Betrachtungen

über

einige Gegenstände

der

Elementargeometrie

von

Bernard Bolzano.

Τας ἐπιδοσεις ὁρωμεν γιγνομενας, και των τεχνων, και των ἀλλων ἁπαντων, οὐ δια τους ἐμμενοντας τοις καθεστωσιν, ἀλλα δια τους ἐπανορθουντας, και τολμωντας ἀει τι κινειν των μη καλως ἐχοντων. Isocr. Evag.

Prag, 1804.
in Commission bey Karl Barth.

Considerations

on

Some Objects

of

Elementary Geometry

by

Bernard Bolzano

*τὰς ἐπιδόσεις ἴσμεν γιγνομένας καὶ τῶν
τεχνῶν καὶ τῶν ἄλλων ἁπάντων οὐ διὰ τοὺς
ἐμμέν οντας τοῖς καθεστῶσιν, ἀλλὰ διὰ τοὺς ἐπα-
νορθοῦντας καὶ τολμῶντας ἀεί τι κινεῖν τῶν μὴ
καλῶςἐχόντων.*

—*Isocrates*, Evagoras.

Prague, 1804
In Commission with Karl Barth

Title page quotation:

Progress is made, not only in the arts, but in all other activities, not through the agency of those who are satisfied with things as they are, but through those who correct, and have the courage constantly to change, anything which is not as it should be.

Isocrates, Evagoras, Vol. III, p.9, Loeb Classical Library,

Tr. L.van Hook, Heinemann, 1945.

Dem

Hochwürdigsten, Hochgelehrten und
Wohlgebornen

Herrn, Herrn

Stanislaus Wydra,

Director und Professor der Mathematik, emeri-
tirten Rector Magnificus, Domherrn bey
Aller Heiligen etc. etc.

zum Beweise

einer unbegränzten Hochachtung und Dankbarkeit

gewidmet

von seinem ehemaligen Schüler
dem Verfasser.

Dedicated to the
most worthy, learned and noble Gentleman

Herr

Stanislaus Wydra[a]

Director and Professor of Mathematics,
Emeritus Rector Magnificus, Canon of
All Saints, etc., etc.

as Proof
of an unbounded Respect and Gratitude

by his former Pupil,
the Author

[a] Stanislaus Wydra (1741–1804) was a Czech Professor of Mathematics at Prague University from 1772 until 1803 when he became blind. He published works (in Latin) on differential and integral calculus and a history of mathematics in Bohemia and Moravia. He also published a work on arithmetic in Czech.

Preface

It is well known that in addition to the widespread usefulness provided by its *application* to practical life, mathematics also offers a second use which, while not so obvious, is no less beneficial. This is the use which derives from the exercise and sharpening of the mind, from the beneficial promotion of a *thorough way of thinking*. It is this use that is chiefly intended when the state requires every student to study this science [*Wissenschaft*].[b] As I could no longer restrain the ambition to contribute something to the constant progress of this splendid science, in my spare time—and following my personal preferences—I have been considering, on the whole, only the improvement of theoretical [*spekulative*] mathematics, i.e. mathematics in so far as it will bring about the second benefit mentioned above.

It is necessary here to mention some of the rules which, among others, in my opinion apply to this matter.

Firstly, I propose for myself the rule that the *obviousness of a proposition* does not free me from the obligation to continue searching for a proof of it, at least until I clearly realize that absolutely no proof could ever be required, and why. If it is true that ideas are easier to grasp when they are everywhere clear, correct and connected in the most perfect order than when they are to some extent confused and incorrect, then we must regard the effort involved in tracing back all truths of mathematics to their ultimate foundations, and thereby endowing all concepts of this science with the greatest possible clarity, correctness and order, as an effort which will not only promote the *thoroughness* of education but will also make it *easier*. Furthermore, if it is true that if the first ideas are clearly and correctly grasped then much more can be deduced from them than if they remain confused, then this effort can be credited with a *third* possible use—the *enlargement* of the science. The whole of mathematics offers the clearest examples of this. At one time something might have seemed superfluous, as when Thales (or whoever discovered the first geometric proofs) took much trouble to prove that the angles at the base of an isosceles triangle are equal, for this is obvious to common sense. But Thales did not doubt *that* it was so, he only wanted to know *why* the mind makes this necessary judgement. And notice, by drawing out the elements of a hidden argument and making us clearly aware of them, he thereby obtained the key to new truths which were not so clear to common sense. The application is easy.

[b] On the translation of *Wissenschaft* see the remarks in the *Note on the Translations*.

Secondly, I must point out that I believed I could never be satisfied with a completely strict proof *if it were not derived from the same concepts* which the *thesis* to be proved contained, but rather made use of some fortuitous, alien, *intermediate concept* [*Mittelbegriff*], which is always an erroneous μεταβασις ἐις ἀλλο γενος.[c] In this respect I considered it an error in geometry that all propositions about angles and ratios [*Verhältnissen*] of straight lines to one another (in triangles) are proved by means of *considerations of the plane* for which there is no cause in the *theses* to be proved. I also include here the concept of *motion* which some mathematicians have used to prove purely geometrical truths. Even *Kästner* is one of these mathematicians (e.g. *Geometrie, II. Thl., Grundsatz von der Ebne*).[d] *Nicolaus Mercator*, who tried to introduce a particularly systematic geometry, included the concept of motion in it as essential. Finally, even *Kant* claimed that motion as the *describing* of a space belonged to geometry. His distinction (*Kritik der reinen Vernunft*, S. 155)[e] in no way removes my doubt about the necessity, or even merely the admissability of this concept in pure geometry, for the following reasons.

Firstly, I at least cannot see how the idea of motion is to be possible without the idea of a *movable object* in space (albeit only imagined) which is to be distinguished from space itself. Because, in order to obtain the idea of motion we must imagine not only infinitely many *equal* spaces next to one another, but we must assume *one and the same thing* being successively in different spaces [*Räume*] as its *locations* [*Orte*]. Now if even *Kant* regards the concept of an object as an *empirical* concept, or if it is admitted that the concept of a thing *distinguished* from space is alien to a science which deals *merely* with space itself, then the concept of motion should not be allowed in geometry.

On the other hand, I think the theory of motion already presupposes that of space, i.e. if we had to prove the *possibility* of a certain motion which had been assumed for the sake of a geometrical theorem, then we would have to have recourse

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[c] *Translation*: crossing to another kind. Bolzano uses this phrase in *BD* II §29 and in *RB Preface* (see p. 126 and p. 254 respectively of this volume). It is a near quotation of a phrase used by Aristotle in the *Posterior Analytics* at 75^a38. Bolzano has ἐις (to), where the text in Barnes has εξ (from). The complete sentence reads: ‘One cannot, therefore, prove by crossing from another kind—e.g. something geometrical by arithmetic.’ (Aristotle–Barnes, 1975, p. 13).

[d] The axiom of the plane [*Grundsatz von der Ebne*] referred to here is as follows. ‘A straight line, of which two points are in a plane, lies completely within this plane (Theorem 1, Definition 7, and Axiom 1). But since the plane, in which this straight line is, can rotate around it as an axis, three points determine the position of a plane, and therefore every plane angle, and every triangle is in a plane.’ Kästner (1792), I (iv), p. 350.

[e] The distinction referred to occurs in a footnote on p. 155 of the second German edition (Kant, 1787). ‘Motion of an object in space does not belong to pure science and consequently not to geometry. For the fact that something is movable cannot be known *a priori*, but only through experience. Motion, however, considered as the describing of a space, is a pure act of the successive synthesis of the manifold in outer intuition, in general by means of the productive imagination, and belongs not only to geometry, but even to transcendental philosophy.’ (Kant, 1929, p. 167)

to precisely this geometrical proposition. An example is the above-mentioned axiom of the plane (due to *Kästner*). Now because the assumption of any motion presupposes for the proof of its possibility (which one has a duty to give), particular theorems of space, there must be a science of the latter which precedes all concepts of the former. This is now called pure geometry.

In favour of my opinion there is also *Schultz* who, in his highly-regarded *Anfangsgründe der reinen Mathesis*, Königsberg, 1790, did not assume any idea of motion.[f]

In the present pages I am not providing any *œuvre achevé* but only a small sample of my investigations to date, which concerns only the *very first propositions of pure geometry*.

If the reception of this work is not wholly unfavourable then a second might follow it shortly on the first principles of mechanics.[g] I would especially like to have the judgement of those well-informed about contemporary geometrical ideas. That is the reason, as a more specific motive, that I have chosen to put something into print on this difficult material straight away rather than on another subject (as would certainly have been possible). Now something more about this material.

It is obvious that for a proper theory [*richtige Theorie*] of the straight line—I am thinking of the proofs of propositions such as: the possibility of a straight line, its determination by two points, the possibility of being infinitely extended, and some others—no considerations of *triangles or planes* can be used. On the contrary, the latter theory [*Lehren*] must only be based on the former. So I have set out in the *first part* an attempt to prove the *first propositions of the theory of triangles and parallels* only on the assumption of *the theory of the straight line*. As far as I am aware this has not been done before, because in all other places various *axioms of the plane* have been assumed, axioms which, if they had to be proved,[h] would require precisely that theory of triangles. Therefore in my view the first theorems of geometry have been proved only *per petitionem principii*;[i] and even if this were not so, a *probatio per aliena et remota*[j] has still been given which (as already mentioned) is absolutely not permissible.

[f] Johann Schultz (1739–1805) was a Professor of Mathematics at Königsberg and a friend of Kant. The work Schultz (1790) expresses several of the methodological principles, which were to be espoused by Bolzano, but it does not contain Bolzano's repudiation of the plane for the theory of the straight line and triangles.

[g] This almost certainly refers to a paper on the composition of forces not published until 1842. This is the work *ZK* listed in the Selected works of Bernard Bolzano on p. 679. Bolzano explains in the *Preface* of that paper that he had been working on the material it contains forty years earlier.

[h] It was common at this time to regard an axiom simply as a self-evident statement. Bolzano did not think this was generally true and often sought for other kinds of justification. See, for example, *BD* II §21.

[i] *Translation:* by 'begging the question'.

[j] *Translation:* proof by alien and remote [ideas].

I regard the theory of the straight line itself, although *provable* independently of the theory of triangles and planes, yet still so little *proved*, that in my view it is at present the most difficult matter in geometry. In the *second part* I shall present extracts from my own considerations on the matter which seem to me to be the most fundamental, although they still do not reach the foundation. I only do this to find out whether I should continue on this path that I have taken.

I Attempt to Prove the First Theorems Concerning Triangles and Parallel Lines Assuming the Theory of the Straight Line

§ 1

Definition. *Angle* is that predicate of two straight lines *ca*, *cb* (Fig. 1), having one of their extreme points *c* in common, which is shared by every other system of two lines $c\alpha$, $c\beta$, which are *parts* of the former with the same initial point *c*. *c* is called the *vertex* of the angle, and the lines *ca*, *cb*, in so far as their length is disregarded so that the lines $c\alpha$, $c\beta$ could be taken instead, are called its *arms* [*Schenkel*].[k]

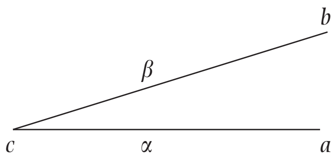

Fig. 1.

§ 2

Note. The phrasing which makes this a long definition is caused by the usage by which the angle *acb* is called *equal* to the angle $\alpha c\beta$, and accordingly *angle* is really a property of two *directions* (as I define the word in Part II[l]) and not two *lines*.[m] Others say: 'the angle is independent *of the magnitude* [*Größe*] of the arms,' which must, however, be understood with the qualification that the magnitude of an arm is never *negative*. Moreover, with some thought it becomes clear that the expression I have chosen is complete. For example, in order to prove that the angle $acb = \alpha c\beta$ (Fig. 2) we conclude directly *ex definitione*, $acb = ac\beta$, $ac\beta = \alpha c\beta$, therefore $acb = \alpha c\beta$.

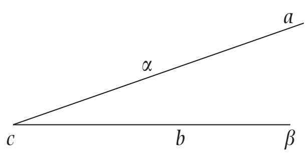

Fig. 2.

[k] *Schenkel* is consistently used for the lines of indeterminate length bounding an angle, in contrast to *Seite* (side) for the determinate lines bounding a figure such as a triangle. See, for example, the proof in I §12. *Schenkel* is literally 'thigh' or 'leg', and although some authors have used the phrase 'the legs of an angle', it is perhaps in deference to a Victorian sense of propriety that 'arm' has prevailed in English as the most common term for this purpose and has been adopted here.

[l] See II §6.

[m] In II §12 angle is simply identified with 'the system of two directions proceeding from a point'. Compare Hilbert (1971), p. 11: 'Let α be a plane and *h*, *k* any two distinct rays emanating from *O* in α and lying on two distinct lines. The pair of rays *h*, *k* is called an *angle* . . .'.

§ 3

Theorem. Every angle determines [*bestimmt*] its adjacent angle.

Proof. From the definition (§1) an angle is determined if its arms are determined. Now the arms of the given angle determine at the same time the arms of the adjacent angle. For these are: one, an arm of the given angle itself, the other, an extension of the other arm of the given angle beyond its vertex. Now we know from the theory of the straight line that this extension (viewed apart from its length in the above sense (§1)) is determined [*gegeben*].[n]

§ 4

Corollary. Therefore if two angles are equal, their adjacent angles are equal. For *things which are determined in the same way are equal*.[o]

§ 5

Theorem. Vertically opposite angles[p] are equal, $ac\beta = bc\alpha$ (Fig. 3).

