Implicit Active Model using Radial Basis Function Interpolated Level Sets

Xianghua Xie and Majid Mirmehdi
Department of Computer Science
University of Bristol, Bristol BS8 1UB, England.
{xie,majid}@cs.bris.ac.uk

Abstract

Building on recent work by others that introduced RBFs into level sets for structural topology optimisation, we introduce the concept into active models and present a new level set formulation able to handle more complex topological changes, in particular perturbation *away from the evolving front*. This allows the initial contour or surface to be placed arbitrarily in the image. The proposed level set updating scheme is efficient and does not suffer from self-flattening while evolving which will cause large numerical error. Unlike conventional level set based active models, periodic re-initialisation is also no longer necessary and the computational grid can be much coarser, thus, it has great potential in modelling in high dimensional space. We show results on synthetic and real data for active modelling in 2D and 3D.

1 Introduction

The application of the level set method [7] to the active contour model has enabled the latter to adapt to complex topologies. It avoids the need to reparameterise the curve and the contours are able to split or merge in order to capture an unknown number of objects. However, the original level set based active contour [1] proved to be of limited use in real applications as it assumes that contours reach the object boundaries at roughly the same time. Thus, it often suffers from weak edge leakage. The development of improved external forces, in particular region based methods, such as [10, 8], have greatly improved the performance of level set based snakes. They are generally less initialisation dependent and exhibit better ability in handling textures and image noise interference. Among many others, some realised, practical applications can be found in [10, 9].

The extension of the active contour model into the active surface model is relatively straightforward due to their implicit representation in the level set scheme. However, this implicit representation embeds the contour or surface into a higher dimensional space which needs to be updated iteratively as a whole, becoming more computationally expensive than traditional parametric approaches. The evolution of the embedded contour or surface is solved using partial differential equations (PDEs) which in most cases involves costly finite difference methods (FDM).

More importantly, in conventional level set methods, active contours or surfaces are not able to create topological changes *away from the zero level set* where the deformable contours or surfaces are embedded [9, 11]. This means, for example, that the level sets would miss holes inside objects. In order to accurately solve the associated PDEs using FDM, a local method, it requires the implicit function to be smooth and maintained to be so while evolving. Thus, re-initialisation is usually necessary in order to achieve numerical stability. Although alternative methods without re-initialisation are available, they often require dedicated extension of the speed function defined on the contour.

As a primary interpolation tool, radial basis functions (RBFs) have received increasing attention in solving PDE systems in recent years. For example, Cecil *et al.* [2] used RBFs to generalise conventional FDM on a non-uniform (unstructured) computational grid to solve the high dimensional Hamilton-Jacobi PDEs with high accuracy. Very recently, Wang *et al.* [11] interpolated level set functions using RBFs and transformed the Hamilton-Jacobi PDEs into a system of ordinary differential equations (ODEs) for structural topology optimisation in 2D.

In this paper, we adapt the approach presented for structure design in [11] to apply to active modelling and show how our proposed model greatly enhances the performance of active models. Following [11], we interpolate the initial level set function using RBFs and treat the implicit contour/surface propagation as an ODE problem, which is much easier and more efficient to solve. However, the updating scheme proposed in [11] was found to be unsuitable for active modelling. A simple yet effective normalisation scheme is proposed to resolve this issue. This new active model exhibits significant improvements in initialisation invariancy, convergence ability, and topology adaptability. The initial contour or surface is embedded into an implicit function derived from the distance transform in the way same as the conventional level set approach. However, we then interpolate it using RBFs which can be placed on a much coarser grid. The interpolation is characterised by its expansion coefficients. Thus, deforming the original implicit function is achieved by updating the expansion coefficients. Re-initialisation is found no longer necessary and perturbation away from the zero level is allowed to obtain more sophisticated topological changes. The contour or surface can therefore be initialised anywhere in the image. We show an implementation of this RBF level set method in a region based active contour model. The extension of this method to 3D on synthetic data is also demonstrated.

