Computing linear sections of varieties: quantum entanglement, tensor decompositions and beyond

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Product tensors: \[ X_{\text{Sep}} = \{ u \otimes v : u, v \in \mathbb{C}^n \} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n \]

Problem: Given a basis for a linear subspace \( U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n \), determine if \( U \) is entangled, i.e. if \( U \cap X_{\text{Sep}} = \{0\} \).

Applications: Quantum Information

- Range criterion: For a density operator \( \rho \in D(\mathbb{C}^n \otimes \mathbb{C}^n) \),
  \[ \text{Im}(\rho) \text{ entangled } \Rightarrow \rho \text{ entangled} \]
- Entangled subspaces can be used to construct entanglement witnesses and quantum error-correcting codes
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Outline:
1. Algorithm (Nullstellensatz Certificate)
2. Algorithm to recover elements of \( U \cap X_{\text{Sep}} \), with applications to tensor decompositions
3. Generalization to arbitrary conic variety \( X \)
4. Robust generalization of Hilbert’s Nullstellensatz for this problem
Product tensors: \( X_{\text{Sep}} = \{ u \otimes v: \ u, v \in \mathbb{C}^n \} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n \)

Problem: Given a basis for a linear subspace \( U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n \), determine if \( U \) is entangled, i.e. if \( U \cap X_{\text{Sep}} = \{0\} \).

[Buss et al 1999]: This is NP-Hard in the worst case.

[Barak et al 2019]: Best known algorithm takes \( 2^{\tilde{O}(\sqrt{n})} \) time.

[Classical AG, Parthasarathy 01]: \( \dim(U) > (n-1)^2 \implies U \) is not entangled

\( U \) generic and \( \dim(U) \leq (n-1)^2 \implies U \) is entangled

Algorithm (deg. 2 N.C.): Takes \( \text{poly}(n) \)-time and outputs either: “Hay in a haystack problem”

1. Fail, or
2. A certificate that \( U \) is entangled
Product tensors: \( X_{\text{Sep}} = \{ u \otimes v : u, v \in \mathbb{C}^n \} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n \)

**Problem:** Given a basis for a linear subspace \( U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n \), determine if \( U \) is entangled, i.e. if \( U \cap X_{\text{Sep}} = \{0\} \).

[Buss et al 1999]: This is NP-Hard in the worst case.

**Works-Extremely-Well Theorem [JLV 22]:**

\( U \) generic and \( \dim(U) \leq \frac{1}{4} (n - 1)^2 \Rightarrow \) Algorithm outputs a certificate that \( U \) is entangled

Algorithm (deg. 2 N.C.): Takes \( \text{poly}(n) \)-time and outputs either: “Hay in a haystack problem”

1. Fail, or
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The Algorithm
(Nullstellensatz Certificate)
Product tensors: \( X_{\text{Sep}} = \{ u \otimes v: \ u, v \in \mathbb{C}^n \} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n \)

Problem: Given a basis for a linear subspace \( U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n \), determine if \( U \) is entangled, i.e. if \( U \cap X_{\text{Sep}} = \{0\} \).

Idea: Problem is difficult because it’s non-linear

\( (X_{\text{Sep}} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n \) isn’t a linear subspace).)

Make it linear: Instead check if \( U \cap \text{Span}(X_{\text{Sep}}) = \{0\} \).

Works extremely well already for \( d = 2 \)!

Lift it up: Let \( I(X_{\text{Sep}})_{d}^\perp = \text{Span}\{(u \otimes v)^\otimes d: \ u, v \in \mathbb{C}^n \} = S^d(\mathbb{C}^n) \otimes S^d(\mathbb{C}^n) \)

Check if \( S^d(U) \cap I(X_{\text{Sep}})_{d}^\perp = \{0\} \).
Product tensors: $X_{\text{Sep}} = \{u \otimes v: u, v \in \mathbb{C}^n\} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$

Problem: Given a basis for a linear subspace $U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$, determine if $U$ is entangled, i.e. if $U \cap X_{\text{Sep}} = \{0\}$.

