Introduction

Bernard Bolzano was a Christian humanist who devoted a lifetime of thought and writing to a far-reaching and wide-ranging reform of the representation, organization, and discovery of knowledge. He was ordained as a Catholic priest in 1805 and thereafter, for almost fifteen years, he held a university post in Prague lecturing in theology and giving regular 'edifying discourses' to both the students and the public. He was eventually dismissed from his post by the Hapsburg Emperor, Franz I, because his public views, on social and political issues were deemed dangerously liberal, and his theological views even heretical. Nevertheless, he was popular as an educator and unflagging in his zeal as a reformer, driven in these roles, as in all parts of his life, by an ethical principle that he called the 'highest moral law', namely, 'always to behave in a way which will best promote the common good'. The well-being and progress—in the broadest sense—of humanity was his lifelong and overriding concern.

His working life occupied, almost exactly, the first half of the nineteenth century and was spent in the midst of a strong resurgence of Czech culture within a society dominated by German and Austrian influences. Born on 5 October 1781 of an Italian father and German mother, Bolzano clearly identified himself with the people and culture of his adopted country; he chose to describe himself as a 'Bohemian of the German tongue'. In spite of numerous obstacles, recurring illnesses, and persecution, his extraordinary energy, determination and hard work resulted in a prodigious output embracing philosophy, logic, mathematics, physics, politics, education, theology, and ethics. Evidence for the sheer scale of his writings lies in the monumental and meticulous complete edition [*Bolzano Gesamtausgabe*] (*BGA*) that is being prepared by Frommann-Holzboog of Stuttgart with a planned total of 120 volumes.

One result of his project for the reform of knowledge, and one which had a central place in Bolzano's thinking, was the four-volume work *Wissenschaftslehre* (*WL*) published in 1837, which was essentially his own novel reformulation and development of logic. It provoked Edmund Husserl to declare in 1900, 'we must count him one of the greatest logicians of all time, . . . logic as a science must be based upon Bolzano's work' (Husserl, 2001, i, p. 142). Substantial parts of *WL* have already been given independent English translations, one by Berg (1973) and one by George (1972).

Another result of his reform programme was a large amount of mathematics. Written throughout his lifetime and of varying quality, this is now generally

recognized as containing within it substantial contributions that were original, profound, prescient of what would become fruitful and familiar to much later generations of mathematicians. His mathematics is closely allied to, and sometimes even promoted by, his own distinctive philosophical views about concepts and proofs. The material is in three main forms and complicated by the censorship imposed on Bolzano as a result of his dismissal in 1821, which prevented him publishing even mathematics within the Habsburg Empire, at least until the death of Franz I in 1835. First there are the mathematical works published in his lifetime (i.e. the works included here as *BG*, *BD*, *BL*, *RB*, *DP*); they were published in Prague or Leipzig and manuscripts for them do not survive. Then there is the *Nachlass*, the material remaining after his death, and this is in two very different forms. There is writing intended for publication and in a reasonable state of preparation but which was not published until after his death, either very soon as *PU*, or much later and in different forms as *RZ* and *F*. Finally, there is the material recorded throughout his lifetime in diaries. This is in the *BGA* Series II B *Miscellanea Mathematica* of which eighteen volumes have already appeared. The huge amount of diary material has been transcribed by Bob van Rootselaar and Anna van der Lugt from a handwriting that is difficult to read and full of personal abbreviations and corrections. They date from 1803 and provide a rich source for future scholars seeking to trace the origins and development of Bolzano's ideas.

We summarize here the main technical and mathematical contributions of Bolzano that are easily identified and recognized today. He gave the first topological definitions of line, surface, and solid (in 1817), and stated the Jordan curve theorem as a result requiring proof. Banishing the usual ill-defined infinitesimals of his time and skilfully employing an arithmetic limit concept, he defined and used concepts of convergence, continuity, and derivatives in a way very similar in detail to that found in modern textbooks. In the 1830s, he began an elaborate and original construction of a form of real numbers—his so-called 'measurable numbers'. Much of the work associated with definitions or constructions of real numbers (such as those by Weierstrass, Méray, Dedekind, and Cantor) dates from several decades later. He formulated and proved (1817) the greatest lower bound property of real numbers which is equivalent to what was to be called the Bolzano–Weierstrass theorem. He later gave a superior proof with the aid of his measurable numbers. In the course of an extensive treatment of functions of real variable he also constructed, also in the early 1830s, a function everywhere continuous and nowhere differentiable. He proved the function had these properties on a dense subset of argument values. The discovery of such surprising functions is usually attributed to Weierstrass whose examples again date from at least thirty years later. He developed the first elementary theory of infinite collections, putting great emphasis on the concept of a 1-1 correspondence and clearly understanding that it is characteristic of infinite multitudes that they always have a 1-1 correspondence with some proper subcollection of themselves. Such fruitful, new mathematical concepts and theories were for Bolzano simply the consequences to be expected of what he regarded as his main contribution—namely,