Proof. Their determining pieces are equal. The angle $ac\beta$ is an adjacent angle of acb, the angle $bc\alpha$ (in that order) is an adjacent angle of bca. Therefore, if $acb = bca$,[q] then also $ac\beta = bc\alpha$ (§§ 3, 4).

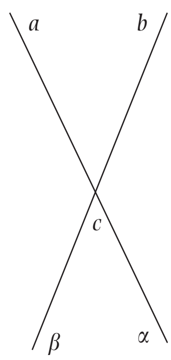

Fig. 3.

[n] Perhaps Bolzano has in mind II §15 where the concept of 'opposite direction' is introduced.

[o] The words 'same' and 'equal' here both translate *gleich*.

[p] The German *Scheitelwinkel*, is literally 'vertex angles'.

[q] In II §14 Bolzano says he still has 'no satisfactory proof' of this.

§ 6

Note. This is no different from the ordinary way of demonstrating the equality of two things: we conclude *ex datis* that their determining pieces are equal (§4). The Euclidean proof of the present theorem does not follow this method. But it has, in my view, two further *defects*. Firstly, the alien consideration of a *plane* has already been introduced here; for angles are added, which is only possible on the condition (albeit tacit) that the angles are in the same plane. Secondly, it assumes that angles are *quantities*, and on this assumption they are added and subtracted subject to the purely arithmetic axiom: 'equals taken from equals gives equal remainders'. A thing is called a *quantity* [*Größe*] in so far as it is regarded as consisting of a *number* [*Anzahl*] (plurality [*Vielheit*]) of things which are equal to the *unit* (or the measure). Therefore if I were to consider an angle as a *quantity* I must as a consequence imagine it as composed of several individual equal angles in one plane, which—it can be said explicitly or not—is really nothing but the idea of the area contained within the arms. So Schultz would be justified when he considers this infinite area as an essential property of angle. The author of *Bemerkungen über die Theorien der Parallelen des H. Hofpr. Schultz etc.* (Libau 1796) opposes this assumption of Schultz at length, yet does no better because he still considers angles as quantities and even brings in the concept of motion in defining (S. 55) angle as the concept of the ratio of the uniform motion of a straight line about one of its points to a complete rotation. But he thereby shows us clearly the true origin of all ideas of angles as quantities, which in my opinion is nothing but the empirical concept of motion. Now it is obvious that I can think of two lines with a common endpoint, therefore an *angle* (§1), without *having* to think of a surface, or of other lines drawn between them (component angles), or of a motion by which one of these lines comes from the other's position into its own position. Consequently, the angle in its essence is not a quantity. This was something the thorough [scholar] *Tacquet* surely already realised (*Elementa Geometriae I, Prop.* 3, *Schol.* 16). I am only surprised that he explains the usual, and contrary, kind of presentation as merely an abbreviation which, though improper, is a harmless *way of speaking* and may therefore be retained. If angle is a mere *quality* then one can only speak of *equality* or *inequality* of angles (or as *Tacquet* would have it: similarity or dissimilarity), but not of their being greater or smaller.[r] These denote two special *kinds* of inequality, which are really only valid for *quantities*, or at least one must first be agreed on their meaning. I shall therefore nowhere treat angles as *quantities*, and I shall reject as unusable all proofs in Euclid in which they are so regarded.[s] Nevertheless this makes no difference to the whole algebraic part of

[r] Tacquet does indeed begin his work as Bolzano describes but soon systematically adopts a quantitative treatment which is hard to explain as merely a 'way of speaking'.

[s] In other words, Bolzano is rejecting the Euclidean development of elementary (plane) geometry almost entirely.

geometry because here (as is well known) what we have in mind are *arcs* and not angles.

§ 7

Definition. Let us suppose the two points *a*, *b* in the arms *ca*, *cb* of an angle (Fig. 4) are different from the vertex *c*, and that through them there is a straight line *ab*, then the system of straight lines *ca*, *cb*, *ab* is called a *triangle*.

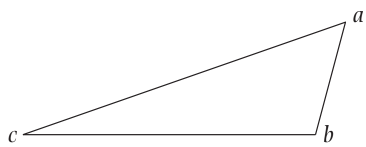

Fig. 4.

§ 8

Note. There is therefore no mention of any area.

§ 9

Corollary. In every triangle there are three angles. Each of these is *included* by two sides (i.e. it has them as arms) and stands *opposite* the third (i.e. it does not have it as an arm). Every side *lies on* two angles (i.e. it supplies one arm of the two angles).

These are in fact proper theorems,[t] but they are so easy to prove that I may save space by stating them in this way.

§ 10

Theorem. In two equal triangles, I. the *sides* are equal which stand across two equal sides, or which are opposite an equal angle, or which lie on two equal angles; II. the *angles* are equal which stand across two equal angles, or which are included by two equal sides, or which are opposite an equal side.

Proof. These sides (I), and angles (II), are determined in their triangles by the data [*Angabe*] given. For it follows from §9 that there is only one side or only one angle which belongs to these data. Now since the triangles themselves and their data are equal, the determining pieces of these sides and angles are equal.

[t] Bolzano distinguishes between theorems and corollaries in *BD* II §§24, 25 on pp. 120–121.

§ 11

Note. I call triangles which are otherwise called *equal and similar* simply *equal*.[u] According to usage the word *equal* says more than the word *similar*, so that if two objects are called equal they must already be *similar*. But *one property* of these objects (which does not determine them), e.g. the magnitude of two areas, can be equal without the objects, the areas themselves, being equal. This one property should not be given the name of the object itself; therefore one should not say, 'two triangles are equal' if actually one only wants to say that the magnitudes of their areas are equal. If one avoids this rather unmathematical metonymy then the addition of *similar* to the word *equal* is superfluous. But if some people want the word *equal* to be used of nothing but the property of *quantity*, then I ask them for another word which could be used generally to denote this concept? This word is not *identity*, for one and the same thing is only called identical in so far as it is compared with itself.

§ 12

Theorem. Two sides and the angle included by them determine[v] the triangle to which they belong.

Proof. From the definition (§1) it follows directly that the angle and the definite lengths of the pieces *ca*, *cb* of its arms together contain all predicates of the system of two lines *ca*, *cb*. For the angle alone contains that which is independent of the definite lengths of the lines *ca*, *cb*; therefore if this is included everything in the system is determined. Now we understand by sides [*Seiten*] (of a triangle)—in contrast to arms—determinate lines. Therefore, everything in the system of two lines *ca*, *cb* is determined; consequently also the points *a*, *b* and the straight line *ab* which is drawn through them (§7) are determined, as well as the angles which the line *ab* forms with the two other sides.

§ 13

Note. From this proposition two corresponding theorems will now follow, one about the equality, and the other about the similarity, of triangles.

§ 14

Theorem. Two triangles in which two sides and the included angle are equal, are themselves equal.

[u] According to Vojtěch (1948), p. 190, Note 14 the phrase *gleich und ähnlich* (equal and similar) was used up to the end of the eighteenth century in German writings, and 'congruere' in Latin writings, to describe structures with equal boundaries. The word *congruiren* was used in German from around the end of the eighteenth century. Bolzano objects to it as empirical and superfluous (see I §49).

[v] Each occurrence of the words 'determine', 'definite', 'determined', and 'determinate' in this paragraph translates forms of the German *bestimmen*.

Proof. For their determining pieces are equal (§12).

§ 15

Corollary. This gives rise through mere negation of the conclusion (*conclusio hypothetica in modo tollente*)[w] to several propositions. For example, if two sides are equal but the third side is unequal, then the included angle must also be unequal. And so on.

§ 16

Definition. Two spatial objects are called *similar* if *all the characteristics* which arise from the comparison of the parts *of each one among themselves*, are *equal* in both; or if through every possible comparison of the parts of each one among themselves, no *unequal* characteristics can be perceived.

§ 17

Theorem. Objects whose determining pieces are similar are themselves similar.

Proof. Suppose they are not similar, then by making comparisons among the parts of one of them, an unequal characteristic must be perceived (i.e. one which is not present in the other). This inequality requires a basis [*Erkenntnisgrund*] in the objects themselves, and so in their determining pieces (for from these everything which is in the object itself must be perceived). There would therefore have to exist a difference in the determining pieces recognizable from a comparison among themselves, consequently they would not be similar (§16).

§ 18

Note. This proposition lies at the basis of the theorems of similarity, in the same way as the proposition 'objects whose determining pieces are equal, are themselves equal' (§4) lies at the basis of the theorems of equality (§6).

§ 19

Axiom. There is no special idea given to us *a priori* of any *determinate distance* (or absolute length of a line), i.e. of a determinate kind of separation of two points.[x]

[w] This medieval term occurs several times in this work and refers to an argument of the form: If *A* then *B*, but not-*B*; therefore not-*A*. The exact relationship of this principle to *reductio ad absurdum* (which is not mentioned her by Bolzano) is subtle, see Coburn and Miller (1977).

[x] It is pointed out in Gray (1989), p. 72 that Lambert had noticed the asymmetry between length and angle in that the latter has a natural absolute value of one revolution while the former has no such corresponding value (in Euclidean geometry). It appears that Lambert would have been aware, as Bolzano evidently was not, that the axiom of this section is equivalent to the parallel postulate. It is discussed further in I §24, but its origin in Bolzano's thinking seems to be a complicated matter.

§ 20

Theorem. All straight lines are similar.

Proof. Straight lines are determined by their two end-points. Now we have (§19) no special idea of any determinate separation of two points. Therefore, every separation of two points is similar to every other one. So too, all straight lines are themselves similar (§17).

§ 21

Theorem. Two triangles in which two sides enclosing an equal angle are in proportion, are themselves similar.

Proof. The determining pieces of these triangles are similar. These are (§12) an angle with the sides which include it, or (because the ratio of one line to another determines the former from the latter), an angle, a side and the ratio of the other side to the first. Now the angle and the ratio in both triangles are *equal* (and consequently also similar), but one side is similar, therefore the determining pieces are similar.

§ 22

Theorem. Similar angles are equal.

Proof. The word *angle* designates (§1) that which determines everything perceivable in the system of two directions *am*, *an* (Fig. 5). This is the distance *mn*, which every two points *m*, *n* have from one another, where *m*, *n* are determined by arbitrary distances *am*, *an* in the two directions; and the distance *pr*, which every point *p* in the one direction *am* has from a point *r* in *mn*; etc. If then in the two given angles all these perceivable pieces are *equal*, this means nothing other than that the angles themselves are equal. Now let *a* and α be *similar* angles (Fig. 6), then if $am : an = \alpha\mu : \alpha\nu$ we must also find (§16) $am : mn = \alpha\mu : \mu\nu$, otherwise the comparison of the parts of the angle α among themselves would not give rise to exactly the same idea as that of the parts of the angle *a*. Now consider αo (in direction $\alpha\mu$) = *am* and αu (in direction $\alpha\nu$) = *an*; then $\alpha\mu : \alpha\nu = \alpha o : \alpha u$. Therefore,

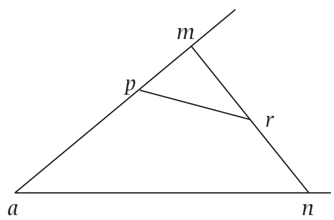

Fig. 5.

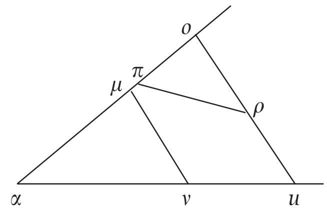

Fig. 6.

(if *ou* is drawn) (§21) $\triangle o\alpha\mu \backsim \triangle\mu\alpha\nu$,[y] whence (§16)

$$ou = \frac{\mu\nu.\alpha o}{a\mu} = \frac{mn.\alpha o}{am} = mn.$$

In the same way it can be shown that if π, ρ are taken so that $o\pi = mp$, $o\rho = mr$, then $\pi\rho = pr$. And so on. So therefore the condition mentioned above, of the equality of all characteristics, holds for angles a and α; accordingly these angles are equal.

§ 23

Corollary. Therefore in similar triangles, the angles opposite proportional sides are equal (§§ 16, 10, 22).

§ 24

Note. This theory of similarity, like its subsequent application, is a result of my own reflection, although *Wolff* has already put forward the same theory in detail in his *Philosophia prima seu Ontologia, Sect. III., Cap. I, de Identitate et Similitudine*, and also in the *Elementis Matheseos universae*, and thus it has been known to the academic world for a long time.[z] I myself have briefly read through the first work only recently, but I read the other several years ago, not in order to learn from them about the elementary theory but only with a view to finding in them some unknown problem. In doing this I skimmed the *Arithmetic* and *Geometry* so carelessly that I was not aware of this important change which is

[y] Such a symbol for similarity, (or else the symbol $\sim$ as used in *BG*(2)) seems to have been first used by Leibniz, and then by Wolff. See Cajori (1929), i, §372 for further details.

[z] The concept of similarity introduced by Wolff in these works is very general but vague. For example, in the section of the *Philosophia* to which he refers, at §195, appears the definition ‘Those things are similar in which the things by which they ought to be distinguished are the same . . . Similitude is the sameness [*identitas*] of those things by which entities ought to be distinguished from each other.’ What Bolzano’s theory of similarity has in common with that of Wolff is the central role played by the idea that a mathematical object may be ‘determined’ in a certain way by its component pieces. Both the pieces and the manner of being determined are important.