Hilbert’s Nullstellensatz:

$U \cap X = \{0\} \iff \text{For some } d \in \mathbb{N} \text{ it holds that }$ 

$$I(U)_d + I(X)_d = \mathbb{C}[x_{1,1}, \ldots, x_{n,n}]_d$$

$$\iff$$ 

$$S^d(U) \cap I(X_{\text{Sep}})_d^\perp = \{0\}$$

Works extremely well already for $d = 2!$
Product tensors: \[ X_{\text{Sep}} = \{ u \otimes v : \ u, v \in \mathbb{C}^n \} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n \]

Problem: Given a basis for a linear subspace \( U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n \), determine if \( U \) is entangled, i.e. if \( U \cap X_{\text{Sep}} = \{0\} \).

\[ I(X_{\text{Sep}})_2^\perp = \text{Span}\{(u \otimes v)^{\otimes 2} : \ u, v \in \mathbb{C}^n\} = S^2(\mathbb{C}^n) \otimes S^2(\mathbb{C}^n) \]

Takes \( \text{poly}(n) \) time to check

Algorithm (2\textsuperscript{nd} level of Nullstellensatz certificate):
If \( S^2(U) \cap I(X_{\text{Sep}})_2^\perp = \{0\} \), output \( U \) is entangled
Otherwise, output \text{Fail}

Correctness: \( u \otimes v \in U \Rightarrow (u \otimes v)^{\otimes 2} \in S^2(U) \cap I(X_{\text{Sep}})_2^\perp \)
\[ \Rightarrow \text{Algorithm outputs Fail.} \]
Product tensors: \( X_{\text{Sep}} = \{ u \otimes v : u, v \in \mathbb{C}^n \} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n \)

Problem: Given a basis for a linear subspace \( U \subseteq \mathbb{C}^n \otimes \mathbb{C}^n \), determine if \( U \) is entangled, i.e., if \( U \cap X_{\text{Sep}} = \{0\} \).

Works-Extremely-Well Theorem [JLV 22]:
\( U \) generic and \( \dim(U) \leq \frac{1}{4} (n - 1)^2 \Rightarrow S^2(U) \cap I(X_{\text{Sep}})^\perp_2 = \{0\} \).

Algorithm (2\textsuperscript{nd} level of Nullstellensatz certificate):
If \( S^2(U) \cap I(X_{\text{Sep}})^\perp_2 = \{0\} \), output \( U \) is entangled
Otherwise, output \( \text{Fail} \)

Correctness: \( u \otimes v \in U \Rightarrow (u \otimes v)^\otimes 2 \in S^2(U) \cap I(X_{\text{Sep}})^\perp_2 \Rightarrow \) Algorithm outputs \( \text{Fail} \).
Algorithm runtime to certify $U \cap X_{\text{Sep}} = \{0\}$

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<th>time</th>
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<tr>
<td>10</td>
<td>63</td>
<td>5.56 s</td>
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Analogous hierarchies for other notions of entanglement (any conic variety)
Let $X \subseteq \mathbb{C}^N$ be any conic variety (for example, $X = X_{\text{Sep}} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$).

**Problem:** Given a basis for a linear subspace $U \subseteq \mathbb{C}^N$, determine if $U$ avoids $X$, i.e. if $U \cap X = \{0\}$. 
Let $X \subseteq \mathbb{C}^N$ be any conic variety (for example, $X = X_{\text{Sep}} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$)

**Problem:** Given a basis for a linear subspace $U \subseteq \mathbb{C}^N$, determine if $U$ avoids $X$, i.e. if $U \cap X = \{0\}$.

$I(X)^{\perp_d} := \text{Span}\{v^\otimes d : v \in X\}$

**Algorithm $d$:**

If $S^d(U) \cap I(X)^{\perp_d} = \{0\}$, output $U$ avoids $X$

Otherwise, output Fail

**Completeness [Hilbert]:** For $d = 2^{O(N)}$, Fail $\iff$ $U$ intersects $X$
Examples

**WEW Theorem [JLV 22]:** For generic $U$ of dimension $\dim(U) \leq \bigotimes$ it holds that $S^d(U) \cap I(X)^\perp_d = \{0\}$, for $d = \bigotimes$.

