Factoring, Root finding, and several other things

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Outline of the Talk

- Multivariate Polynomial Factoring
  - Background and Motivation.
  - Factoring Algebraic Branching Programs.
- Multivariate Factoring and PIT.

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Multivariate Polynomial Factoring: Background

- We encounter integer and polynomial factoring in school.
- Polynomials can be factored in polynomial time.
- Factor $f(x) \in \mathbb{Q}[x]$ using LLL algorithm in deterministic polynomial time.
- Factor $f(x) \in \mathbb{F}_q[x]$ using Berlekamp's algorithm.

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- Polynomials are often easier cases than integers.
- Squarefree: Test if a given integer or polynomial has a factor that repeats.
- For integers, no polynomial time algorithm for this is known.
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- Degree of each variable in $f(x_1, \ldots, x_n)$ is $\leq d$.
- Apply Kronecker substitution $\phi : x_i \mapsto z^{D^{i-1}}$ where $D = d + 1$.
- Each monomial in f uniquely maps to a monomial in $\phi(f)$. Thus, we can invert the map ϕ.

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- If $f = gh$, then $\phi(f) = \phi(g)\phi(h)$.
- Factorize $\phi(f)$ into univariate irreducible factors.
- Though g is irreducible, $\phi(g)$ may not be irreducible.
- Product of a subset of the factors of $\phi(f)$ would correspond to $\phi(g)$.
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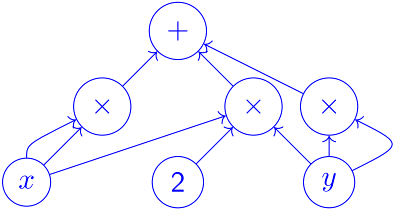
- Kaltofen (1982): Efficient reduction of bivariate to univariate factoring.
- Tools: Newton iteration/ Hensel lifting, Linear System Solving.
- We have to define the size of input and output polynomials in the multivariate setting to talk about time complexity.

- Dense: List all the coefficients of $\binom{n+d}{d}$ many monomials up to degree d.
- Sparse: List only the monomials with nonzero coefficients. Eg. $x_1^2 + x_2 x_3 + 5x^4$.
- Formula: $(1 + x_1)(1 + x_2)x_3 - (1 + x_1)^2$. Reuse of computation not allowed. Structurally, looks like a tree.
- Straight-Line Programs or Arithmetic Circuits.

Two circuits for computing $x^2 + 2xy + y^2$



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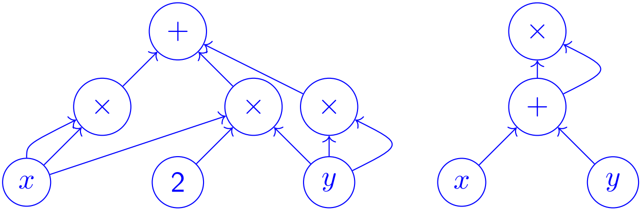

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Complexity of Factors

Factorization of a polynomial

Let f be a polynomial of degree d that has size s in some model.

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{m} f_i^{e_i}$$

Let f_i's be its irreducible factors over $\mathbb{F}$.

Factor size bound question: Do all its factors have $\text{poly}(s, d)$ size in the same model?

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FACTOR SIZE BOUND QUESTION: Do all its factors have $\text{POLY}(s, d)$ size in the same model?

- Let $\mathcal{C}$ be a class of polynomials.
- Closure under multiplication: $f, g \in \mathcal{C} \implies f \times g \in \mathcal{C}$.
- Closure under factoring: If $f \times g$ is in $\mathcal{C}$, are f, g also in $\mathcal{C}$?
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- Sparsity of factors can be superpolynomial wrt input polynomial's sparsity.

- Kaltofen 1986: If size denotes arithmetic circuit size, $g \mid f$ $\implies$ $\text{size}(g) \leq \textsc{poly}(\text{size}(f), \deg(f))$.

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- Multivariate Polynomial Factoring has applications in coding theory and various other problems.

- Helps to bridge two central questions in algebraic complexity: VP vs VNP and polynomial identity testing (PIT).

