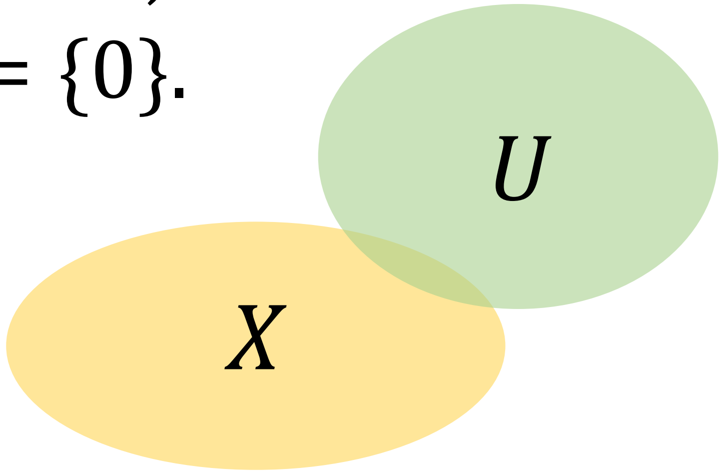

$$I(X)_d^{\perp} := \mathrm{Span}\{v^{\otimes d} : \ v \in X\}$$

Takes $N^{O(d)}$ time to check

<u>Algorithm d:</u>
If $S^d(U) \cap I(X)_d^{\perp} = \{0\}$, output U avoids X
Otherwise, output Fail

<u>Completeness [Hilbert]:</u> For $d = 2^{O(N)}$, Fail $\Leftrightarrow$ U intersects X

Examples

WEW Theorem [JLV 22]: For generic U of dimension $\dim(U) \leq$ 👽 it holds that $S^d(U) \cap I(X)_d^{\perp} = \{0\}$, for $d =$ 🐜

Schmidt rank $\leq r$ tensors

$X_r = \{v \in \mathbb{C}^n \otimes \mathbb{C}^n : \text{Schmidt}-\text{rank}(v) \leq r\}$

👽 $= \Omega_r(n^2)$
🐜 $= r + 1$

Product tensors in-X_{Sep}-arable $\leftrightarrow$ Completely entangled

$X_{\text{Sep}} = \{v_1 \otimes \cdots \otimes v_m : v_i \in \mathbb{C}^n\}$

👽 $\sim(1/4)n^m$
🐜 $= 2$

Biseparable tensors in-X_B-arable $\leftrightarrow$ Genuinely entangled

$X_B = \{v \in (\mathbb{C}^n)^{\otimes m}$: Some bipartition of v has rank 1$\}$

👽 $\sim(1/4)n^m$
🐜 $= 2$

Slice rank 1 tensors

$X_S = \{v \in (\mathbb{C}^n)^{\otimes m}$: Some 1 v.s. rest bipartition of v has rank 1$\}$

👽 $\sim(1/4)n^m$
🐜 $= 2$

Matrix product tensors of bond dimension $\leq r$

$X_{MPS} = \{v \in (\mathbb{C}^n)^{\otimes m}$: Every left-right bipartition has rank $\leq r\}$

👽 $= \Omega_r(n^m)$
🐜 $= r + 1$

Examples

<u>WEW Theorem [JLV 22]</u>: For generic U of dimension $\dim(U) \leq$ 👽 it holds that $S^d(U) \cap I(X)_d^{\perp} = \{0\}$, for $d =$ 🐜

Schmidt rank $\leq r$ tensors

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Product tensors in-X_{Sep}-arable ↔ Completely entangled

$X_{\text{Sep}} = \{v_1 \otimes \cdots \otimes v_m : v_i \in \mathbb{C}^n\}$

👽 $\sim(1/4)n^m$
🐜 $= 2$

Bisepara

$X_B = \{v$

n^m

<u>Takeaway:</u> Algorithm certifies entanglement of subspaces of dimension a constant multiple of the maximum possible in polynomial time.

Slice rank 1 tensors

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Derksen's proof (sketch)

*A slightly weaker WEW Theorem appears in [JLV 22] with a different proof.

<u>WEW Theorem [Derksen]*:</u> If $I \subseteq \mathbb{C}[x_1, \ldots, x_N]$ is a homogeneous ideal and R is a non-negative integer such that

$$\dim I_d^{\perp} < \binom{N-R+d}{d},$$

then there exists an R-dimensional subspace $U \subseteq \mathbb{C}^D$ such that $S^d(U) \cap I_d^{\perp} = \{0\}$.

Proof sketch: By a theorem of Galligo, after a linear change of coordinates wma $J \coloneqq \mathrm{lm}(I)$ is Borel-fixed with respect to the reverse lexicographic monomial order.