Notably, very recently in [5], Morse *et al.* placed RBFs at contour landmarks to implicitly represent the active contour, thereby avoiding the manipulation of a higher dimensional function. However, the method requires dynamic insertion and deletion of landmarks which is non-trivial. Similar to the parametric representation, the resolution and position of the landmarks can affect the accuracy of contour representation.

In the next section we present a brief review of the conventional level set method, RBF interpolation, the proposed RBF level set evolution, and its application to a region based active contour model. The extension to 3D is presented in Section 3. Conclusions and future work are discussed in Section 4.

2 Proposed Method

2.1 Level Set Representation

Using level sets [7], a contour or surface is implicitly represented by a multi-dimensional scalar function with the moving front embedded at the zero level set. Let C and Φ denote the moving front and the level set function respectively. The relationship between these two can be expressed as: $C = \{\mathbf{x}|\Phi(\mathbf{x}) = 0\}$ where $\mathbf{x} \in \mathbb{R}^n$, and subject to $\Phi(\mathbf{x}) > 0$ for

$\mathbf{x}$ inside the front and $\Phi(\mathbf{x}) < 0$ for $\mathbf{x}$ outside. This representation is parameter free and intrinsic. Considering the front (contour or surface) evolving according to $dC/dt = F\mathcal{N}$ for a given function F (where $\mathcal{N}$ denotes the inward unit normal), then the embedding function should deform according to $\partial\Phi/\partial t = F|\nabla\Phi|$, where F is computed on the level sets. By embedding the evolution of C in that of Φ, topological changes of C, such as split and merge, are handled automatically.

The level set function is commonly initialised using the signed distance transform and its evolution numerically solved using FDM with the upwind scheme [7]. The numerical error using this local approximation method may gradually accumulate and can contaminate the solution. Thus, periodic re-initialisation of the level set function is usually applied to maintain numerical stability. The conventional level set method generally prevents topological changes taking place away from the developing front which restricts other forms of topological changes, such as developing holes inside objects. The method presented here will allow the level set contour or surface to deal with regions away from the evolving front by initiating new fronts in the level set and thus capture holes or inner boundaries of objects. This makes the active contour or surface framework not only much more successful but also initialisation invariant.

2.2 RBF Interpolated Level Set Function

Similar to recent works by Cecil *et al.* [2] and Wang *et al.* [11], we interpolate the level set function $\Phi(\mathbf{x})$ using a certain number of RBFs. Each RBF, ψ_i, is a radially symmetric function centred at position $\mathbf{x}_i$. Only a single function ψ is used to form this family of RBFs. The multiquadric spline, found to be one of the best for RBF interpolation [3] is used here, with the RBFs then written as:

$$\psi_i(\mathbf{x}) = \psi(||\mathbf{x} - \mathbf{x}_i||) = \sqrt{(\mathbf{x} - \mathbf{x}_i)^2 + c_i^2}, \tag{1}$$

where c_i is usually treated as a constant for all RBFs. The interpolation is expressed as:

$$\Phi(\mathbf{x}) = p(\mathbf{x}) + \sum_{i=1}^{N} \alpha_i \psi_i(\mathbf{x}), \tag{2}$$

where N denotes the number of RBFs, α_i are the expansion coefficients of the corresponding RBF, and $p(\mathbf{x})$ is a first-degree polynomial, which in the 2D case can be written as $p(\mathbf{x}) = p_0 + p_1 x + p_2 y$.[1] To ensure a unique solution to this RBF interpolation, the expansion coefficients must satisfy $\sum_{i=1}^{N} \alpha_i = \sum_{i=1}^{N} \alpha_i x_i = \sum_{i=1}^{N} \alpha_i y_i = 0$. These N number of RBFs are uniformly distributed in the domain and their centre values, denoted by $f_1, ..., f_N$, are given by the level set function. The RBF interpolant then can be obtained by solving the following linear system:

$$\mathbf{H}\boldsymbol{\alpha} = \mathbf{f}, \quad \text{where} \quad \mathbf{H} = \begin{pmatrix} \mathbf{A} & \mathbf{P} \\ \mathbf{P}^T & 0 \end{pmatrix}, \tag{3}$$
$$\boldsymbol{\alpha} = [\alpha_1 \ \cdots \ \alpha_N \ p_0 \ p_1 \ p_2]^T, \qquad \mathbf{f} = [f_1 \ \cdots \ f_N \ 0 \ 0 \ 0]^T,$$

and $\mathbf{A}_{i,j} = \psi_j(\mathbf{x}_i), i, j = 1, \ldots, N$, $\mathbf{P}_{i,j} = p_j(\mathbf{x}_i), i = 1, \ldots, N, j = 1, 2, 3$, and p_j are the basis for the polynomial. Thus, the RBF interpolation of the level set function in (2)

[1] For simplicity, we present the solution in 2D. Its solution in higher dimension is straightforward.

can be written as:

$$\Phi(\mathbf{x}) = \mathbf{\Psi}^T(\mathbf{x})\boldsymbol{\alpha}, \tag{4}$$

where $\mathbf{\Psi}(\mathbf{x}) = [\psi_1(\mathbf{x}) \ \cdots \ \psi_N(\mathbf{x}) \ 1 \ x \ y]^T$.

2.3 Active Modelling using RBF Level Set

As stated in Section 2.1, the deformation of the active contour is achieved by propagating the level sets along their normal directions according to a localised speed which is usually image dependent. It can be expressed as the following PDE:

$$\frac{\partial \Phi}{\partial t} + F|\nabla\Phi| = 0, \tag{5}$$

where F is the speed function along the normal direction. Unlike the conventional level set method, here we have a level set function interpolated by RBFs. Following [11], we assume that time and space are separable and the time dependence of the level set function is now due to the RBF interpolation, i.e. the expansion coefficients. Updating the level set function is now considered as updating the RBF expansion coefficients. In other words, the expansion coefficients become time dependent: $\Phi = \mathbf{\Psi}^T(\mathbf{x})\boldsymbol{\alpha}(t)$. Thus, the level set updating function (5) can be re-written as:

$$\frac{\partial \Phi}{\partial t} + F|\nabla\Phi| = \mathbf{\Psi}^T \frac{d\boldsymbol{\alpha}}{dt} + F|(\nabla\mathbf{\Psi})^T\boldsymbol{\alpha}| = 0. \tag{6}$$

This indicates the original PDE problem can now be treated as an ODE problem. The spatial derivative $\nabla\mathbf{\Psi}$ can be solved analytically using the first order Euler's method, also adopted in [11]. Substituting (3) into (6) we have,

$$\mathbf{H}\frac{d\boldsymbol{\alpha}}{dt} + \mathbf{B}(\boldsymbol{\alpha}) = 0, \tag{7}$$

where

$$\mathbf{B}(\boldsymbol{\alpha}) = [F(\mathbf{x}_1)|(\nabla\mathbf{\Psi}^T(\mathbf{x}_1))\boldsymbol{\alpha}| \ \ \ldots \ \ F(\mathbf{x}_N)|(\nabla\mathbf{\Psi}^T(\mathbf{x}_N))\boldsymbol{\alpha}| \ \ 0 \ \ 0 \ \ 0]^T \tag{8}$$

The solution can be obtained by iteratively updating the expansion coefficients:

$$\boldsymbol{\alpha}(t^{n+1}) = \boldsymbol{\alpha}(t^n) - \Delta t\mathbf{H}^{-1}\mathbf{B}(\boldsymbol{\alpha}(t^n)). \tag{9}$$

The updating of the level set function starts from interpolating its initial state using RBFs. As usual, the initial level set function is obtained from the signed distance transform. Then RBFs are uniformly spread across the domain and the interpolation takes place which gives us the initial value of the expansion coefficients, $\boldsymbol{\alpha}$. The interpolated level set then is evolved according to (9) and (2). Unlike conventional level set approaches where the upwind scheme [7] is often used and re-initialisation is applied to maintain numerical stability, the coefficient updating is much simpler and efficient and does not require re-initialisation.