a fundamental reorganization of the principles and foundations of mathematics that placed practical demands on the concepts and proofs that were acceptable or appropriate as a theory was developed. Some details of these foundational views occur in all his works, but most particularly in the work *BD* of 1810.

It is one of the fascinations of Bolzano's thought that while he was well-read in his predecessors' work he nevertheless espoused a highly individual general philosophy that appeared to have an intimate and unusually formative effect on his own specific mathematical ideas. His philosophical views about mathematics acted as a productive driving force for new mathematics. In this way, even if, as seems to have been the case, he had relatively little influence on later mathematicians the reasons for his innovations, and the reasons for his reformulations of definitions and proofs may continue to be worth our while seeking to understand. Not only may this be valuable for understanding how mathematics has changed in the past, but such lessons may even yet not be irrelevant for suggesting fruitful directions in the future.

His mathematical publications began with the geometrical work *BG* in 1804. This was a bold effort to reorganize elementary geometry. His subsequent works *BD*, *BL*, *RB*, and *DP*, published while he was Professor of *Religionwissenschaft* were, on his own account, instalments intended to gain attention, advice, and criticism (see *RB Preface*). He may have seen these works foremost as an illustration and accompaniment to his developing views on knowledge and logic rather than part of a deliberate contribution to mathematics for its own sake. Later, his extensive mathematical research and achievements appear to have taken on a life and purpose of their own in the unfinished *Theory of Quantity* [*Größenlehre*], encompassing the works *RZ*, *F*, and *PU*, that he began planning and writing in the 1830s. It is only a small proportion of his mathematical work that is presented in translation in the present volume. Any such selection is likely to be somewhat arbitrary. It is not simply the 'highlights'; indeed, it contains some material that is very sketchy, or long-winded and amateurish. But it also contains some of his best insights and most far-reaching contributions. It is hoped that it is reasonably representative of both his published material, and his *Nachlass*. The eight works translated here have been put into three groups according to subject and chronology with a short introduction about the mathematical significance of each group.

We turn now to a brief account of Bolzano's life and work, and his own broad intellectual and moral concerns. His work is remarkable not only for its mathematical content but also for the circumstances, the context, and the significance of its creation.

There is an enigma at the heart of Bolzano's work that immediately strikes the modern reader as soon as she strays only slightly beyond the 'strictly' mathematical writings. But it is an enigma that demands resolution if we are to read, interpret, and judge his mathematics in a way that does justice to his perceptions and values as well as to ours. The chief motive in Bolzano's life and work was a moral one: to promote most effectively the 'common good' [*das allgemeine Wohl*].

Yet throughout his life he was predominantly occupied with philosophy, logic, and mathematics, and not the parts of these subjects that were associated with potential usefulness, but instead his preference was for topics—such as the foundations of mathematics and the nature of proof—that we would probably regard today as the very furthest from possible practical application. The society of early nineteenth century Bohemia had only recently emerged from feudal practices such as widespread serfdom and robot (i.e. forced labour for a landowner); it had suffered decades of unjustly heavy taxation and the frequent threats, and realities, of famine; there was poverty and social injustice on a huge scale. So we need to understand, for example, how the pursuit of the 'correct' or 'objective' proof of the intermediate value theorem, or the 'proper' definition of a line, could, without hypocrisy or blindness, be regarded as best promoting the common good. The entire drive and direction of these two goals—the moral and the mathematical—appear, while not directly opposed to one another, yet to be in such tension that it is hard for us to imagine them coalescing and forming part of the mainspring of the life and work of a single individual. But in one so single-minded of purpose, and so productive as was Bolzano, they must not only have coexisted but must surely have complemented, combined, and reinforced each other in some very constructive kind of way. In order to understand this better we shall seek a broader perspective by briefly surveying some the most memorable and significant of the turning points in the history of Bohemia that would be the natural 'landscape of the mind' for Bolzano as he grew up in late eighteenth-century Prague.