- Kabanets and Impagliazzo (2003): Exponential lower bound for arithmetic circuits $\implies$ Quasi-poly blackbox deterministic PIT for circuits.

- Hardness of multiples: If f is hard for $\mathcal{C}$, all its nonzero multiples are hard for $\mathcal{C}$.

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- Dutta, Saxena, S (2018): If we take formula/ABP, $g \mid f \implies$ size$(g) \leq$ poly(size$(f), d^{O(\log d)})$.
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Factorization of Arithmetic Branching Programs

ABPs are closed under factors

Theorem (S, Thierauf, 2020)

Let polynomial $p(\bar{x})$ over $\mathbb{F}$ have ABP-size s.

All factors of p have ABP-size $\textsc{poly}(s)$

Algorithm: Factors can be efficiently constructed in randomized polynomial time.

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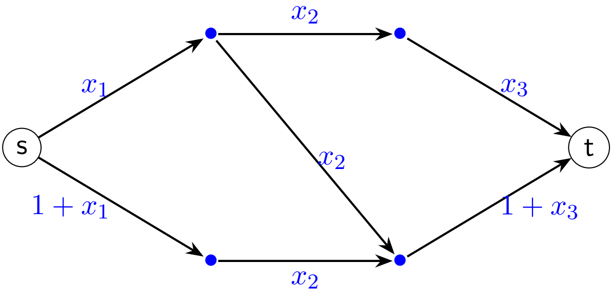

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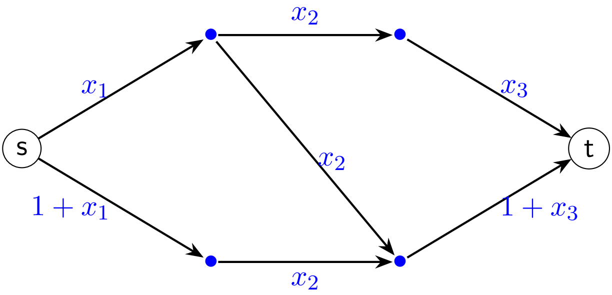

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Computational power of ABPs

Arithmetic Formula $\leq$ ABP $\leq$ Arithmetic Circuit

DET: compute determinant of $n \times n$ matrices

- $\mathrm{DET} \in \mathrm{NC}^2$ [Csanky-Faddeev-LeVerrier 1976] [Berkowitz-Samuelson 1985, Chistov 1985]
- DET by poly(n)-size ABPs [Mahajan-Vinay 1997]

Consequence

Poly-size ABPs can compute solutions of linear systems over $\mathbb{F}(x_1, \ldots, x_n)$.

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Newton Iteration [Oliveira 2016, Dutta, Saxena, S 2018]

Factoring $\leq$ root approximation in power series

- $$p(\boldsymbol{x}, y) \text{ has factor } y - q(\boldsymbol{x}) \iff p(\boldsymbol{x}, q(\boldsymbol{x})) = 0$$

- approximate root via Newton iteration

$$y_{t+1} = y_t - \frac{p(\boldsymbol{x}, y_t)}{p'(\boldsymbol{x}, y_t)}$$

- $\log d$ iterations leads to $d^{\log d}$-size ABPs

Hensel Lifting (Kurt Hensel, 1861 - 1941)

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Factoring via Hensel Lifting

After preprocessing: $f(x, y)$, 2-variate, degree d, *monic* in x (i.e. highest x-power has constant coefficient)

Goal: Compute g s.t. $f = g\,h$

1. Initial step: Factorize univariate polynomial $f(x, 0)$
   - $f(x, 0) = g_0(x)\,h_0(x)$
   - Equivalently: $f(x, y) \equiv g_0(x)\,h_0(x) \pmod{y}$
2. Lifting: compute $g_1(x, y), h_1(x, y)$
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Iterate lifting $t = O(\log d)$ times: $f \equiv g_t\,h_t \pmod{y^{2^t}}$

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Main difference to earlier liftings

Start with g_0, h_0 monic

- standard Hensel Lifting maintains g_k, h_k monic, for all k
- simplified version gives up monicness: saves a division
- ABP-size grows by a constant factor in each iteration

  $\implies$ overall size $\mathrm{poly}(c^{\log d}, s) = \mathrm{poly}(s)$
- Crucial technical part: Jump Step still works!