Although (9) has been proven useful in structure optimisation in [11], a direct application of this updating scheme was found to be unsuitable for active contour models. An example is given in Fig. 1 where a circular shape is embedded in an initial level set function. A constant force is applied to this active contour, i.e. F is a constant. This force

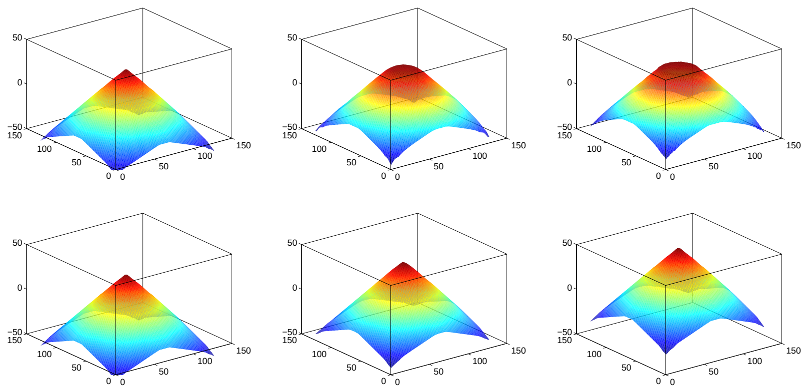

Figure 1: Updating RBF level set using non-normalised and normalised schemes - first row: Non-normalised scheme; second row: proposed normalised scheme.

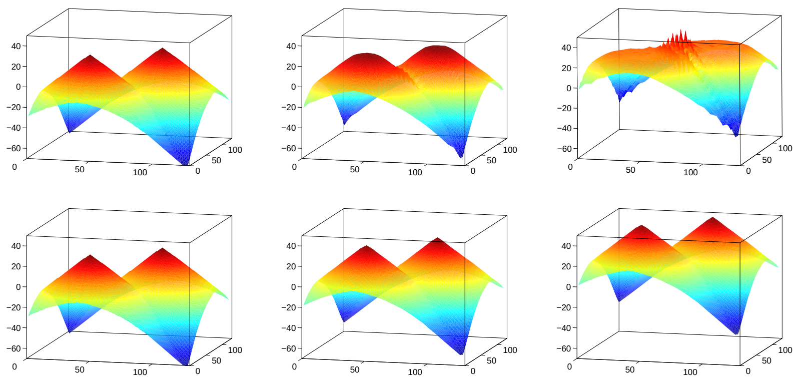

Figure 2: Updating the RBF level set using non-normalised and normalised schemes - first row: Non-normalised scheme; second row: Proposed normalised scheme.

expands the contour outwards which should generally lift the level set function. However, as shown in the first row, the top of the level set function becomes stationary and gradually turns into a flat surface. This is due to the gradient values of the RBF interpolated level set at those points being close to zero ($|(\nabla \mathbf{\Psi}^T(\mathbf{x}_i))\boldsymbol{\alpha}| \rightarrow 0$) and based on (8) and (9) the expansion coefficients at those places would evolve much slower. As a result, the level set function tends to get flattened and this is undesirable when topological changes should be taking place. See for example in Fig. 2 where two circles are expanding due to the same constant force. The valley in the level set function is affected and introduces numerical artifacts, and finally contaminates the solution as shown in the last image in the first row, indicated by the highly irregular spikes in the level set function. Special care is thus necessary, for example using dedicated velocity extension.