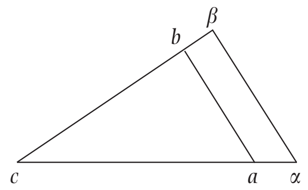

Fig. 7.

introduced in small scholia immediately after the definitions. The first form which I gave to my proof of the proposition §21, before I had read *Wolff's Ontologia*, is briefly as follows: we have no *a priori* idea of any determinate separation of two points, or more generally of any determinate spatial object. If therefore, an *a priori* knowledge of spatial objects is to be possible it must be valid for every unit of measurement adopted. For example, if in the $\triangle acb$ (Fig. 7), $cb = n.ca$, $ab = m.ca$, and in $\triangle\alpha c\beta$ with equal angles in the same way, $c\beta = n.c\alpha$, then $\alpha\beta$ must $= m.c\alpha$ because otherwise we would have to have had an *a priori* idea of the determinate line *ca* for which only the number *m* is valid. My intention on the discovery of this proof was to complete the well-known gap in the theory of parallels by means of the theory of the similarity of triangles. In fact, even if one is not completely satisfied with *Wolff's* proof (or mine) of the theory of the similarity of triangles, it still seems to me that the effort of proving this theory (from a basis independent of parallel lines and considerations of the plane) deserves more attention from geometers.[a] *Kant* has noted (*Von dem ersten Grunde des Unterschiedes der Gegenden im Raume*, 1768, to be found in the collection of some of his writings by *Rink*),[b] and he has repeated these thoughts elsewhere (*Prolegomena* p. 57ff),[c] that there are *differences* in spatial objects (therefore also the properties on which they are based) which cannot be perceived from any comparison among the parts of each one. For there are spatial objects which are completely equal and similar to each other and yet cannot be brought into the same space: therefore they must possess a difference. Such for example are two equal spherical triangles on opposite hemispheres. *Kant* called the basis of this difference the *direction*[d] in which the parts of the one and the other spatial object lie. This Kantian observation is

[a] Bolzano's outline of his 'first form' of proof for I §21 attempts to derive the result only from his axiom in I §19. The need to be independent of 'parallel lines and considerations of the plane' follows from Bolzano's requirement for the correct ordering of concepts, and is also a reference, by way of contrast, to the methods adopted in Book VI of Euclid (1956).

[b] The work referred to is in Kant (1968), ii, pp. 375–84.

[c] The passage referred to is now most easily found in Kant (1968), x, §13, and in English in Kant (1953), §13.

[d] The German here is *Gegend* as in the title of the work by Kant referred to a few lines above in this subsection. Dictionaries from the eighteenth century (such as Wolff (1747) and even the general dictionary Adelung (1793–1801)) make clear that in addition to its meaning as 'region', *Gegend* was commonly used as a synonym for *Richtung* (direction), which is the meaning intended here.

indeed quite correct; however, it is not only the *direction* but also secondly *the determinate kind of separation* (*the distance*) which is a property which cannot be perceived by any comparison of the parts of an object with each other. Indeed, if in two objects all properties which can be observed from the comparison of the parts of each one among themselves are equal, then it only follows that the two things are *similar*. They can still be *unequal*. If two objects are to be recognized as *equal*, then the one must be compared with a part of the other, or generally both must be compared with one and the same third thing (a common unit of measurement).* The reason for this is that we have no *a priori* idea of any determinate separation of two points (distance); so there is nothing we can do but to note the ratios of different distances to one another. This is the truth which I set up as an axiom in §19. As a motivation either for the acceptance of this axiom or the invention of some other way of proving the theory of similarity I may be permitted to recall the following. If a correct methodology requires of every systematic proof that it demonstrates the connection of the subject with the predicate, without the interference of fortuitous intermediate concepts, then our previous proofs of all theorems of similarity cannot stand up to any criticism. Let any expert cast a glance at our textbooks (at *Euclid*) in this respect and I hope to be cleared of any suspicion of slander. There therefore remains the obligation to look for error-free proofs for these theorems. These proofs would have to demonstrate—among other requirements just mentioned here—what is true of the *genus* without using an induction from the individual *species*. 'Volumes of similar solids vary as the cubes of similar sides, or more generally, *as* any other *solid* determined from them in a similar way. Surfaces vary as surfaces; lines (curved) vary as lines.' Where could Euclidean geometry demonstrate these propositions in this generality without resorting to the consideration of individual kinds (such as triangles)? But from §17 these propositions follow in complete generality and quite directly. For let A, a be two similar solids and let the solid B be determined from A in the same way as b is determined from a; consequently B, b are also similar and it is required to prove $A : B = a : b$. The system of solids A and B has the determining pieces: the solid A and the way that B comes from A. These determining pieces are *ex hypothesi* similar to the determining pieces in the system a and b. Therefore (§17) both systems are similar. Consequently everything which can be observed in the one system by the comparison of its parts is also equal in the other. Therefore if the volumes of the solids, A, B and a, b are compared, then it must be that $A : B = a : b$. The same proof can be applied to surfaces and lines. Finally, I may remark that I also use exactly this axiom (§19) in proving the first essential theorems in mechanics and that I believe that it can be usefully applied in all areas of mathematics (except arithmetic and

* *Kant* could therefore have cited in his paper not only the concept of direction but also that of separation as counter-examples to those philosophers who regard space as a pure relationship of coexisting objects.

algebra—because they have no particular thing for their object, but abstract plurality itself).

§ 25

Theorem. In an isosceles triangle the angles on the base are equal: $a = b$ (Fig. 8).

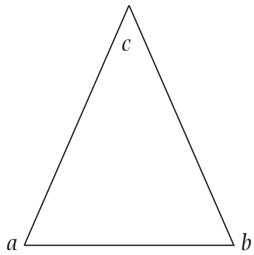

Fig. 8.

Proof. They are determined in the same way. From §14 it follows that $\triangle acb = \triangle bca$ (in the order of the letters). For ca in $\triangle acb = cb$ in $\triangle bca$; cb in $\triangle acb = ca$ in $\triangle bca$; $\angle acb$ in $\triangle acb = \angle bca$ in $\triangle bca$. Consequently (§10) $\angle b$ opposite ac in $\triangle acb = \angle a$ opposite bc in $\triangle bca$.

§ 26

Theorem. It is possible to erect from one point of a straight line another straight line in such a way that the two adjacent angles made with the segments of the former are *equal*.

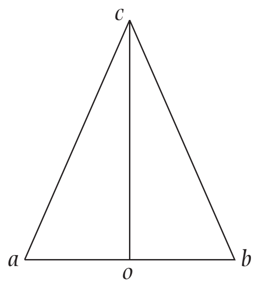

Fig. 9.

Proof. It is possible to think of two equal straight lines $ca = cb$ (Fig. 9) meeting at some angle c. Now if we draw ab and suppose o to be the *centre* [*Mitte*] of ab

(a concept defined in the theory of the straight line),[e] and finally we draw co, then (§25) $a = b$ and *per constructionem* $ac = bc$, $ao = bo$. Therefore (§14) $\triangle cao = \triangle cbo$. Hence (§10) $\angle coa = \angle cob$.

§ 27

Note. Euclid presents this proposition in the form of a *problem*. Now it is well known that theoretical geometry (e.g. that of *Euclid*) intends, by means of its problems, only to show the *possibility* of this or that spatial object. In contrast, the aim of giving a method whereby various spatial objects can be empirically constructed using a few simple instruments (e.g. straight edge and compass), is a *practical* aim. So the problems of *theoretical* geometry are really *theorems*, which is therefore the appropriate form for them. On the other hand, the practical part of geometry could contain the problems separately; and this opinion was also that of the Jesuit, *I. Gaston Pardies*. For this reason, the theoretician must also be allowed (and this is more important) to assume certain spatial objects without explaining the method of their actual construction, provided he has proved their *possibility*. With this in mind, I assumed the centre of ab in the present proposition without showing how it is to be found. And in what follows I shall assume the fourth proportional line for three given lines without showing how it would be constructed: it suffices that it is clear directly from §20 that we can think of a certain line d which has exactly the same ratio to the line c as that which the line b has to a.

§ 28

Theorem. It is possible to construct from one point of a straight line another straight line such that the adjacent angles formed are unequal.

Proof. As in §26, just let ca, cb be unequal, and instead of §14 apply §15.

§ 29

Corollary. From §§ 26, 28 there now follows the possibility of drawing from *any* point of any straight line another straight line so that the adjacent angles formed are either equal (§26) or unequal (§28).

Because any straight line, through being shortened or lengthened can become equal to ab (§§ 26, 28), and the given point in it can become its mid-point o, then what is possible for ab (§§ 26, 28) must be possible for any straight line and point.

[e] See II §30.

§ 30

Theorem. Every system of a straight line produced indefinitely in both directions [*zu beiden Seiten*] and a point outside it, is similar[f] to every other such system (Fig. 10): $o, xy \backsim \omega, \xi\eta$.

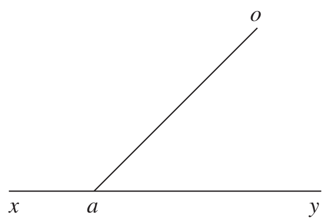

Fig. 10.

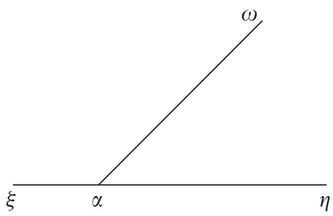

Fig. 10*.

Proof. For no difference can be observed in the two systems from the comparison of their parts (§16). Since the straight lines xy, $\xi\eta$ are produced indefinitely in both directions no point on these lines can be determined by the position which it has on them. Now since the points o, ω lie outside these straight lines, then it is essential that every line drawn from o, ω to a point a, α of the infinite line forms adjacent angles with the latter. The angles so formed in both systems can now either be unequal or equal. If they are *unequal*, I cannot conclude from this any difference in the two systems because the points a, α, on which these angles depend, are indeterminate. But if these angles are equal, $oax = \omega\alpha\xi$, $oay = \omega\alpha\eta$, then because the arms ax, ay; $\alpha\xi$, $\alpha\eta$ are indeterminate there can be no comparison of these with the lines oa, $\omega\alpha$. But in themselves these lines are similar, therefore o, xy; ω, $\xi\eta$ are similar systems in which oa, $\omega\alpha$ are similarly situated lines.

§ 31

Theorem. From any point o (Fig. 10) outside a straight line xy, produced indefinitely in both directions, it is possible to draw another straight line so that one angle which it forms with the line xy is *equal* to some given angle $\omega\alpha\xi$.

Proof. Consider a point ω in one of the arms of the given angle $\omega\alpha\xi$, then this is a point *outside* the other arm $\alpha\xi$. Now if the latter is produced indefinitely in both directions then one has a system ω, $\xi\eta$ of a point and an indeterminate

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[f] The 1804 text has *gleich* (equal) here, which is clearly an error for *ähnlich* (similar). This is pointed out in *BG*(2).

straight line outside it, which is therefore *similar* (§30) to the given system o, xy. Consequently, because in the former system a line can be drawn from ω to $\xi\eta$ so that it forms the angle $\omega\alpha\xi$, then in the given system it must also be possible to draw a line from o to xy which forms an angle $oax = \omega\alpha\xi$.

§ 32

Theorem. From every point o (Fig. 11) outside a straight line xy, one and only one straight line can be drawn to the latter so that it forms *equal* adjacent angles on it.

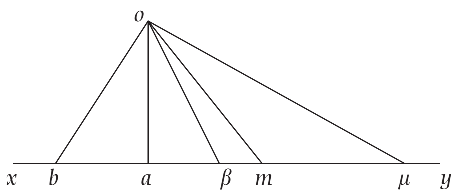

Fig. 11.

Proof. That one line can be drawn follows from §26 together with §31. Therefore let $oax = oay$. Likewise it follows from §28 together with §31 that a line om can be drawn from o to xy which forms *unequal* adjacent angles, omx not $= omy$. Now suppose there were another line ob which made $obx = oby$, then one may take $a\beta = ab$ (in the opposite direction to ab) and draw $o\beta$. Therefore (§14) $\triangle oab = \triangle oa\beta$, and (§10) $ob = o\beta$, $\angle oba = \angle o\beta a$; consequently (§4) $\angle obx = \angle o\beta y$. But *ex hypothesi* $\angle obx = \angle oba$, therefore $\angle oba = \angle o\beta y$. If one now assumes $\beta\mu = bm$ and draws $o\mu$, then (§14) $\triangle obm = \triangle o\beta\mu$. Therefore (§10) $om = o\mu$; $\angle omb = \angle o\mu\beta$. In $\triangle mo\mu$ (§25) $\angle om\mu = \angle o\mu m = \angle o\mu\beta$. Consequently $\angle omb = \angle om\mu$, i.e. $\angle omx = \angle omy$, *contra hypothesim*.

§ 33

Corollary. Therefore, the point o determines the line oa with the property of forming equal adjacent angles with xy. Consequently it also determines the nature of the angles oax, oay themselves.

§ 34

Theorem. All angles which are equal to their adjacent angles are also equal to each other.

Proof. In Fig. 12 let $\angle oax = \angle oay$, $\angle\omega\alpha\xi = \angle\omega\alpha\eta$; and if $\angle oax$ is not $= \angle\omega\alpha\xi$, then a line could be drawn from o which forms with xy an angle $= \omega\alpha\xi$ (§31). This forms equal adjacent angles (§4), therefore it cannot be different from oa (§32).

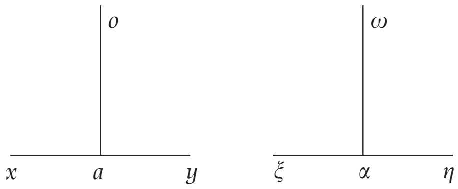

Fig. 12.