### Schmidt rank $\leq r$ tensors

$X_r = \{v \in \mathbb{C}^n \otimes \mathbb{C}^n: \text{Schmidt–rank}(v) \leq r\}$

- $\bigotimes = \Omega_r(n^2)$
- $\bigoplus = r + 1$

### Product tensors

**in-$X_{\text{Sep}}$-arable $\leftrightarrow$ Completely entangled**

$X_{\text{Sep}} = \{v_1 \otimes \cdots \otimes v_m: v_i \in \mathbb{C}^n\}$

- $\bigotimes \sim (1/4)n^m$
- $\bigoplus = 2$

### Biseparable tensors

**in-$X_B$-arable $\leftrightarrow$ Genuinely entangled**

$X_B = \{v \in (\mathbb{C}^n)^\otimes m: \text{Some bipartition of } v \text{ has rank } 1\}$

- $\bigotimes \sim (1/4)n^m$
- $\bigoplus = 2$

### Slice rank 1 tensors

$X_S = \{v \in (\mathbb{C}^n)^\otimes m: \text{Some } 1 \text{ v.s. rest bipartition of } v \text{ has rank } 1\}$

- $\bigotimes \sim (1/4)n^m$
- $\bigoplus = 2$

### Matrix product tensors of bond dimension $\leq r$

$X_{\text{MPS}} = \{v \in (\mathbb{C}^n)^\otimes m: \text{Every left-right bipartition has rank } \leq r\}$

- $\bigotimes = \Omega_r(n^m)$
- $\bigoplus = r + 1$
Examples

**WEW Theorem [JLV 22]:** For generic $U$ of dimension $\dim(U) \leq \underline{\Omega}(n^2)$, it holds that $S^d(U) \cap I(X)^\perp_d = \{0\}$, for $d = \underline{\Omega}(n)$. 

**Schmidt rank $\leq r$ tensors**

$X_r = \{v \in \mathbb{C}^n \otimes \mathbb{C}^n: \text{Schmidt-rank}(v) \leq r\}$

$\Rightarrow \exists \Omega_r(n^2)$

$\Rightarrow r + 1$

**Product tensors**

$X_{\text{Sep}} = \{v_1 \otimes \cdots \otimes v_m: v_i \in \mathbb{C}^n\}$

$\Rightarrow \Omega_r(1/4)n^m$

$\Rightarrow 2$

**Biseparable tensors**

$X_B = \{v \in \mathbb{C}^n \otimes \mathbb{C}^n: \text{Some 1 v.s. rest bipartition of } v \text{ has rank 1}\}$

$\Rightarrow \Omega_r(1/4)n^m$

$\Rightarrow 2$

**Slice rank 1 tensors**

$X_S = \{v \in (\mathbb{C}^n)^\otimes m: \text{Some 1 v.s. rest bipartition of } v \text{ has rank 1}\}$

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$\Rightarrow 2$

**Matrix product tensors of bond dimension $\leq r$**

$X_{\text{MPS}} = \{v \in (\mathbb{C}^n)^\otimes m: \text{Every left-right bipartition has rank } \leq r\}$

$\Rightarrow \Omega_r(n^m)$

$\Rightarrow r + 1$

**Takeaway:** Algorithm certifies entanglement of subspaces of dimension a constant multiple of the maximum possible in polynomial time.
Derksen’s proof (sketch) *A slightly weaker WEW Theorem appears in [JLV 22] with a different proof.

**WEW Theorem [Derksen]*: If $I \subseteq \mathbb{C}[x_1, ..., x_N]$ is a homogeneous ideal and $R$ is a non-negative integer such that
\[
\dim I^\perp_d < \binom{N - R + d}{d},
\]
then there exists an $R$-dimensional subspace $U \subseteq \mathbb{C}^D$ such that $S^d(U) \cap I^\perp_d = \{0\}$.

**Proof sketch:** By a theorem of Galligo, after a linear change of coordinates wma $J := \text{Im}(I)$ is Borel-fixed with respect to the reverse lexicographic monomial order.

If $x_R^d \not\in J_d$, then $J_d \subseteq \langle x_1, ..., x_{R-1} \rangle_d$. But then
\[
\dim(I^\perp_d) = \dim(J^\perp_d) \\
\geq \dim (\mathbb{C}[x_1, ..., x_N]_d / \langle x_1, ..., x_{R-1} \rangle_d) \\
= \binom{N-R+d}{d}, \text{ a contradiction.}
\]
So $x_R^d \in J_d$. But this implies all monomials in $x_1, ..., x_R$ of degree $d$ lie in $J$.

It follows that $S^d(U) \cap I^\perp_d = \{0\}$ for $U = \text{span}\{e_1, ..., e_R\}$. 
Lifted Jennrich’s algorithm to recover elements of $U \cap X$ (with applications to tensor decompositions)
Suppose $U \subseteq \mathbb{C}^N$ has a basis $\{v_1, \ldots, v_R\}$ such that each $v_i \in X$.