Fortunately, in active contours the direction of the speed along the normal has dominant effect on the final segmentation, not its magnitude. Since the gradient of the level

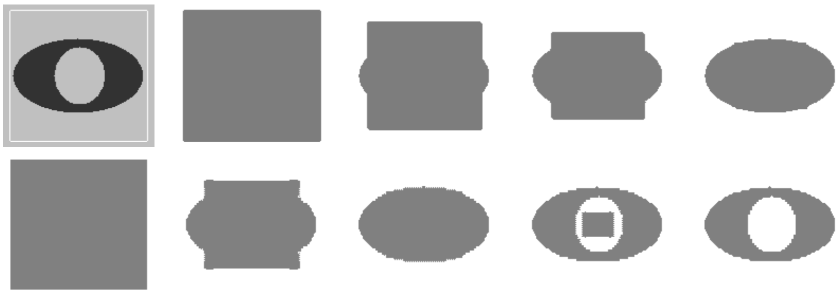

Figure 3: More complex topological changes are readily achievable - first row: initial snake and recovered shape using conventional level set method; second row: recovered shape using proposed method. The final images in both rows show the stabilised results.

set function is generally smoothly varying, a simple yet highly effective solution can be devised to solve this problem. We modify the speed function by "normalising" it against the local gradient estimated from the RBF interpolants, i.e.

$$F'(\mathbf{x}_i) = \frac{1}{|(\nabla \mathbf{\Psi}^T(\mathbf{x}_i))\boldsymbol{\alpha}|} F(\mathbf{x}_i). \tag{10}$$

Note that due to this global modelling using RBFs, the gradient is dependent on all the RBF centres across the domain, instead of local neighbours. Thus, the gradient near the advancing front is unlikely to be zero, i.e. this normalisation will be unlikely to disturb the developing front. Eq. (8) then simplifies to:

$$\mathbf{B}(\boldsymbol{\alpha}) = [F(\mathbf{x}_1) \ \ \ldots \ \ F(\mathbf{x}_N) \ \ 0 \ \ 0 \ \ 0]^T. \tag{11}$$

Updating the expansion coefficients and hence the level set are now even simpler and more efficient. The second row in Fig. 1 shows the results using the normalised approach. The level set function does not get flattened while updating the expansion coefficients. Topological changes, for example merging shown in the second row in Fig. 2, can be conveniently handled, in contrast to the non-normalised scheme (shown in the first row).

One of the main advantages of using a RBF interpolated level set to represent an active contour is that more sophisticated topological changes, besides merging and splitting, can be readily achieved. Let F be a region indication function, i.e. $F < 0$ for points inside an object of interest and $F > 0$ for the rest. In Fig. 3, the object of interest is shown in dark gray, and the initial snake is drawn in white. The snake using the conventional level set scheme with re-initialisation failed to recover the hole in the object as periodic re-initialisation prevents it from doing so. The proposed RBF based level set method successfully recovered the shape without dedicated effort in monitoring the front propagation. This occurs because the proposed method uses RBF interpolants to estimate the level set gradient, a global estimation instead of a local one. Front propagation is then unlikely to introduce oscillation around the zero level set. Thus re-initialisation is not necessary to maintain stability. The proposed RBF expansion coefficients updating scheme prevents other level sets, away from the evolving front, from flattening themselves so that these level sets are sensitive enough to sufficient gradient changes for the RBF interpolated front to grow new fronts (i.e contours or surfaces).