If $x_R^d \notin J_d$, then $J_d \subseteq \langle x_1, \ldots, x_{R-1} \rangle_d$. But then

$$\begin{aligned} \dim\left(I_d^{\perp}\right) &= \dim\left(J_d^{\perp}\right) \\ &\geq \dim\left(\mathbb{C}[x_1, \ldots, x_N]_d / \langle x_1, \ldots, x_{R-1} \rangle_d\right) \\ &= \binom{N-R+d}{d}, \text{ a contradiction.} \end{aligned}$$

So $x_R^d \in J_d$. But this implies all monomials in $x_1, \ldots, x_R$ of degree d lie in J.

It follows that $S^d(U) \cap I_d^{\perp} = \{0\}$ for $U = \mathrm{span}\{e_1, \ldots, e_R\}$. ∎

Lifted Jennrich's algorithm to recover elements of $U \cap X$ (with applications to tensor decompositions)

Suppose $U \subseteq \mathbb{C}^N$ has a basis $\{v_1, \dots, v_R\}$ such that each $v_i \in X$.

Problem: Given some other basis $\{u_1, \dots, u_R\}$ of U, recover $\{v_1, \dots, v_R\}$ (up to scale).

Example: Jennrich's Algorithm: If $U' \subseteq S^d(\mathbb{C}^N)$ is spanned by $\{v_1^{\otimes d}, \dots, v_R^{\otimes d}\}$ with $\{v_1, \dots, v_R\}$ linearly independent, then $\{v_1^{\otimes d}, \dots, v_R^{\otimes d}\}$ can be recovered from any basis of U' in $n^{O(d)}$- time.

Lifted Jennrich's Algorithm [JLV 2022]: Run Jennrich on $U' = S^d(U) \cap I(X)_d^{\perp}$.

$$v^{\otimes d} \in U' \iff v \in U \cap X$$

Generalizes FOOBI algorithm [DLCC '07]

For this to work, need:

1. $\{v_1^{\otimes d}, \dots, v_R^{\otimes d}\}$ spans U'.
2. $\{v_1, \dots, v_R\}$ is linearly independent.

Suppose $U \subseteq \mathbb{C}^N$ has a basis $\{v_1, \ldots, v_R\}$ such that each $v_i \in X$.

Works-Extremely-Well Theorem [JLV 22]:
If $d \geq 2$, X is irreducible, cut out in degree d, and has no equations in degree $d-1$, then (1) and (2) hold for generic $v_1, \ldots, v_R \in X$ as long as $R \leq \frac{\dim(I(X)_d)}{d!\binom{N+d-1}{d}}(N+d-1)$

Pro le).

Example: Jennrich's Algorithm: If $U' \subseteq S^d(\mathbb{C}^N)$ is spanned by $\{v_1^{\otimes d}, \ldots, v_R^{\otimes d}\}$ with $\{v_1, \ldots, v_R\}$ linearly independent, then $\{v_1^{\otimes d}, \ldots, v_R^{\otimes d}\}$ can be recovered from any basis of U' in $n^{O(d)}$- time.

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Exa

$\{v_1$ y

bas

Works-Extremely-Well Theorem [JLV 22]:

If $d \geq 2$, X is irreducible, cut out in degree d, and has no equations in degree $d-1$, then (1) and (2) hold for generic $v_1, \dots, v_R \in X$ as long as $R \leq \frac{\dim(I(X)_d)}{d!\binom{N+d-1}{d}}(N+d-1)$

$\Delta = \{(1, \dots, 1), \dots, (R, \dots, R)\}$

Proof technique: Show that

$$\operatorname{span}\{v_{i_1} \cdots v_{i_d} : (i_1, \dots, i_d) \notin \Delta\} \cap I(X)_d^{\perp} = \{0\}$$

for generic $v_1, \dots, v_R \in X$. This is equivalent to (1).

Lifted Jennrich's Algorithm [JLV 2022]: Run Jennrich on $U' = S^d(U) \cap I(X)_d^{\perp}$.

Compare with Derksen's result:

$S^d(U) \cap I(X)_d^{\perp} = \{0\}$ for generic $v_1, \dots, v_R \in \mathbb{C}^N$

For this to work, need:

1. $\{v_1^{\otimes d}, \dots, v_R^{\otimes d}\}$ spans U'.
2. $\{v_1, \dots, v_R\}$ is linearly independent.

Q: Clean algebraic proof?

Similar WEW Theorems were claimed in [DL 06, DLCC 07] for the special case $X = X_{\mathrm{Sep}}$, but their proofs are incorrect.

Application: $(X, \mathbb{C}^k)$-decompositions

For $T \in V \otimes \mathbb{C}^k$, an $(X, \mathbb{C}^k)$-decomposition is an expression $T = \sum_{i=1}^{R} v_i \otimes z_i \in V \otimes \mathbb{C}^k$

where $v_1, \dots, v_R \in X$

$\mathrm{rank}_X(T) := \min\{R : \text{there exists an } (X, \mathbb{C}^k)\text{–decomposition of T of length R}\}$

<u>Example:</u> When $X = X_{\mathrm{Sep}} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, an $(X, \mathbb{C}^k)$-decomposition is just a tensor decomposition.