§ 35

Note. Since such angles are all equal to each other they may therefore be designated by the common name of *right angles*.

§ 36

Theorem. If from the point o (Fig. 13) oa is perpendicular to xy and a is the centre of mn, then I. the lines $om = on$, II. the angles $aom = aon$, III. the angles $amo = ano$.

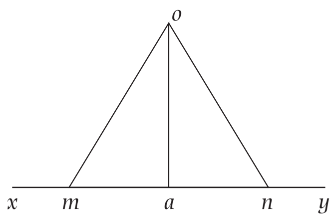

Fig. 13.

Proof. These follow directly from §14 and §10.

§ 37

Theorem. Conversely, if for the perpendicular oa either I. the lines $om = on$, or II. the angles $aom = aon$, or III. the angles $amo = ano$; then a is the centre of mn.

Proof. I. If (Fig. 14) $om = on$ and one takes p as the centre of mn, then it follows, as in §26, that op is perpendicular to mn, therefore (§32) p must be the same as [*einerlei*] a.[g] II. If (Fig. 15) $\angle aom = \angle aon$ and if one supposes $om : oa = on : o\alpha$ (the latter taken from o on oa) then (§21) $\triangle moa \backsim \triangle no\alpha$; consequently (§23) $\angle mao = \angle n\alpha o$, therefore = R.[h] Therefore α is the same as a (§32). Thus $oa = o\alpha$, therefore on account of the proportionality, also $om = on$ and (§10) $am = an$. III. If (Fig. 16)

[g] The word *einerlei* is translated by either 'identical' or 'the same as', but never by 'equal'. See II §1.

[h] It is not mentioned in Cajori (1929) but the letter R was in common use to denote a right angle, for example, in works by Schultz and Klügel.

$\angle amo = \angle ano$ and if one supposes $mo : ma = no : n\alpha$ (the latter taken from n on na) then (§21) $\triangle oma \backsim \triangle on\alpha$; consequently (§23) $\angle oam = \angle o\alpha n$ therefore = R. Therefore α is the same as a (§32). Thus, since (§21) $ma : ao = n\alpha : ao$, and $ao = \alpha o$, also $ma = n\alpha = na$.

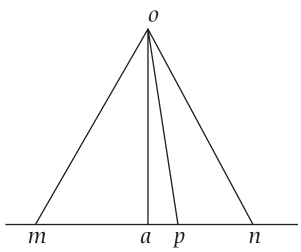

Fig. 14.

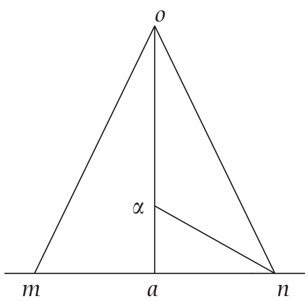

Fig. 15.

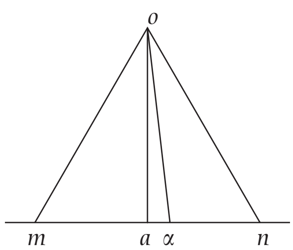

Fig. 16.

§ 38

Corollary. There are therefore no more than two lines om, on from o (Fig. 13) to xy for which either I. $om = on$, or II. $\angle aom = \angle aon$, or III. $\angle amo = \angle ano$.

For whichever of these occurs, $am = an$ (§37); now there are only two points in the straight line xy which are at an equal distance from the same point a.

§ 39

Theorem. From the same point o (Fig. 17) only one line oa can be drawn to the unbounded line xy so that it makes a given angle oax with the indefinitely extended part ax of this line.

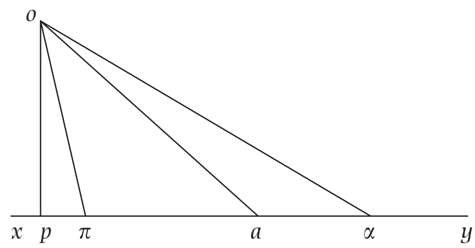

Fig. 17.

Proof. That one such line can be drawn follows from §31. That *only* one can be drawn is made clear as follows. Put $\angle o\alpha x = \angle oax$. Now consider the perpendicular op from o to xy. If p is at a or α then $\angle o\alpha x = \angle oax = \mathrm{R}$, therefore a, α are the same point (§32). If p is not at a or α, then consider $ao : ap = \alpha o : \alpha\pi$ (the latter taken in that arm of the two adjacent angles at α which is $= oap$ (§4)). Then (§21) $\triangle oap \backsim \triangle o\alpha\pi$. Therefore (§23) $\angle opa = \angle o\pi\alpha = \mathrm{R}$. Consequently, (§32) π is the same as p therefore $op = o\pi$ and by the ratio $op : pa = o\pi : \pi\alpha$ also $pa = \pi\alpha = p\alpha$. Therefore, (from the theory of the straight line) either p is the centre of $a\alpha$ or a is the same as α. The first alternative cannot hold because then (from the theory of the straight line) ap, αp would not be contained in one and the same indefinitely extended arm ax. Therefore the second alternative holds.

§ 40

Corollary. Conclusio in modo tollente.[i] If therefore two lines (Fig. 18) ac, bd form equal angles $cax = dbx$ with arms which are contained in the same indefinitely produced part of xy, then these lines at c, d cannot intersect anywhere. Likewise their extensions beyond a, b cannot intersect with one another, $a\gamma$ with $b\delta$. (But ac could indeed intersect $b\delta$ as in Fig. 18*.) If the angles $cax = dbx = \mathrm{R}$ (Fig. 18**) then $c\gamma$, $d\delta$ definitely cannot intersect; also ac cannot intersect with $b\delta$ towards c, δ because likewise $\angle cax = \angle\delta bx$ (§39).

[i] See the footnote about this on p. 40.

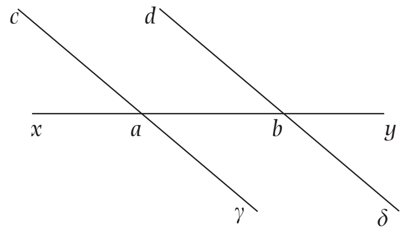

Fig. 18.

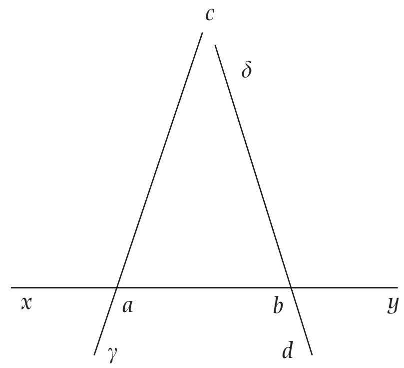

Fig. 18*.

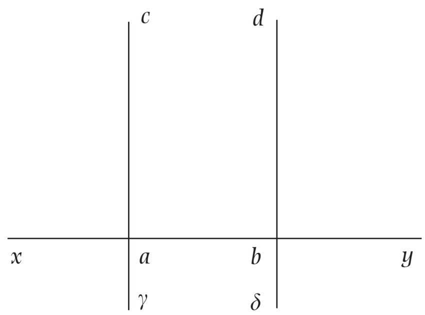

Fig. 18.**

§ 41

Theorem. One side and the two angles lying on it determine the triangle to which they belong.

Proof. If they did not determine it, there would have to be able to be another one. Therefore let (Fig. 19) *bac*, *bam* be two different triangles on the same angle *bac* and with the same side ab in which the second angle $abc = abm$. Suppose $ac : ab = am : an$ (the latter taken in ab from a) then (§21) $\triangle cab \backsim \triangle man$; consequently (§23) $\angle mna = \angle cba = (ex\ hyp.)\ \angle mba$. Now since an lies in ab, the lines na, ba, produced beyond a, contain the same infinite part of the line ab (from the theory of the straight line). Hence (§39) n must be the same as b and consequently, because $ac : ab = am : an$, $ac = am$, therefore (§14) $\triangle mab = \triangle cab$.

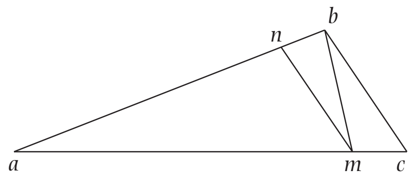

Fig. 19.

§ 42

Theorem. If in two triangles one side with the two angles lying on it are equal, then the two triangles themselves are equal.

Proof. Their determining pieces are equal (§41).

§ 43

Theorem. If in two triangles two angles are equal, then the two triangles themselves are similar.

Proof. Their determining pieces (§41): the side on which those angles lie (§20) and these angles themselves, are similar (§17).

§ 44

Theorem. In any triangle the sum of two sides is never equal to the third.

Proof. This sum is represented by extending ac beyond c so that $c\beta = cb$ (Fig. 20). Now if $a\beta = ab$ then (§25) it would follow in $\triangle\beta ab$ that $\angle\beta = \angle ab\beta$, and in $\triangle\beta cb$, that $\angle\beta = \angle cb\beta$. Therefore (§42) $\triangle c\beta b = \triangle a\beta b$. Consequently (§10) $a\beta = c\beta$, which is contradictory.

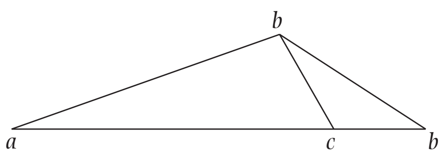

Fig. 20.

§ 45

Theorem. In a right-angled triangle, and only in such a triangle, the square (in the *arithmetic* sense) of the hypotenuse equals the sum of the squares of the other two sides[j] (Fig. 21): $ab^2 = ac^2 + bc^2$.

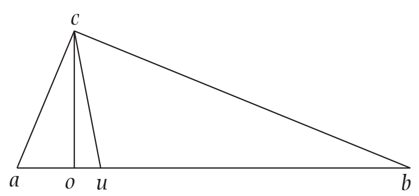

Fig. 21.

Proof. [k] I. To prove that (as *lines*)

$$ab = ac.\frac{ac}{ab} + bc.\frac{bc}{ab}$$

we construct lines of magnitude

$$ac.\frac{ac}{ab} \text{ and } bc.\frac{bc}{ab}.$$

Take $ab : ac = ac : ao$ (the latter in ab from a) and $ba : bc = bc : bu$ (the latter in ba from b) in order to obtain (§21) similar triangles, namely $\triangle bac \backsim \triangle cao$, $\triangle abc \backsim \triangle cbu$ (in the order of the letters). Consequently (§23) $\angle acb = \angle aoc$, $\angle bca = \angle buc$, and since $\angle c = \mathrm{R}$, the points o, u must be the same.[l] Now since *ex constructione* ao lies in the arm ab from a, $bu = bo$ in the arm ba from b, then o is in between a and b (from the theory of the straight line) and

$$ab = ao + ob = ac.\frac{ac}{ab} + bc.\frac{bc}{ab}.$$

II. If acb is *not* a right angle then o cannot be the same as u, otherwise aoc, buc would be equal adjacent angles; consequently it cannot also be that

$$ab = ao + bu = \frac{ac^2}{ab} + \frac{bc^2}{ab}.$$

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[j] The word 'sides' here is translating *Katheten*, which corresponds to the now obsolete term 'cathetus' meaning 'perpendicular' (*OED*).

[k] According to Simon (1906), p. 109, this is a 'completely original' proof of Pythagoras theorem. It is certainly a muddled proof with the theorem being explicitly arithmetic, and the proof being explicitly to do with lines. Bolzano begins to address the underlying problem in II §33 ff. For a recent useful discussion of this important issue as it arises in Euclid, see Grattan-Guinness (1996).

[l] By I §32.

§ 46

Theorem. Three sides determine the triangle to which they belong.

Proof. If I just prove that three sides determine an angle, then it follows (by §12) that they also determine the triangle itself. I can only *infer* the determination of an angle (which is not given) in a triangle from the previous paragraphs (§§ 12, 41) if either (§12) two sides with the included angle, or (§41) one side and the two angles lying on it, are given. Therefore one given *angle* is always required. Therefore I form an angle (Fig. 22) in the $\triangle acb$, whose sides are given to me. Now since I can calculate the sides in right-angled triangles (§45), I will form a *right* angle and because it must be in a *triangle*, I will form it by dropping a perpendicular from a vertex c of the triangle on to the opposite side ab, thereby producing two right-angled $\triangle\triangle$ adc, bdc which (§45) give the equations $b^2 = x^2 + y^2$, $a^2 = x^2 + (\pm c \mp y)^2 = x^2 + (c - y)^2$ (according to whether d lies inside or outside ab). From these equations, x and y can be determined unambiguously. But these are two sides of $\triangle cda$ whose included angle $cda = \mathrm{R}$ is given. Hence (§12) the angle a is determined and thereby also the $\triangle abc$.

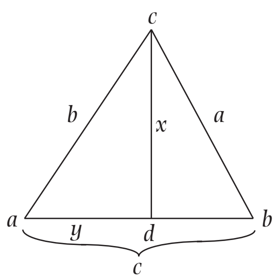

Fig. 22.

§ 47

Theorem. If in two triangles the three sides are equal then the triangles themselves are equal.

Proof. For their determining pieces are equal (§46).

§ 48

Theorem. If in two triangles the three sides are in proportion then the triangles themselves are similar.

Proof. For their determining pieces are similar (§§ 46, 17).