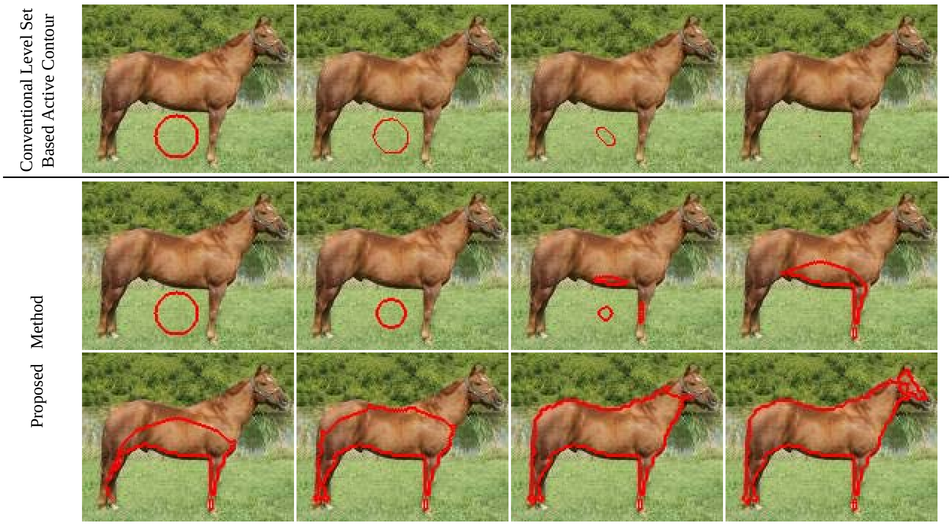

Figure 4: Comparative result on real image - first row: Segmentation result using conventional level set with the initial snake forced to shrink; second and third rows: Results using proposed RBF level set method.

2.4 A Region Based Active Contour Model using RBF Level Sets

We now present a region based active contour model as a demonstration of the proposed RBF level set method. As mentioned earlier in Section 1, region based methods generally perform better in the presence of weak edges and image noise interference. More importantly in relevance to this work, region based methods are considered much less initialisation dependent. There are two classes of popular region based approaches. One is based on the well-known Mumford-Shah formulation [6], where the contours compete with each other while preserving the piecewise constant assumption. The other, such as the works in [10, 8], globally model the image data and the active contour evolves to maximise its posterior. We opt for the second approach and model the image data using Gaussian Mixture Models (GMM). Note we are not advocating a region based approach or this particular GMM based method, but we employ these to demonstrate the performance of our proposed RBF level set method. Our aim is to give a comparative study of the proposed RBF level set method with the conventional level set approach *in the same active contour framework*.

The colour (or intensity) histogram of a given image is modelled using GMM. Each pixel is then assigned posterior probabilities for each class. Let u denote the posterior probability of the class of interest. The GMM region based active contour can be formulated as:

$$\frac{dC}{dt} = (1 - \frac{1}{m})u\mathcal{N}, \tag{12}$$

where m is the number of classes and $\frac{1}{m}$ is the average expectation of a class probability. Its level set representation takes the following form:

$$\frac{\partial\Phi}{\partial t} = (1 - \frac{1}{m})u|\nabla\Phi|. \tag{13}$$

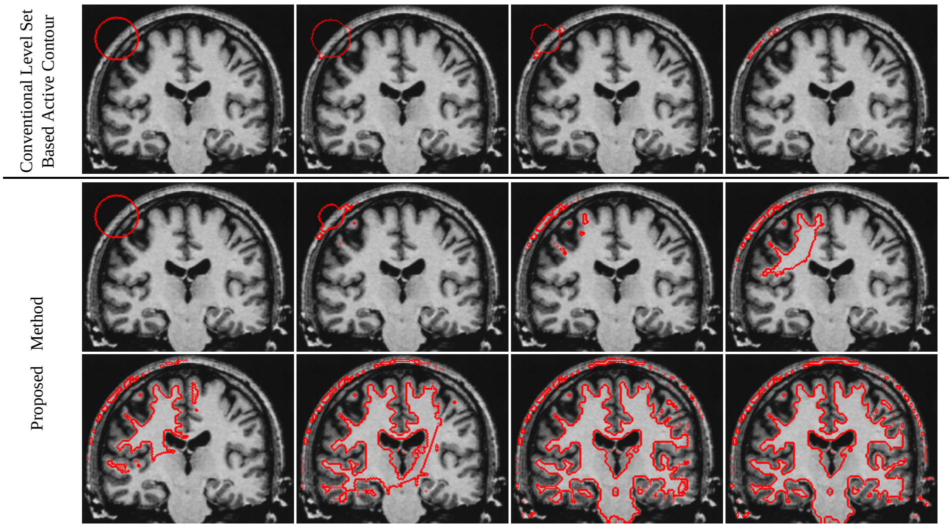

Figure 5: Comparative result on real image - first row: Segmentation result using conventional level set; second and third rows: Results using proposed RBF level set method.