Viewing T as a map $\mathbb{C}^k \to V$, each $v_i \in T(\mathbb{C}^k) \cap X$,

so computing $T(\mathbb{C}^k) \cap X \leftrightarrow (X, \mathbb{C}^k)$-decomposing T

(Assuming that $\{z_1, \dots, z_R\}$ is linearly independent)

Corollary to WEW Theorem [JLV 22]: A generic tensor $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^k$ with

$$\mathrm{rank}(T) \leq \min\{\frac{1}{4}(n-1)^2, k\}$$

has a unique rank decomposition, which is recovered in POLY(n)-time by applying our algorithm to $T(\mathbb{C}^k)$.

In particular, a generic $n \times n \times n^2$ tensor of rank $\sim \frac{1}{4} n^2$ is recovered by algorithm.

Corollary to WEW Theorem [JLV 22]: A generic tensor $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^k$ of $(X_r, \mathbb{C}^k)$-rank

$$\mathrm{rank}_{X_r}(T) \leq \min\{\Omega_r(n^2)\,, k\}$$

has a unique tensor rank decomposition, which is recovered in $n^{O(r)}$-time by applying our algorithm to $T\left(\mathbb{C}^k\right)$.

$T = \sum_i v_i \otimes w_i\,,$ where $v_i \in X_r$

$(X_r, \mathbb{C}^k)$-rank $\Leftrightarrow$ r-aided rank $\Leftrightarrow$ $(r, r, 1)$-multilinear rank

Corollary to WEW Theorem [JLV 22]: A generic tensor $T \in (\mathbb{C}^n)^{\otimes m}$ of tensor rank

$$\operatorname{rank}(T) = O(n^{\lfloor m/2 \rfloor})$$

has a unique tensor rank decomposition, which is recovered in $n^{O(m)}$-time by applying our algorithm to $T\left((\mathbb{C}^n)^{\otimes \lfloor m/2 \rfloor}\right)$.

(This is new when m is even. When m is odd you can just use Jennrich directly.)

Robust generalization of the entanglement certification hierarchy

Robust generalization:

Instead of determining whether U avoids X,

Compute $h_X(U) \coloneqq \max_{\substack{v \in X \\ \|v\|=1}} \langle v, P_U v \rangle$

$P_U = \mathrm{Proj}(U)$

U avoids $X \quad \Leftrightarrow \quad h_X(U) < 1$

Theorem/Robust Hierarchy [JLV 23+]:

Let $X \subseteq \mathbb{C}^N$ be nice*, $U \subseteq \mathbb{C}^N$ linear, and $P_U = \mathrm{Proj}(U)$.

*Any conic variety

For each d, let $\mu_d = \lambda_{\max}\left(P_X^d \left(P_U \otimes I^{\otimes d-1}\right) P_X^d\right)$. $P_X^d = \mathrm{Proj}(I(X)_d^{\perp})$

Then the μ_d form a non-increasing sequence converging to $h_X(U) := \max\limits_{\substack{v \in X \\ \|v\|=1}} \langle v, P_U v \rangle$.

Robust generalization:

Instead of determining whether U avoids X,

Compute $h_X(U) := \max\limits_{\substack{v \in X \\ \|v\|=1}} \langle v, P_U v \rangle$

$P_U = \mathrm{Proj}(U)$

U avoids $X \iff h_X(U) < 1$

Theorem/Robust Hierarchy [JLV 23+]:

Let $X \subseteq \mathbb{C}^N$ be nice*, $W \in \mathrm{Herm}(\mathbb{C}^N)$ Hermitian.

*Any conic variety

For each d, let $\mu_d = \lambda_{\max}\left(P_X^d\left(W \otimes I^{\otimes d-1}\right)P_X^d\right)$. $P_X^d = \mathrm{Proj}(I(X)_d^\perp)$

Then the μ_d form a non-increasing sequence converging to $h_X(W) \coloneqq \max_{\substack{v \in X \\ \|v\|=1}} \langle v, Wv \rangle$.

Robust generalization:

Instead of determining whether U avoids X,

Compute $h_X(U) \coloneqq \max_{\substack{v \in X \\ \|v\|=1}} \langle v, P_U v \rangle$

$P_U = \mathrm{Proj}(U)$

Theorem/Robust Hierarchy not only holds for P_U, but for any Hermitian W!

U avoids $X \quad \Leftrightarrow \quad h_X(U) < 1$

Conclusion

1. **Complete hierarchies** of linear systems to certify entanglement of a subspace. These work extremely well already at early levels.

Title: Complete hierarchy of linear systems for certifying quantum entanglement of subspaces

2. **Poly-time algorithms** to find low-entanglement elements of a subspace. These also work extremely well.

Title: Computing linear sections of varieties: quantum entanglement, tensor decompositions and beyond

3. **Robust version** of certification hierarchies to compute the distance between a variety and a linear subspace.

Title: TBD

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