§ 49

Note. Why have I abandoned the usual proofs of the three propositions on the equality of triangles? I have not essentially altered the proof of the first theorem (§14) except for its presentation. As far as this *presentation* is concerned I have indeed omitted the concept of covering [*Decken*][m] which is usually used here and for several other theorems. I do not want to criticize here the inappropriateness of the choice of the German *word*, *Decken*, which easily misleads the beginner into thinking of a *lying-on-top-of-one-another* [*Übereinanderliegen*], and not of the identity of boundaries; instead of this it would be more appropriate to use 'coinciding' [*Ineinanderfallen*], or *congruence* [*Congruiren*] ($\sigma\upsilon\mu\pi\iota\pi\tau\varepsilon\iota\nu$).[n] But the concept of congruence is itself both empirical and superfluous. It is *empirical*: for if I say A is congruent to B, I imagine A as an *object* which I *distinguish* from the *space* which it occupies (likewise for B). It is *superfluous*: one uses the concept of covering to deduce the equality of two things if they are shown to cover each other in a certain position, according to the axiom 'spatial things which cover each other are *equal* to each other'. (In this way, one actually proves identity when only equality had to be shown.) Now one could never conclude that two things are congruent, i.e. that their boundaries are identical, until one had shown that all determining pieces are identical. But if this is proved one can also deduce without covering that these determining things themselves are identical. Therefore *Schultz* omitted the concept of covering throughout his *Anfangsgründe* without needing to alter much on this account. As for the proofs of the second and third theorems[o] these (even as more recent geometries have rearranged them) are completely based on theorems of the plane. Any expert will see this for himself. Therefore, I could not be satisfied with them because of the principles stated in the *Preface*. But do my own proofs indeed meet the requirement made in the *Preface*—to avoid *all fortuitous intermediate concepts*? I believe so. However, to avoid extensive detours in this small work I have not been able to give the detailed deduction of the necessity of every intermediate concept introduced. I therefore ask the learned reader to provide this by some thinking of his own.

§ 50

Theorem. If in the angle $x\,cy$ (Fig. 23), $ca : cb = cd : ce$ then ab, de never intersect, however far they are produced.

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[m] An example of the 'usual' use of the concept of covering to which Bolzano refers is the following axiom: 'Figures which cover one another are equal to one another, and figures which are equal and similar cover one another' (Wolff, 1717, §50).

[n] On the word *Congruiren* see the footnote to I §11 on p. 39. The main meanings of the Greek term are 'to fall together', 'to coincide', 'to agree' (*LSJ*).

[o] That is, I §42 and I §47.

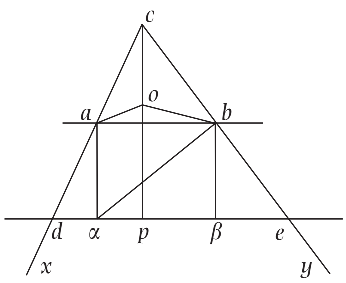

Fig. 23.

Proof. (§21) $\triangle acb \backsim \triangle dce$; therefore

$$ab = de.\frac{ac}{dc}.$$

Consider cp perpendicular to de and $cd : cp = ca : co$ (taken in cp from c); therefore (§21)

$$ao = dp.\frac{ca}{cd} \quad \text{and} \quad bo = ep.\frac{cb}{ce} = ep.\frac{ca}{cd}.$$

Hence

$$ao \pm ob = (dp \pm pe).\frac{ca}{cd} = de.\frac{ca}{cd},$$

therefore $ao \pm ob = ab$. From which (by §44 *in modo tollente*), it may be concluded that o is in the straight line ab. Also (§23) $\angle aoc = \angle dpc = \mathrm{R}$. Consequently, ab, de never intersect (§40).

§ 51

Theorem. Also (Fig. 23) the perpendiculars from a and b to de, $a\alpha = b\beta$, as also their distances $ab = \alpha\beta$. And the angles at a, b are also right angles.

Proof. I. By §43 $\triangle ad\alpha \backsim \triangle cdp$; $\triangle be\beta \backsim \triangle cep$. Consequently

$$a\alpha = cp.\frac{da}{dc} \quad \text{and} \quad b\beta = cp.\frac{eb}{ec} = cp.\frac{da}{dc}.$$

Therefore $a\alpha = b\beta$.

II. Furthermore

$$d\alpha = dp.\frac{da}{dc} \quad \text{and} \quad e\beta = ep.\frac{eb}{ec} = ep.\frac{da}{dc}.$$

Therefore

$$\begin{aligned}\alpha\beta &= de - d\alpha - e\beta = de - (dp + pe).\frac{da}{dc} \\ &= de.\left(1 - \frac{da}{dc}\right) = de.\frac{ac}{dc} = ab \quad \text{(as proved in §50).}\end{aligned}$$

III. If $b\alpha$ is drawn, then (§47) $\triangle ba\alpha = \triangle\alpha\beta b$, therefore (§30) $\angle ba\alpha = \angle a\beta b = \mathrm{R}$. Similarly $\angle ab\beta = \mathrm{R}$.

§ 52

Theorem. If at four points, a, b, c, d (Fig. 24) the four angles $abc = bcd = cda = dab = \mathrm{R}$, then the line between any two of these four points equals the line between the other two: $ab = cd$, $ac = bd$, $ad = bc$.

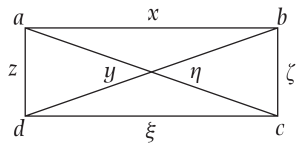

Fig. 24.

Proof. Let $ab = x$, $cd = \xi$, $ac = y$, $bd = \eta$, $ad = z$, $bc = \zeta$, then (§45)

$$y^2 = z^2 + \xi^2 = x^2 + \zeta^2$$
$$\eta^2 = z^2 + x^2 = \xi^2 + \zeta^2 .$$

Therefore $\xi^2 - x^2 = x^2 - \xi^2$, $\xi^2 = x^2$, or (in respect of length) $\xi = x$. Therefore also $y = \eta$, $z = \zeta$.

§ 53

Theorem. If (Fig. 25) $\angle a = \angle b = \angle\alpha = \angle\beta = \mathrm{R}$; then of the three conditions: I. $\angle m$ or $\angle\mu = \mathrm{R}$, II. $am = \alpha\mu$ and $bm = \beta\mu$, III. $m\mu = a\alpha$, each one has the others as consequences.

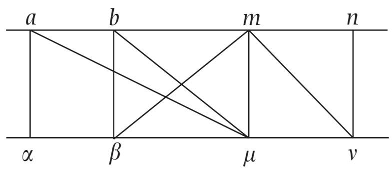

Fig. 25.

Proof. I. Let $m = \mathrm{R}$ and call $a\alpha = b\beta = a$, $ab = \alpha\beta = b$, $bm = x$, $\beta\mu = y$, $m\mu = z$. Now (§45), $a\mu^2 = a^2 + b^2 + 2by + y^2 = b^2 + 2bx + x^2 + z^2$ and $b\mu^2 = a^2 + y^2 = x^2 + z^2$, whence it follows that $2by = 2bx$, $y = x$ and $am = b + x = b + y = \alpha\mu$. Then from the right-angled $\triangle bm\mu$, $z^2 = a^2 + x^2 - x^2 = a^2$, therefore $z = a$.

Finally from (§47) $\triangle\beta bm = \triangle m\mu\beta$, therefore $\angle m\mu\beta =$ R. II. Let $am = \alpha\mu$ and $bm = \beta\mu$. Now if m were not a right angle there would still be a perpendicular from μ to ab, and for this (*ex dem.* I.) $am = \alpha\mu$, $bm = \beta\mu$; now there is only one point m in ab which has these two definite distances from a and b. Therefore m is a right angle. III. Let $m\mu = a\alpha$. Now if m were not a right angle there would still be a perpendicular from μ to ab whose length would be (*ex dem.* I.)$=a\alpha$. Now the hypotenuse μm cannot equal this side (§45); consequently this perpendicular is μm itself.

§ 54

Note. Therefore because all perpendiculars from points on one of the two lines ab, $\alpha\beta$ to the other are *equal*, they are called parallel lines. *Wolff* assumed this property for the definition of parallels without considering the obligation to prove the possibility of this property. This was a very unphilosophical error for that wise man.

§ 55

Corollary. The distances between any two perpendiculars $m\mu$, $n\nu$, are equal, $mn = \mu\nu$. This follows from the theory of the straight line, because (§53) it must also be that $am = \alpha\mu$, $bm = \beta\mu$, and these double distances from a, b determine the points m, n.

§ 56

Theorem. Every line $m\nu$, which intersects both parallels ab, $\alpha\beta$ (Fig. 25), forms equal adjacent angles with them.

Proof. Draw perpendiculars $m\mu$, νn from m, ν to the opposite parallel. Then (§47) $\triangle m\mu\nu = \triangle\nu nm$. Therefore (§10) the angles $nm\nu = \mu\nu m$.

§ 57

Corollary. The equal angles $m\nu\mu$, νmn are called *alternate angles*.[p] I would like to call the infinite parts mx, $\nu\eta$ (Fig. 25*) of the parallels which form the arms of the two associated alternate angles, *alternate arms*. It will be seen that the perpendicular from the initial point m of the one arm mx, or from any point r *outside this arm* meets the other alternate arm $\nu\eta$. For the perpendicular from m this is obvious from §56; for the perpendicular from r, I show this as follows. Because r is outside mx, and therefore (*ex demonstratione*) outside mn, then (by virtue of the properties of the straight line) nr is $> nm$, and $> mr$. Therefore also (by §55) $\nu\rho$ $(=nr) > \nu\mu$ $(=nm)$ and $> \mu\rho$ $(=mr)$. Consequently (by the

[p] The term translated here as ‘alternate angles’ [*Wechselwinkel*] was also applied to the (unequal) angles arising from an intercept on non-parallel lines (see I §65).

theory of the straight line) μ, ρ lie in the identical direction from ν, or $\nu\mu$, $\nu\rho$ are identical arms.

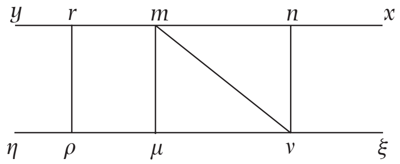

Fig. 25*.

§ 58

Theorem. The diagonals (Fig. 26) $a\beta$, $b\alpha$ in a rectangle intersect in their mid-point.

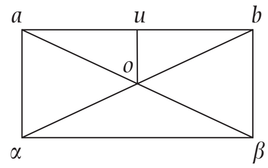

Fig. 26.

Proof. Take o, u the mid-points of $a\beta$, ab; then $\triangle oau \backsim \triangle \beta ab$ and $ou = \frac{1}{2}\beta b = \frac{1}{2}a\alpha$, and $\angle auo = \angle ab\beta = \mathrm{R} = \angle buo$. Therefore (§21) $\triangle buo \backsim \triangle ba\alpha$, therefore $bo = \frac{1}{2}b\alpha$. In the same way, $\alpha o = \frac{1}{2}\alpha b$. Consequently (§44 *in modo tollente*) o is in αb, and therefore $a\beta$ and $b\alpha$ bisect each other.

§ 59

Theorem. Through the same point o (Fig. 27), outside the straight line xy, there is only one straight line parallel to xy.[q]

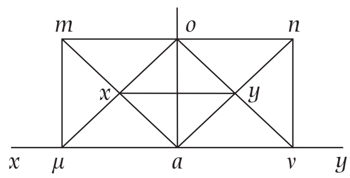

Fig. 27.

[q] On this apparent statement and proof of Euclid's parallel postulate, see the remarks on p. 19.

Proof. Let om, on be two parallels to xy. Take (for brevity) $om = on$ and drop perpendiculars to xy from m, n, then (§55) $om = a\mu$, $on = a\nu$. If am, $o\mu$ are drawn, these equal lines must intersect in their mid-point x (§58); similarly an, $o\nu$ in y. If the straight lines xy, mn are now drawn, then (§21) $\triangle\mu o\nu \backsim \triangle xoy$, from which $yx = \frac{1}{2}\nu\mu$. Furthermore (§47) $\triangle xoy = \triangle xay$. Also $\triangle man \backsim \triangle xay$, from which it follows that $mn = 2xy = \nu\mu$. Now since ν, a, μ are in the same straight line [*Gerade*] and $a\mu = a\nu$ then *either* ν, μ are the identical point and therefore also m, n are the identical point; hence om, on are the same lines, *or* ν, μ lie on opposite sides of a and therefore also n, m lie on opposite sides of o; consequently om, on are again in the same straight line [*geraden Linie*].[r]

§ 60

Theorem. If in each of the parallels ab, $\alpha\beta$ (Fig. 28) two points are given at equal distances $mn = \mu\nu$, then the lines $m\mu$, $n\nu$ which join these points in a particular way, are parallel and equal.

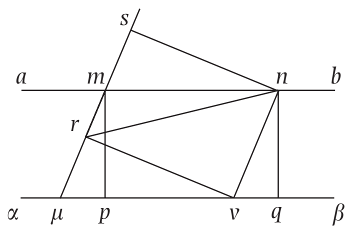

Fig. 28.

Proof. If the perpendiculars mp, nq are drawn, then (§55) $pq = mn$. But $mn = \mu\nu$, so $mn = pq$. Now it follows from the theory of the straight line that there is another combination of the four points μ, ν, p, q which occur in the straight line $\alpha\beta$ for which two distances are equal to each other. If this is $\mu p = \nu q$ then m, μ; n, ν are the points that must be joined to obtain the parallels. Now because also $mp = nq$ (§53), by (§14) $\triangle mp\mu = \triangle nq\nu$. Consequently $m\mu = n\nu$. If one draws perpendiculars to μm from n, ν, then it follows easily (from §21) that because one of the adjacent angles nms, $nm\mu$, must be $= \nu\mu r$ (§56), this is the angle nms in whose arm the perpendicular ns falls. Therefore (§43[s]) $\triangle nms \backsim \triangle\nu\mu r$. And since $nm = \nu\mu$, then $ns = \nu r$, $ms = \mu r$, consequently $\mu m = rs$, and since $\mu m = \nu n$, $rs = \nu n$. From this (§47) $\triangle rsn = \triangle n\nu r$, therefore $\angle n\nu r = \angle s = \text{R}$. Hence $m\mu$, $n\nu$ are parallels (§53).