For simplicity we ignore the internal contour regularisation terms, but use the image dependent force term alone to deform the active contour. The contour is supposed to expand and shrink to maximise the posterior of the regions of interest. With the proposed method, new contours can even grow out in regions away from existing contours, which is not possible for the conventional level set approach. Equally importantly, the initial contour can be forced to vanish from the image domain while newly appearing fronts are able to localise the regions. This gives significant improvement in initialisation invariancy and achieves global minimum, instead of local minimum (as demonstrated earlier in Fig. 3).

Fig. 4 shows the comparative results of the GMM region snake using the conventional level set approach (top row) and the proposed RBF level set method (rows 2 and 3). The initial snake was placed outside the object of interest and was forced to *shrink*. The conventional method failed to localise the object while the proposed method succeeded by growing out new contours inside the object. In this case, the conventional method requires the initial snake to be specifically placed overlapping or inside the object. Another example is given in Fig. 5, where multiple regions exist. The proposed method could localise all the regions that were indicated by the function u, while conventional level sets could only capture those that the initial contour had touched.

3 Extension to 3D

Similar to the conventional level set method, the extension of the proposed method to higher dimensions is straightforward. Even better, the proposed method demands only a much coarser mesh grid. The RBF centres can be more loosely placed in 3D, instead of the full pixel grid often used in conventional level set approaches. Also, solving the ODE system in 3D is much easier than solving the PDE system. The updating of the expansion coefficients are efficient and again does not require re-initialisation of the level set function. The main computation cost comes from interpolating the initial level set and

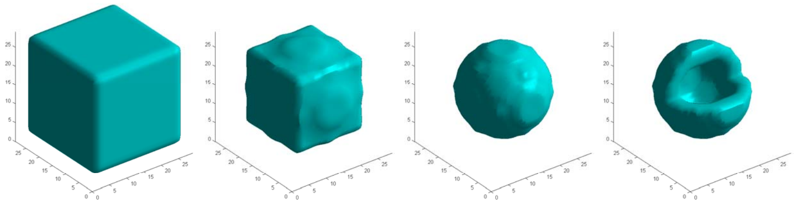

Figure 6: Recovering a hollow sphere using proposed method - from left: Initial deformable surface, evolving deformable surface, stabilised surface, and the stabilised surface with a section cut away to show the hole captured inside.

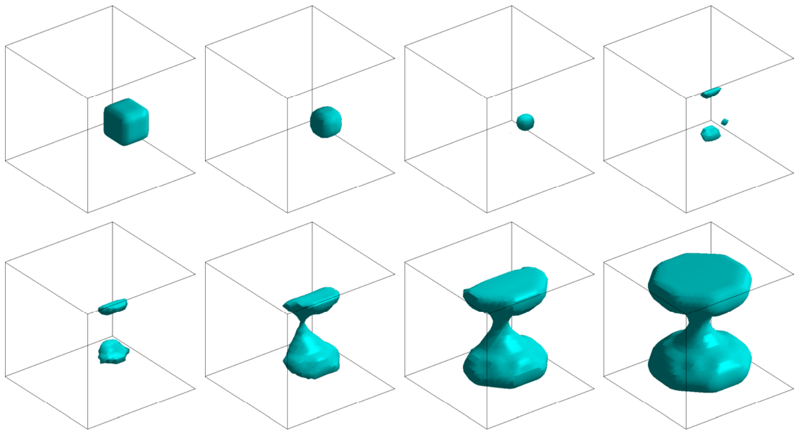

Figure 7: Arbitrary initialisation - The initial surface is placed outside the object and is forced to shrink, but the proposed method allows the level set to deform further to develop a zero level set outside the initial surface and recover the object.

reconstructing the level set function after it stabilises. However, there are several methods available to speed up the process, such as the Fast Multipole Method (FMM) [4].