**Problem:** Given some other basis $\{u_1, \ldots, u_R\}$ of $U$, recover $\{v_1, \ldots, v_R\}$ (up to scale).

**Example:** **Jennrich’s Algorithm:** If $U' \subseteq S^d(\mathbb{C}^N)$ is spanned by $\{v_1^d, \ldots, v_R^d\}$ with $\{v_1, \ldots, v_R\}$ linearly independent, then $\{v_1^d, \ldots, v_R^d\}$ can be recovered from any basis of $U'$ in $n^{O(d)}$-time.

**Lifted Jennrich’s Algorithm [JLV 2022]:** Run Jennrich on $U' = S^d(U) \cap I(X)^\perp_d$.

For this to work, need:

1. $\{v_1^d, \ldots, v_R^d\}$ spans $U'$.
2. $\{v_1, \ldots, v_R\}$ is linearly independent.

Generalizes FOOBI algorithm [DLCC ‘07]
Suppose $U \subseteq \mathbb{C}^N$ has a basis $\{v_1, \ldots, v_R\}$ such that each $v_i \in X$.

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**Example:** Jennrich’s Algorithm: If $U' \subseteq S^d(\mathbb{C}^N)$ is spanned by $\{v_1 \otimes d, \ldots, v_R \otimes d\}$ with $\{v_1, \ldots, v_R\}$ linearly independent, then $\{v_1 \otimes d, \ldots, v_R \otimes d\}$ can be recovered from any basis of $U'$ in $n^{O(d)}$-time.

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**Works-Extremely-Well Theorem [JLV 22]:**

If $d \geq 2$, $X$ is irreducible, cut out in degree $d$, and has no equations in degree $d - 1$,

then (1) and (2) hold for generic $v_1, \ldots, v_R \in X$ as long as $R \leq \frac{\dim(I(X)_{d})}{d! \left(\frac{N + d - 1}{d}\right)} (N + d - 1)$
Suppose $U \subseteq \mathbb{C}^N$ has a basis $\{v_1, \ldots, v_R\}$ such that each $v_i \in X$. 

**Works-Extremely-Well Theorem [JLV 22]:**

If $d \geq 2$, $X$ is irreducible, cut out in degree $d$, and has no equations in degree $d - 1$, then (1) and (2) hold for generic $v_1, \ldots, v_R \in X$ as long as $R \leq \frac{\dim(I(X)_d)}{d!(N+d-1)}(N + d - 1)$

**Example:**

Let $\Delta = \{(1, \ldots, 1), \ldots, (R, \ldots, R)\}$ and consider the span of $v_i$ for $i \notin \Delta$. Then, $\text{span}\{v_1, \ldots, v_d : (i_1, \ldots, i_d) \notin \Delta\} \cap I(X)^\perp_d = \{0\}$ for generic $v_1, \ldots, v_R \in X$. This is equivalent to (1).

**Lifted Jennrich's Algorithm [JLV 2022]:** Run Jennrich on $U' = S^d(U) \cap I(X)^\perp_d$.

For this to work, need:

1. $\{v_1 \otimes d, \ldots, v_R \otimes d\}$ spans $U'$.
2. $\{v_1, \ldots, v_R\}$ is linearly independent.

**Q:** Clean algebraic proof?
Application: \((X, \mathbb{C}^k)\)-decompositions

For \(T \in V \otimes \mathbb{C}^k\), an \((X, \mathbb{C}^k)\)-decomposition is an expression

\[ T = \sum_{i=1}^{R} v_i \otimes z_i \in V \otimes \mathbb{C}^k \]

where \(v_1, \ldots, v_R \in X\)

\[ \text{rank}_X(T) := \min \{ R : \text{there exists an } (X, \mathbb{C}^k)\text{-decomposition of } T \text{ of length } R \} \]

**Example:** When \(X = X_{\text{Sep}} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n\), an \((X, \mathbb{C}^k)\)-decomposition is just a tensor decomposition.

Viewing \(T\) as a map \(\mathbb{C}^k \to V\), each \(v_i \in T(\mathbb{C}^k) \cap X\), so computing \(T(\mathbb{C}^k) \cap X \leftrightarrow (X, \mathbb{C}^k)\text{-decomposing } T\)

(Assuming that \(\{z_1, \ldots, z_R\}\) is linearly independent)
Corollary to WEW Theorem [JLV 22]: A generic tensor 
\( T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^k \) with
\[
\text{rank}(T) \leq \min \left\{ \frac{1}{4} (n - 1)^2, k \right\}
\]
has a unique rank decomposition, which is recovered in 
\( \text{POLY}(n) \)-time by applying our algorithm to 
\( T(\mathbb{C}^k) \).