[r] See the footnote on this on p. 64.

[s] Both German editions have §44 by mistake.

§ 61

Theorem. If (Fig. 28*) ab is parallel to[t] cd, ac is parallel to bd, then also $ab = cd$ and $ac = bd$.

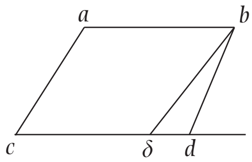

Fig. 28*.

Proof. If ab were not $= cd$ and one takes $c\delta = ab$ in the unbounded part of cd which is not an alternate arm of ab, then $b\delta$ would be parallel to ac (§60). Therefore (§59) d is identical with δ. (Because bd can have only one point in common with cd.)

§ 62

Theorem. If (Fig. 28*) ab is parallel to cd and $\angle acx = \angle bdx$ then also ac is parallel to bd.

Proof. For if we suppose $b\delta$ (drawn from b) is parallel to ac, then $\angle acx = \angle b\delta x$ (§60). Therefore $\angle bdx = \angle b\delta x$; hence (§39) d is identical with δ.

§ 63

Theorem. If the lines (Fig. 29) ab, de intersect the arms of the angle xcy in such a way that either $ca : cb$ not $= cd : ce$, or the angle cab not $= cde$ (where a, d are taken in one arm, b, e in the other) then ab, de intersect somewhere.

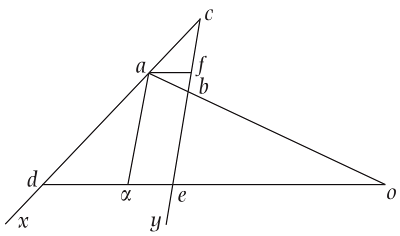

Fig. 29.

[t] Bolzano often abbreviated words, even in publications, so the original *par.* here, and in numerous other places, is better regarded as an abbreviation for German *parallel zu* than a piece of notation. Hence it has been translated.

Proof. Of these two conditions each one has the other as consequence as is clear from §§ 23, 24 *in modo tollente*. Now let

$$\frac{ca}{cb} < \frac{cd}{ce}$$

and $ca < cd$ (without loss of generality). Consider $cd : ce = ca : cf$ (the latter taken in ce from c). Therefore $cf < ce$, as well as $< cb$. Consequently f is in cb as well as in ce; hence $\angle afe = \angle afb$. Now (§23) $\angle cfa = \angle ced$, therefore also (§4) $\angle afe (= \angle afb) = \angle beo$. Consider $fb : fa = eb : eo$ (the latter taken from e in the arm of the last-named angle beo which $= \angle afb$). Therefore (§21) $\triangle afb \backsim \triangle oeb$. And

$$bo = ab.\frac{eb}{fb}.$$

Consider $dc : de = da : d\alpha$ (the latter drawn in de) then *subtrahendo* $dc : de = ac : \alpha e$. But $dc : de = ac : af$ therefore $\alpha e = af$. Similarly $fe = a\alpha$. Since $\triangle ad\alpha \backsim \triangle caf$ (§23), $\angle a\alpha d = \angle cfa$; therefore (§4) $\angle a\alpha o = \angle afb$. Furthermore

$$\begin{aligned} \alpha a : \alpha o &= ef : \alpha e \pm eo = ef : af \pm af.\frac{eb}{fb} \\ &= ef : af.\left(\frac{fb \pm eb}{fb}\right) = ef : \frac{af.fe}{fb} = fb : af. \end{aligned}$$

Therefore (§21) $\triangle a\alpha o \backsim \triangle bfa$. Hence

$$ao = ab.\frac{a\alpha}{fb} = ab.\frac{ef}{fb}.$$

Therefore (in Fig. 29)

$$ab + bo = ab + \frac{ab.eb}{fb} = ab.\frac{ef}{fb} = ao.$$

Hence o is in the straight line ab (§44). Or (in Fig. 29*)

$$ao + ob = ab.\frac{ef}{fb} + \frac{ab.eb}{fb} = ab.\left(\frac{be + ef}{fb}\right) = ab.$$

Consequently o is again in the straight line ab.

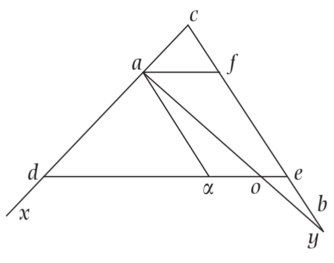

Fig. 29*.

§ 64

Theorem. If (Fig. 30) ab, cd are parallel, and the pieces ab, cd are unequal or the angles cax, dbx are unequal, then ac, bd intersect.[u]

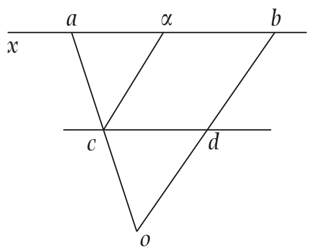

Fig. 30.

Proof. The first condition has the other as its consequence. For if $b\alpha$ is taken $= dc$ in the arm bx which is not an alternate arm to dc, then $c\alpha$, db are parallels (§60), and $\angle dbx = \angle c\alpha x$. Now $c\alpha x$, cax cannot be equal angles (§39). Therefore $\angle cax$ not $= \angle dbx$. Now suppose $a\alpha : ac = ab : ao$ (the latter drawn in that part of ac which forms an angle with ab equal to αac). Draw bo, do. Now $\angle dco$ is always $= \angle \alpha ac$. For if, for example, in one case the direction $a\alpha$ is identical to that of ab, then (*per constructionem*) ac also has the identical direction to that of ao. Now since (*ex hypothesi*) bx, dc are not alternate arms, neither are ab, cd; hence, since these parallels are cut by ac, $\angle bao (= \angle \alpha ac) = \angle dco$. Similarly in the other case. Furthermore, $a\alpha : ac = ab : ao = \alpha b : co$ (adding or subtracting) $= cd : co$. Therefore (§21) $\triangle dco \backsim \triangle \alpha ac$. Whence

$$do = c\alpha . \frac{cd}{a\alpha} = bd . \frac{\alpha b}{a\alpha}.$$

But from $\triangle bao \backsim \triangle \alpha ac$, it follows that

$$bo = c\alpha . \frac{ab}{a\alpha} = bd . \frac{ab}{a\alpha}.$$

From these equations, by comparison with bd it follows that bd, bo are identical straight lines.[v]

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[u] Other cases of what is described in the theorem are illustrated in Fig. 30*and Fig. 30**on the next page.

[v] They are identical when viewed as unbounded straight lines. Bolzano uses the word *Gerade* here for straight line. The first such use is in I §59. Previously he has usually used *gerade Linie* when referring to a bounded line segment, unless otherwise qualified. For example, in I §30, 'a straight line [*gerade Linie*] produced indefinitely in both directions'. However, the distinction of usage between *Gerade* and *gerade Linie* is not preserved consistently.

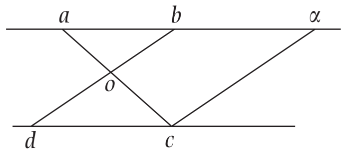

Fig. 30*.

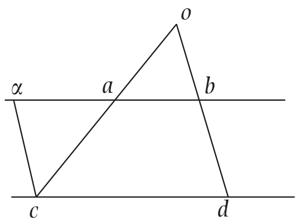

Fig. 30.**

§ 65

Theorem. If the two lines (Fig. 31) *ab*, *de* cut two other lines *ad*, *be* which are either parallel or intersecting, then the first are either parallel or intersecting depending on whether the fifth line *mn* by which they are cut forms equal or unequal alternate angles.

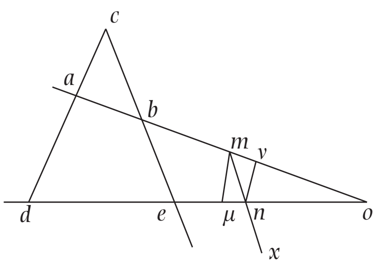

Fig. 31.

Proof. I. If these angles are *equal*, then *ab*, *de* cannot intersect. For if this happened in *o* and the perpendicular from *n* to *mo* fell inside *om*, then the perpendicular from *m* to *no* would fall outside *on*, because the angles $nm\nu = mn\mu$ are supposed to be *alternate angles* (§57). Therefore $\angle omx = \angle mn\mu =$ (§5) $\angle onx$ (if *nx* is an extension of *mn* beyond *n*), contradicting (§39). But if $\angle bac$, $\angle edc$ were unequal, then *ab*, *de* would have to intersect (§§ 63, 64). Therefore these angles are equal; therefore (§§ 50–54) *ab*, *de* are parallel. II. If the alternate angles are *unequal* then $\angle bac$, $\angle edc$

must also be unequal because otherwise these alternate angles would be equal (§§ 50, 57, 62). But if $\angle bac$, $\angle edc$ are unequal then *ab*, *de* intersect (§§ 63, 64).

§ 66

Theorem. If two lines (Fig. 32) *ab*, *cd* which are either parallel or intersecting have a third line *ac* cutting both, and the fourth line *bo*, which is either parallel to the third or intersects it, cuts the first *ab*, then it also cuts the other *cd* or is parallel to it.

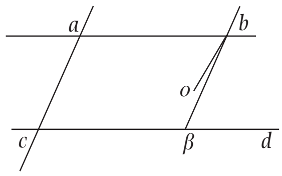

Fig. 32.

Proof. I. If *ab* is parallel to *cd* and *bo* is parallel to *ac* then *bo*, which cuts *ab*, must necessarily also cut *cd*. For if we take $c\beta = ab$, then $b\beta$ is parallel to *ac*, therefore *bo* is the same as $b\beta$ (§59). II. If *ab*, *cd* (Fig. 32*) intersect in *x*, but *ac* is parallel to *bo*, then take $xa : xb = xc : x\beta$; consequently (§§ 50, 54) $b\beta$ is parallel to *ac*. Therefore (§59) $b\beta$ is the same as *bo*. III. If *ab*, *cd* intersect as well as *ac*, *bo*, then *bo*, *cd* do not necessarily have to cut each other.[w] But one of the following two cases must hold: either *bo* is parallel to *cd* or they intersect. For here it is the case that in the angle *bac* two lines *cy*, *bx* cut both arms, and §62 can easily be applied.

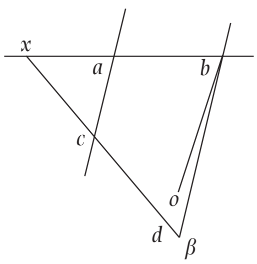

Fig. 32*.

§ 67

Note. These are perhaps the most important propositions of the theory of parallels expressed here without the concept of the plane (which is equivalent to the

[w] Case III in fact refers to Fig. 32**.

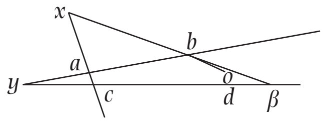

Fig. 32.**

condition: *that two lines either intersect or are parallel*), and from which, several other propositions, particularly trigonometric ones, can now be derived in the usual way. It will have been noticed here and there that I have assumed certain propositions from the theory of the straight line which are not explicitly mentioned in the usual textbooks of geometry. Yet they are indispensable to making geometrical language precise, and I would have wished to have been able to make stricter use of them, but I could not do so from fear that it would be interpreted as being over-pedantic. It is generally acknowledged that the efforts of geometers in the theory of parallels, up to the most recent ones of *Schultz*, *Gensich*, *Bendavid*, *Langsdorf*, have still all been inadequate. Now other people have already made the objection to the proof of *Schultz* (and the proof by the Frenchman *Bertrand* is essentially the same as that of *Schultz*) that it is based on a consideration of a different kind,[x] namely on the *infinite surface of the angle*, as well as on axioms of infinity, which have not yet gained general approval. *Gensich* aims, very ingeniously, to remove only the difficulties of infinity, but nothing is altered concerning the first matter. The fact that I cannot therefore be satisfied with this proof of the theory of parallels arises from the principles already stated (*Preface* and §6). *Bendavid's* proof is so hasty (somewhat unexpected for the author of *Die Auseinandersetzung des mathem. Unendlichen*) that it is rendered completely invalid. The proof offered by *Langsdorf* (*Anfangsgründe der reinen Elementar- und höheren Mathematik*, Erlangen, 1802) cannot satisfy me, nor all those who have not yet been convinced of the possibility and necessity of his *spatial points* (which are supposed to be *simple elements* in space, and from their accumulation together in finite numbers lines, surfaces and solids are formed). However, even if this truth-seeking scholar has not changed his conviction about spatial points, this does not necessarily prevent him, since he also accepts geometrical points and lines, from giving some approval to the methods of proof (if nothing else) in my present writing. Other *more recent* attempts concerning parallels are not known to me. Since I have now ventured on a goal which has so often proved unsuccessful in the past, it would be immodest if I claimed the discovery for myself after ἕυρηκα has so often been uttered too hastily. I prefer to leave it to the *judgement* of the reader and of the future.

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[x] Here, and in the last sentence of II §18, Bolzano uses the word *heterogen* to refer to the idea of one concept being of a different kind to another. More often he uses *fremdartig* (alien) for this purpose (e.g. in *Preface* on p. 32).