We examine the ability of the proposed method in handling complex 3D topologies and initialisation invariancy. We apply the 3D RBF level set method on synthetic data and evolve the active surface according to (5), where $F < 0$ for regions inside the 3D objects and $F > 0$ otherwise, as before. In Fig. 6, the target object was a hollow sphere. The initial surface was placed to surround the object and was forced to shrink to capture the object boundaries. With the proposed RBF level set method, not only was the outer boundary localised, but also the boundary inside was captured, i.e. as the active surface was deforming, a new zero level set developed inside the object. The next example given in Fig. 7 shows that the region indication function shrinks the active surface that initialised outside the target object. There was no intersection between the initial surface and the object, neither when the initial surface deformed and disappeared. However, the proposed method allows the level set to deform further to "grow" outside the initial surface and finally recovers the object. This again demonstrates the method's initialisation independence feature. In the final example shown in Fig. 8, we demonstrate the ability of the proposed method in modelling very complex geometry in 3D.

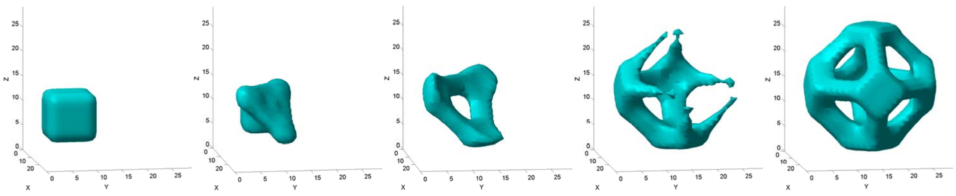

Figure 8: Recovering a complex 3D shape.

4 Conclusion

We have presented a novel method to perform implicit modelling using RBFs. The proposed method has a number of advantages over the conventional level set scheme: (a) The evolution of the level set function is considered as an ODE problem rather than a much more difficult PDE problem; (b) Re-initialisation of the level set function was found no longer necessary for this application; (c) More complex topological changes, such as holes within objects, are comfortably found; (d) The active contour and surface models using this technique are initialisation independent; (e) The computational grid can be much coarser, hence it is more computationally cheaper when updating the level set function, particularly in high dimensional spaces. Future work includes implementing a fast implementation of RBF fitting and reconstruction, and applying this method to large scale 3D segmentation problems.

References

[1] V. Caselles, R. Kimmel, and G. Sapiro. Geodesic active contour. *IJCV*, 22(1):61–79, 1997.

[2] T. Cecil, J. Qian, and S. Osher. Numerical methods for higher dimensional Hamilton-Jacobi equations using radial basis functions. *Journal of Computational Physics*, 196:327–347, 2004.

[3] R. Franke. Scattered data interpolation: Tests of some methods. *Mathematics of Computation*, 38:181–200, 1982.

[4] L. Greengard and V. Rokhlin. A fast algorithm for particle simulations. *Journal of Computational Physics*, 73:325–348, 1987.

[5] B. Morse, W. Liu, T. Yoo, and K. Subramanian. Active contours using a constraint-based implicit representation. In *CVPR*, pages 285–292, 2005.

[6] D. Mumford and J. Shah. Optimal approximations by piecewise smooth functions and associated variational problems. *Commun. Pure Appl. Math.*, 42(5):577–685, 1989.

[7] S. Osher and J. Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations. *Journal of Computational Physics*, 79:12–49, 1988.

[8] N. Paragios and R. Deriche. Geodesic active regions and level set methods for supervised texture segmentation. *IJCV*, 46(3):223–247, 2002.

[9] R. Ramlau and W. Ring. A Mumford-Shah level-set approach for the invesion and segmentation of X-ray tomography data. *Journal of Computational Physics*, 221:539–557, 2007.

[10] J. Suri. Two-dimensional fast magnetic resonance brain segmentation. *IEEE Engineering in Medcine and Biology*, 20(4):84–95, 2001.

[11] S. Wang, K. Lim, B. Khoo, and M. Wang. An extended level set method for shape and topology optimization. *Journal of Computational Physics*, 221:395–421, 2007.