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- standard Hensel Lifting maintains g_k, h_k monic, for all k
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Hensel Lifting: Definition of Lift

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- Compute the following.
- $e = f - gh$
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- $c = ag' + bh' - 1$
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- Additionally compute polynomials q and r such that $g' - g = qg + r$, where $\deg_x(r) < \deg_x(g)$.
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Lifted Factor and Actual Factor

Lemma (Actual factor vs lifted factor)

$g \equiv g_t h'_t \pmod{y^{2^t}}$ for some polynomial h'_t.

Proof Idea

Inductively apply Hensel lifting to both f and factor g starting from $f = g_0 h_0 \pmod{y}$ and $g = g_0 h'_0 \pmod{y}$ respectively.

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- Compare the coefficients of each monomial $x^i y^j$ in LHS and RHS of Eqn. $g \equiv g_t h'_t \pmod{y^{2^t}}$.
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- First prove that $\tilde{g}$ and the factor g have nontrivial gcd by showing that their Resultant is zero.
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GCD and Resultants

- The resultant $r(y) = \mathrm{Res}_x(g, \tilde{g})$ is a polynomial (of degree $\leq 2d^2$) in y defined via determinant of Sylvester matrix.

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- Thus $g_t w \bmod y^{2^t}$ is not free of x. Contradiction. Thus, $r(y)$ must be 0 mod y^{2^t}. If $2^t \geq 2d^2 + 1$, then $r(y) = 0$. QED.

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- Let $f(x, z_1, \ldots, z_n)$ be the given polynomial to be factored
- Create a new polynomial $\widehat{f}(x, y, \boldsymbol{z}) = f(x, yz_1, \ldots, yz_n)$
- Consider $\widehat{f}$ as a bivariate in x and y with coefficients in $\mathbb{F}[\boldsymbol{z}]$.
- To get back f from $\widehat{f}$, simply put y to 1 in $\widehat{f}$.
- Putting y to 0 in $\widehat{f}$, we get univariate $\widehat{f}(x)$.

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Linear System over a big field

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- But we have to pay the price in the jump step. Our linear system now has coefficients from $\mathbb{F}(z_1, \ldots, z_n)$.
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Factorization and PIT

- Test if $g(x_1, \ldots, x_n)$ divides $f(x_1, \ldots, x_n)$.
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- Whether a linear polynomial divides a sparse polynomial can be tested in polynomial time.
- $f(x_1, \ldots, x_n)$ is divisible by $x_1 - \ell(x_2, \ldots, x_n)$ iff $f(\ell, x_2, \ldots, x_n) = 0$.
- Semidiagonal model: $\Sigma m_i \ell^{e_i}$ where m_i is a monomial and ℓ are linear polynomials [Saha-Saptharishi-Saxena 2010].
- Testing if a quadratic polynomial divides a s-sparse polynomial in $s^{\log s}$ time [Forbes 2015]. Reduces to PIT of sums of monomials times powers of quadratics.
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- Given sparse polynomials $f, g_1, \ldots, g_k$, test if $f = \prod_{i=1}^{k} g_i$. Or more generally, $f = \prod_i g_i^{e_i}$.
- Testing in deterministic polynomial time open, even when g_i have bounded degree.
- More general question: Given sparse polynomials $f_1, \ldots, f_k$ and $g_1, \ldots, g_r$, test if $\prod_{i=1}^{r} f_i = \prod_{i=1}^{k} g_i$.
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- Given a black box computing a multivariate polynomial f, black boxes of the irreducible factors of f can be computed in randomized polynomial time [Kaltofen-Trager 1991].

- Dimension reduction: Randomly project to bivariates.

- This works due to an effective version of Hilbert's irreducibility theorem.

- If $f(x, z_1, \ldots, z_n)$ is irreducible, then $f(x, \beta_1 + \alpha_1 y, \ldots, \beta_n + \alpha_n y)$ is irreducible with high probability if β_i, α_i picked at random.

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Thank You!