II

Thoughts Concerning a Prospective Theory of the Straight Line

§ 1

First of all let me say something about the concepts of identity [*Einerleiheit*] and equality [*Gleichheit*]. In a geometrical investigation it is not my job to look for perfectly correct definitions of these two words. But I must state the *specific meaning* which I attach to them (because otherwise it would be ambiguous). Therefore, I understand by *identity* (*identitas*) the concept which arises from the comparison of a thing (solely) *with itself*. Identity I put as contradictory to *difference*. Difference I divide again into the two contradictory *species*: *equality* and *inequality*. Consequently equality presupposes difference and it is correct to say, 'everything is identical with itself' (but not *equal* to itself), 'two *different* things are either *equal* or *unequal*' (never *identical*). Yet if one says, 'The thing A is identical with the thing B' this really means: A and B have been assumed, hypothetically, to be two different things and it has been proved that in fact they are not different but an *identical* thing. *Properties* of things can be called identical or different. But in so far as they are hypostasized and considered as *things* themselves, they are *eo ipso* different and can now be called equal or unequal.

§ 2

These theorems can now be proposed: things whose determining pieces are identical are themselves an identical thing, and conversely (*conversio simplex*). Things whose determining pieces are equal are themselves equal things, and conversely. I also take this opportunity to submit for criticism the following two propositions which I have often been inclined to assume in mathematical proofs. If, among the determining pieces of two things one is different (but the rest are identical), then there must also be a difference in the pieces determined. If, among the determining pieces of two things one is unequal (but the rest are equal), then there must also be an inequality in the pieces determined.

§ 3

Before the geometer applies the concept of equality to spatial things he should first demonstrate the *possibility of equal spatial things*. Perhaps the axiom proposed in I §19 may be used here, which may be expressed generally: we do not have an *a priori* idea of any determinate spatial thing (not even of a *point*). Therefore *several* completely *equal* spatial things must be possible for which all equal predicates hold. Therefore if some spatial thing A is possible at a point a, then also an equal spatial thing $B = A$, must be possible at the different point b.

§ 4

A genuine definition must contain only those characteristics of the concept to be defined which constitute its *essence* [*Wesen*], and without which we could not even conceive of it. Therefore we should regard as very artificial the definitions of the scholastic *Occam*, for solid, surface, line and point, according to which, solid is that kind of extension which cannot be the boundary of anything else, but surface is the boundary of a solid etc., because in order to conceive of only a point or a line they require in each case the idea of a *solid*. (*Langsdorf* also remarks on this, see the *Preface* of his *Anfangsgründe der Mathematik*.) However, it is obvious that we can perfectly well conceive of a surface, a line or a point, and that we do so without a solid which they bound. In my opinion, it would not be so very objectionable if someone were to turn this round and put forward definitions which required the idea of points for lines, and of lines for surfaces, etc.

§ 5

This much is, I hope, unobjectionable: that the *concept of point*—as a mere *characteristic of space* ($\sigma\eta\mu\varepsilon\iota o\nu$)[y] *that is itself no part of space—cannot be dispensed with* in geometry. This point is indeed a merely imaginary object as I gladly concede to *Langsdorf.* Lines and surfaces are also like this, and indeed all three are so, in yet a different sense from the geometrical *solid*. Something adequate can be given for the latter in the intuition, but not for the former, (indeed everything that is given in the intuition is solid). And perhaps for this very reason any attempted pure *intuition* of *lines* and *surfaces* (say by the motion of a point) must be impossible. The definitions attempted in this paper of the straight line §26, and the plane §43, are made on the assumption that both are simply *objects of thought* [*Gedankendinge*].

§ 6

Since a point, considered in itself alone, offers nothing distinguishable, as we have no determinate *a priori* idea of it, it follows that the simplest object of geometrical consideration is a *system of two points*. From such a *simultaneous conception of two points* there arise certain predicates for these (concepts) which were not present with the consideration of a single point. Everything which can be perceived in the relation of these two points to one another, and indeed in the relation of b to a (Fig. 1), I divide into two component concepts [*Theilbegriffe*]: I. That which belongs to point b in relation to a in such a way that it is *independent* of the *specific point a* (*qua praecise hoc est et non aliud*),[z] and which can consequently be present *equally* in relation to *another* point, e.g. α, is called the *distance of point b from a*. II. That which belongs to point b in relation to a in such a way that it is *dependent simply on*

[y] The main meanings given in *LSJ* are 'mark' (by which something is known), or 'sign', but also 'mathematical point', 'instant of time'.

[z] *Translation:* which is precisely this one and not another one.

the specific point a, where we have now separated off what is already present in the concept of distance, i.e. what can belong to point *b* also with respect to another point. This is called the *direction in which b lies from a*.[a]

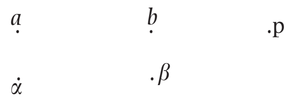

Fig. 1.

§ 7

Now to show the *possibility* of both concepts. I. *Distance*. The mere concept of *being different* [*Verschiedenseyns*] of the point *b* from the point *a* (of their being separate) is no part of the concept of the *relation of the point b to a* (of the *totius dividendi*[b] §6), but is necessarily presupposed by it. If *b* is to be related to *a*, then the idea that *b* is different from *a* must already be assumed. Therefore in order to demonstrate the reality [*Realität*] of the concept of distance as a component of the whole concept mentioned above, one must prove that it contains more than the mere fact that *b* is different from *a*. This I do as follows. If the concept of distance contained nothing else, then the other concept of direction would have to encompass completely the *totum divisum*, i.e. the whole concept of the relation of *b* to *a* would have to contain nothing other than what is dependent simply on the specific point *a*. In other words, the system *ab* could be completely determined by *that* which belongs to point *b* and is purely dependent on *a*, and so can belong to no *other* system. Therefore we would have a special *a priori* idea of *a*, which we would have of no other point, and this is contrary to our axiom. II. The concept of *direction* cannot be completely empty, because otherwise the concept of distance would again have to exhaust completely the concept being analysed. But *ex definitione* the concept of distance contains only what belongs to *b* independently of the *special* [*besonderer*] point *a*, so that it can also belong to the system $b\alpha$. But the whole [concept] being analysed contains only as much as belongs to the system *ab* alone.

§ 8

Therefore since both concepts of distance and direction have a content, each one contains less than the whole concept being analysed. Hence neither distance alone, nor direction alone, determines the point *b*. In other words, there are several

[a] The context and use of the German *die Richtung in welcher b zu a liegt*, here and in subsequent paragraphs, shows it should be understood literally as 'the direction in which *b* lies at *a*', that is, the direction in which *b* lies for an observer at *a*.

[b] *Translation:* the whole [concept] being analysed.

points b, β which are at an *equal distance* from a, and similarly several points b, p which lie in the *same* [*einerlei*] *direction* from a.

§ 9

The assumption of the point a, and the distance and direction of the point b determine the latter *ex definitione*. And conversely the point b determines the distance from a and the direction from a. Therefore two different directions from the same point a can have no single point *in common*, i.e. there is no point to which they both belong.

§ 10

Theorem. For a given point a (Fig. 2) and in a given direction aR there is one and only one point m, whose distance from a equals the given distance of the point y from x.

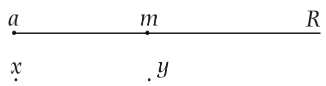

Fig. 2.

Proof. It follows that there is a point at the *given distance* from a, because otherwise there would be a distinction between the unrelated points a, x, which is not permitted. It follows that there is also such a point in the *given direction* from a, because otherwise we would have to have a special idea of a specific direction aR. Finally, it follows that there is only *one*, from §9.

§ 11

Up to now I have looked in vain for a satisfactory proof of the theorem that the distance of b from a is equal to the distance of a from b. Meanwhile the following reasoning may be proposed, although it is unsatisfactory to me, in order perhaps to inspire something better. If two things A and B are *equal*, then it must be *possible* to combine them in such a way that the relation of A to B is *equal* to the relation of B to A. Since the reason for its not being possible would have to lie in the things being unequal. Now since all points are *equal* things, it must be possible for two points to be combined in such a way that the relation of b to a equals the relation of a to b. But if such a combination is simply possible then it is also *actual*, for the combination of two points at a definite distance is a single thing. Now if the relation of b to a is to be equal that of a to b, then the *distance* of b from a must be equal to that of a from b, for the *directions* cannot be compared.

§ 12

The system of two directions proceeding from one point is called an *angle*. I have already indicated in I §2 why the concept of angle really belongs to *directions* and not to lines.

§ 13

However, I also have another definition of angle to offer which is completely analogous to the development of the concepts of direction and distance. Consider two directions R, S from the same point a (Fig. 3), and divide the whole concept of the relation of the direction S to R into the following two component concepts: I. that which belongs to the direction S independently of the specific direction R (and only this direction)—called the *angle which S makes with R;* II. that which belongs to the direction S only with respect to the direction R and to no other, where we have now separated from it what can belong to it equally with respect to another direction—called the *plane* in which S lies with R. (In this sense of the word '*plane*' it would include only that *half* of the *usual* plane through R and S which lies on *that* side of R in which S is.) But for the present I shall keep to the definition §12.

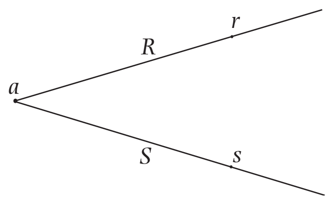

Fig. 3.

§ 14

In each case it needs to be shown that the idea which arises if S is related to R is equal to *that* which arises if R is related to S, i.e. the *angle* $sar = ras$. (Similar to the proposition of §11.) Here also I still have no satisfactory proof.

§ 15

If the angle between the directions R, S (Fig. 4) is such that it determines the direction S by the direction R, i.e. for the identical direction R there can be no direction different from S which forms an equal system with R, then the angle between S and R is called an *angle of two right-angles* (or as *Schultz* calls it) a *straight angle*.[c] The direction S is called *opposite* to the direction R. Therefore according

[c] This is Definition 19 in Schultz (1790), p. 270.

to §14, the direction R is also opposite to that of S. It also follows *ex definitione* that if the directions R, S (Fig. 3) are not opposite to one another then there are always directions different from S, which form an equal angle with R.

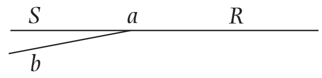

Fig. 4.

§ 16

However, it is something different, which does not follow *ex definitione*, that for any direction R there is only a *single* direction S opposite to it; for there could perhaps be *different* angles which belong to each *single* direction S.

§ 17

Theorem. If in the directions (Fig. 5) aC, $a\Gamma$ the points c, γ are taken at equal distances from a, and in the directions ca, γa the points m, μ are again taken at equal distances from c, γ, then the angles $ca\mu = \gamma am$.

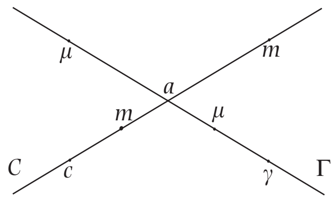

Fig. 5.

Proof. For since the angles $Ca\Gamma = \Gamma aC$, it can easily be shown that the determining pieces of the angle $ca\mu$ are *equal* to the determining pieces of the angle γam.[d]

§ 18

If the directions ab, ac from a (Fig. 6), and the distances ab, ac of the points b, c in these directions are given, then the points b, c are themselves also determined (§8); consequently the system of three points a, b, c is given which is called a *triangle*. The distance bc and the angles at b, c are also determined. Therefore one has a triangle here which consists only of three points and the three angles of the directions of every two points to the third—not of three lines. It may also be

[d] See I §5.

seen how several theorems about triangles, which in Part I are mixed up with the concept of the line, can already be presented here (with small changes). But I retained the concept there, although fundamentally heterogeneous, so that such abstractions should not become a burden.

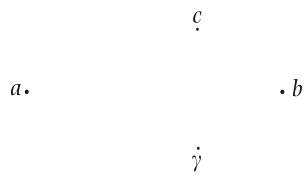

Fig. 6.

§ 19

Theorem. If the direction *ac* of the point *c* from *a* is not determined by its angle with the direction *ab* of the point *b* from *a*, then the direction *bc* is also not determined by the angle which it makes with *ba*.

Proof. By assumption there is at least one other direction different from *ac* which forms an equal angle with *ab*. Now if one takes the point γ in it at the distance $a\gamma = ac$, then the systems γab, *cab* (triangles) are *equal* because their determining pieces are equal. Consequently the angles $abc = ab\gamma$ and the distances $bc = b\gamma$. Hence the direction *bc* is not identical to that of $b\gamma$, for otherwise (§8) c, γ would be an identical point. Therefore there are different directions *bc*, $b\gamma$ which form an equal angle with *ba*.

§ 20

Corollary. The same holds of the angle *acb*. And this is actually the theorem that in every triangle there are three angles (I §9).

§ 21

But if the directions *ac*, *ab* (Fig. 7) are either identical [*einerlei*] or opposite, then the directions at *b*, *c* must also be identical or opposite. For if in one case neither of the two held, then neither of the two could also hold at *a* (§§ 19, 20).

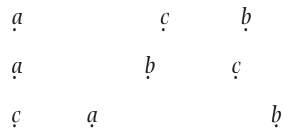

Fig. 7.

§ 22

It is also easy to prove the general proposition: If in a system of any number of points the rule holds that every single point, with another second point, lies in the identical direction, or the opposite direction, from a certain third point, then exactly the same holds of every two points from every third point.

§ 23

Theorem. If the two directions (Fig. 8) *oa*, *ob* are neither identical nor opposite, then the two directions *ao*, *bo* have only the single point *o* in common.