In particular, a generic \( n \times n \times n^2 \) tensor of rank \( \sim \frac{1}{4} n^2 \) is recovered by algorithm.
Corollary to WEW Theorem [JLV 22]: A generic tensor $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^k$ of $(X_r, \mathbb{C}^k)$-rank

$$\text{rank}_{X_r}(T) \leq \min\{\Omega_r(n^2), k\}$$

has a unique tensor rank decomposition, which is recovered in $n^{O(r)}$-time by applying our algorithm to $T(\mathbb{C}^k)$.

$$T = \sum_i v_i \otimes w_i , \text{ where } v_i \in X_r$$

$(X_r, \mathbb{C}^k)$-rank $\iff$ $r$-aided rank $\iff$ $(r, r, 1)$-multilinear rank
Corollary to WEW Theorem [JLV 22]: A generic tensor $T \in (\mathbb{C}^n)^\otimes m$ of tensor rank \[
\text{rank}(T) = O(n^{\lfloor m/2 \rfloor})
\] has a unique tensor rank decomposition, which is recovered in $n^{O(m)}$-time by applying our algorithm to $T \left((\mathbb{C}^n)^\otimes \lfloor m/2 \rfloor \right)$.

(This is new when $m$ is even. When $m$ is odd you can just use Jennrich directly.)
Robust generalization of the entanglement certification hierarchy
Robust generalization:

Instead of determining whether $U$ avoids $X$, compute $h_X(U) := \max_{\|v\|=1} \langle v, P_U v \rangle$

$U$ avoids $X \iff h_X(U) < 1$
Theorem/Robust Hierarchy \([JLV\ 23+]\):

Let \(X \subseteq \mathbb{C}^N\) be nice\(^*\), \(U \subseteq \mathbb{C}^N\) linear, and \(P_U = \text{Proj}(U)\).

For each \(d\), let \(\mu_d = \lambda_{\text{max}}(P_{X}^d (P_{U} \otimes I^\otimes d^{-1}) P_{X}^d)\). \(P_{X}^d = \text{Proj}(I(X)_{\frac{1}{d}})\)

Then the \(\mu_d\) form a non-increasing sequence converging to \(h_X(U) := \max_{\norm{v}=1} \langle v, P_U v \rangle\).

\(^*\)Any conic variety

Robust generalization:

Instead of determining whether \(U\) avoids \(X\),

Compute \(h_X(U) := \max_{\norm{v}=1} \langle v, P_U v \rangle\)

\(U\) avoids \(X\) \iff \(h_X(U) < 1\)
Theorem/Robust Hierarchy [JLV 23+]:
Let $X \subseteq \mathbb{C}^N$ be nice*, $W \in \text{Herm}(\mathbb{C}^N)$ Hermitian.

For each $d$, let $\mu_d = \lambda_{\max} \left( P_X^d \left( W \otimes I^\otimes d-1 \right) P_X^d \right)$. $P_X^d = \text{Proj}(I(X)_{\frac{1}{d}})$

Then the $\mu_d$ form a non-increasing sequence converging to $h_X(W) := \max_{\|v\|=1} \langle v, Wv \rangle$.

*Any conic variety

Robust generalization:
Instead of determining whether $U$ avoids $X$,
Compute $h_X(U) := \max_{\|v\|=1} \langle v, P_Uv \rangle$, $P_U = \text{Proj}(U)$

$U$ avoids $X \iff h_X(U) < 1$

Theorem/Robust Hierarchy not only holds for $P_U$, but for any Hermitian $W$!
Conclusion

1. **Complete hierarchies** of linear systems to **certify** entanglement of a subspace. These **work extremely well** already at early levels.
   
   **Title:** Complete hierarchy of linear systems for certifying quantum entanglement of subspaces

2. **Poly-time algorithms** to **find** low-entanglement elements of a subspace. These also **work extremely well**.
   
   **Title:** Computing linear sections of varieties: quantum entanglement, tensor decompositions and beyond

3. **Robust version** of certification hierarchies to compute the **distance** between a variety and a linear subspace.
   
   **Title:** TBD
Computing linear sections of varieties: quantum entanglement, tensor decompositions and beyond

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