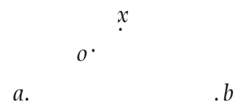

Fig. 8.

Proof. Assuming they had another point *x* in common, then it would follow (§21) that *oa*, *ob* were identical or opposite to *ox* and were consequently also identical or opposite to each other, *contra hypothesim*.

§ 24

As yet I am still not in a position to demonstrate the *possibility* of the concept of opposite direction. In general what I have to put forward as still unproved, as well as what has gone before (§§ 11, 14, 21), can be summed up in the following proposition. 'In a system *of three points* consider the relationship of the directions in which every two lie from the third: if these directions are *identical or opposite at one point* then they are *identical* at *two* points and *opposite at one* point.' A theory of the straight line can be based on these assumptions which must all be proved without the concept of the straight line so that they can be accepted for my purpose *absque petitio principii*. The chief propositions of this theory are as follows.

§ 25

Definition. The point *m* (Fig. 9) may be called (for the sake of brevity) *within* or *between a and b* if the directions *ma*, *mb* are opposite.[e]

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[e] The need for an analysis of the concept of 'betweenness' was to be pointed out by Gauss in a letter to Bolyai in 1832, cited in Kline, 1972, p. 1006. Such an analysis in terms of axiomatic systems was carried out much later by, for example, Pasch, Hilbert, and Huntington.

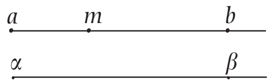

Fig. 9.

§ 26

Definition. An object which contains all and only those points which lie between the two points a and b is called a *straight line between a and b*.

§ 27

Note. The *possibility* of this object follows from what is assumed in §24. From the following it also appears that this object contains an infinite number of points, therefore it must be something qualitatively different from a mere *system of points*.

§ 28

Theorem. Two given points determine the straight line which lies between them.

Proof. For the straight line between a and b should contain all points which lie *between* a and b and no others. Therefore there is only a *single* thing which is called the straight line between a and b.

§ 29

Theorem. If the distances $ab = \alpha\beta$ (Fig. 9) then also the straight lines $ab = \alpha\beta$.

Proof. For their determining pieces (§28) are equal.

§ 30

Theorem. For every two given points (Fig. 10) a, b there is one and only one *mid-point* [*Mittelpunkt*], i.e. a point that is determined from both of them in the same way.

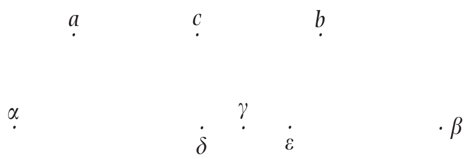

Fig. 10.

Proof. In the opposite directions $\gamma\alpha$, $\gamma\beta$ take the points α, β at arbitrary equal distances $\gamma\alpha = \gamma\beta$; then γ is determined from α, as it is from β. Now, if possible, let δ be another point that is determined from α as it is determined from β. Consequently it must be the case that the distances $\delta\alpha = \delta\beta$. Hence the directions $\delta\alpha$, $\delta\beta$ cannot be identical or α, β would be the identical point. Now if they were different but not opposite, then the directions $\alpha\beta$, $\alpha\delta$ would also be different and not opposite (§20); therefore the direction $\alpha\delta$ would not be determined by $\alpha\beta$, and also the point δ would consequently not be determined by α, β. Accordingly $\delta\alpha$, $\delta\beta$ must be opposite, and so (§24) $\alpha\beta$, $\alpha\delta$ are identical. But also the directions $\alpha\gamma$, $\alpha\beta$ are identical, therefore directions $\alpha\gamma$, $\alpha\delta$ are identical. Consequently (§24) the directions are opposite either at γ or at δ. I assume the first. (The deduction is similar in the other case.) Consider ε in the direction $\gamma\alpha$, which is opposite to that of $\gamma\delta$ or $\gamma\beta$, at the distance $\gamma\varepsilon = \gamma\delta$, then it follows because $\gamma\alpha = \gamma\beta$, that also the distances $\alpha\varepsilon = \beta\delta$, since they are determined in the same way. So as $\beta\delta$, $\beta\alpha$ are identical directions, then also $\alpha\varepsilon$, $\alpha\beta$ must be identical directions. Therefore (*per demonstrationem*) $\alpha\varepsilon$, $\alpha\delta$ are identical directions and also the distances $\alpha\varepsilon = \beta\delta = \alpha\delta$, therefore ε, δ are the identical point, which is contradictory. Thus the point γ is the only one which is determined in the same way from α, β. Now if α and β have a mid-point, then every other two points *a*, *b* must also have one (I §19).

§ 31

Theorem. If the point *c* (Fig. 11) lies *within* the points *a*, *b* then the straight lines between *a*, *c* and between *b*, *c* are *parts* of which the *whole* is the straight line between *a*, *b*.

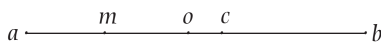

Fig. 11.

Proof. We need to prove that every point of the straight lines *ac* or *bc* is at the same time a point of *ab*, and that every point of *ab* is a point of either *ac* or *bc*. I. Let *m* be a point of *ac*, therefore (§26) the directions *ma*, *mc* are opposite, but those of *cm*, *ca* are identical (§24). And since *ex hypothesi ca*, *cb* are opposite, then so also are *cm*, *cb*. Hence (§24) *bc*, *bm* are identical. But likewise *bc*, *ba* are identical, therefore *bm*, *ba* are identical. Likewise it follows that *am*, *ab* are identical, therefore (§24) *ma*, *mb* are opposite and consequently *m* is a point in the straight line *ab*. II. Let *o* be a point in the straight line *ab*, consequently the directions *ao*, *ab* and *bo*, *ba* are identical. But *ex hypothesi ca*, *cb* are opposite, so *ac*, *ab* are identical. Hence *ac*, *ao* are identical. Consequently (§24) the directions of *ca*, *co* are either identical or opposite. If the former, then *o* therefore lies in the straight line *ac*. If the latter, then it is easy to see that *o* lies in the straight line *bc*.

§ 32

Theorem. If the points m, n (Fig. 12) both lie within a, b then the straight line mn is a *part* of the straight line ab.

Fig. 12.

Proof. In a similar way.

§ 33

Theorem. If the distances (Fig. 13) $ab = bc = cd =$ etc. and the directions ba, bc; and cb, cd; etc. are *opposite*, then the straight line between a and d can be considered as a *magnitude* which, if $n + 1$ is the number of points from a to d, represents the *number* n if its *unit* is the straight line ab.

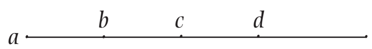

Fig. 13.

Proof. From (§31) it follows that the straight line ac can be viewed as a whole of which the component parts are ab, bc, and the straight line ad can again be viewed as a whole of which the parts are ac, cd, consequently also as a whole of which the component parts are ab, bc, cd, etc. But these parts ab, bc, cd, etc. are equal to each other because the distances $ab = bc = cd =$ etc. (§29). But it is easy to show that they are one fewer in number than the number of points. Consequently etc.

§ 34

Theorem. Every straight line ab (Fig. 14) can be divided into a given number of equal parts which together give the whole ab again.

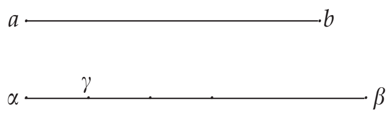

Fig. 14.

Proof. For with the assumption of some straight line $\alpha\gamma$, an $\alpha\beta$ can be conceived which consists of n parts $= \alpha\gamma$ (§33). Hence ab must also be capable of being divided in such a way (I §19).

§ 35

Theorem. The *unit* and the number (therefore the magnitude) determine the straight line to which they belong.

Proof. If two unequal lines were possible to which the same magnitude belonged, then we may conceive of them being from the same point a (Fig. 15) and in the same direction. Their two endpoints b, β must be different (§20). Consequently the directions ba, $b\beta$ are either identical or opposite. If, for example, the latter is the case, then (§31) the straight line $a\beta$ is a whole whose component parts are ab, $b\beta$. Therefore ab alone is not a component part of $a\beta$, consequently $a\beta$ does not have a magnitude which would be equal to that of ab.

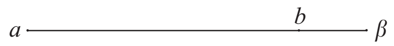

Fig. 15.

§ 36

Corollary. All straight lines which have equal magnitude are therefore equal to one another.

§ 37

Theorem. If the directions ca, cb (Fig. 11) are opposite and the magnitudes of the lines ac, cb are represented, using a common unit, by the numbers m, n, then the magnitude of the straight line ab, using the same unit, is represented by the number $m + n$.

Proof. For according to §31 the line ab is a whole of which the integral [*integrierende*] parts are ac, cb; etc.

§ 38

Theorem. If the directions ab, ac (Fig. 11) are identical and the magnitudes of the lines ab, ac (Fig. 10) are represented using a common unit by the numbers $m + n$, m, then the magnitude of the straight line bc, using the same unit, is represented by the number n.

Proof. For ac, cb are the integral parts of the straight line ab, therefore $ac + cb = ab$, or in numbers, $m + cb = m + n$, hence $bc = n$.

§ 39

Theorem. For every three distances (Fig. 16) *ab*, *cd* and $\alpha\beta$ there is a fourth $\gamma\delta$ with the property that all predicates which arise from the comparison of the two distances *ab*, *cd* are equal to the predicates provided by the comparison of the two distances $\alpha\beta$, $\gamma\delta$.

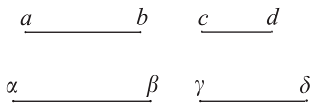

Fig. 16.

Proof. Otherwise we would have to have a special idea *a priori* of the *specific* distance *ab* according to which something would be true of it that is not true of $\alpha\beta$.

§ 40

Corollary. It is easy to demonstrate that the comparison between *ab*, *cd* can be carried on so far that the characteristics resulting from it determine *cd* from *ab*. And if *ab* determines *cd* then $\alpha\beta$ also determines $\gamma\delta$.

§ 41

Theorem. The same (as §39) holds also for straight lines.

Proof. Because these are determined by the distances (§28; I §17).

§ 42

Corollary. Therefore for every three given lines there is one and only one (§40) fourth proportional line.

§ 43

Concluding note. These few propositions are probably sufficient to show how I would think of building up a complete theory of the straight line on the axioms already stated. At the conclusion of this essay I want to add a *definition of the plane* according to which I have already sketched out a large part of a new theory of the plane. The *plane of the angle ras* (Fig. 3) is that object which contains all and only those points which can be determined by their relationship (their angles and distances) to the two directions *R*, *S*.[f]

[f] This definition is taken up again in *DP* §37.

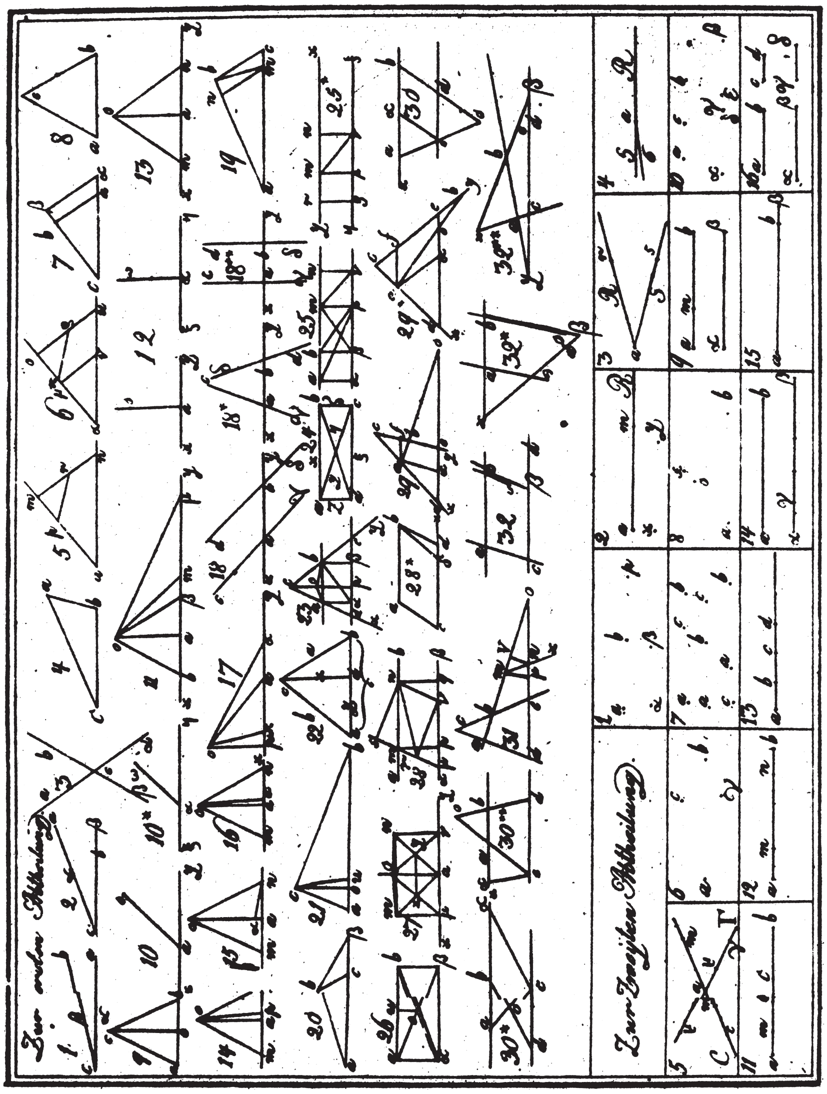

Plate of Figures as it appeared at the end of the first edition.