Of the numerous histories of Bohemia, the Czech peoples, and the Habsburg Empire the following works, in particular, have been consulted for this brief summary of the history and background of the society within which Bolzano grew up and was educated: Betts (1969), Kerner (1932) and Padover (1967). While origins so remote are inevitably uncertain, it was possibly a Roman tribe, the *boii*, that gave rise to the name Bohemia. It is likely that 'Czech' was simply the name of an early Slav leader. But without any doubt Bohemia was the scene, from medieval times to the Enlightenment, of some of the most dramatic and violent struggles in Europe over matters of government, religion, and language.

The Victorian Christmas carol 'Good King Wenceslas' is well-known in the English-speaking world for commemorating the Bohemian Christian Prince, Wenceslas, for bringing justice to 'the poor man gathering wood'. Wenceslas himself was violently killed by his brother after a political intrigue in AD 929. He was soon deemed a martyr and his memory seared into the consciousness of the Czech peoples as a veritable patron saint. In 1348, Charles IV as both King of Bohemia and Holy Roman Emperor founded the University of Prague. It was distinctive in having all four faculties for the arts, medicine, law, and theology, and soon becoming, along with Paris, Oxford, and Bologna one of the key centres of learning in the fourteenth century. Through the influence of John Wycliffe and his realist philosophy and theology there grew a strong relationship between Oxford and Prague. Numerous Czech theology students visited both Oxford and Paris. In 1390, the young Jan Hus, who was inspired and influenced by Wycliffe,

matriculated in Prague. It was a very low point in the history of the Church when, at the Council of Constance in 1415, Jan Hus was condemned to be burned at the stake for heresy. This was taken as a national insult in Bohemia and the subsequent Hussite reform movement, and associated wars, was not an isolated outbreak but a local form of profound revolt that was manifesting itself across the whole of Europe.

The Habsburg dynasty first came to power in Bohemia in 1526 through Ferdinand I. By the late sixteeth century two-thirds of Bohemia was still Protestant and Czech speaking, the other third Catholic and German speaking. In 1620, there occurred perhaps the single most important event in the history of the Czech people, the Battle of the White Mountain. The Protestant Estates army faced a much larger imperial force. The Czechs were routed and the victory of the Catholic forces was devastating. This marked the end of independence for Bohemia. Three quarters of the manors of Bohemia were confiscated, 30,000 families including many estate owners and nobility fled from Bohemia immediately after the Battle and after the Thirty Years war there were less than a third of the original population of three million remaining. The recovery, over the next two centuries, of a sense of nationhood and identity was slow and arduous. The spirit of reform, alongside the face of absolutism and imperial power, had a wide, complex, national presence manifesting itself in political, social, linguistic, and religious ways. Maria Theresa, in power from 1740–80, saw the beginning of some reform and enlightenment, including the suppression of the Jesuits in 1774. Joseph II continued the reforming measures in an extraordinary burst of benevolence and social improvement (not always appreciated by his people) from 1780–90. Perhaps not surprisingly in a Europe convulsed by the French revolution and its aftermath, after Leopold's short reign, the Emperor Franz (from 1792 to 1835) was a conservative and a reactionary, and reversed much of the reforming progress made by his predecessors. In the final decades of the eighteenth century when German became the exclusive language for governmental affairs in Bohemia, and the official language of the university in Prague, the very preservation of the Czech language and culture was fragile and uncertain. What in the end was the mainspring of the recovery of national identity that actually occurred was crucially dependent on the revival and re-telling of Czech national history. The themes of repression and reform, outlined in very brief strokes above, would likely have made a vivid impression and influence on the youthful Bolzano.

The state of almost uninterrupted war which existed in most of Europe between 1789 and 1815 formed the backdrop for Bolzano's childhood and early career. Bohemia's dominant intellectual movement was the so-called 'Catholic Enlightenment', which emphasized the themes of rationality and usefulness in all things and did much to promote education at all levels. Bolzano's father, an Italian art-dealer, had emigrated to Prague in the 1760s and there married a German woman, Cecilia Maurer. Of their twelve children only two survived to adulthood. Bolzano himself was not a strong child, but, in spite of headaches and a weak heart, he wrote, 'I was a very lively boy who never rested for a moment'

(Winter, 1976, p. 56). This disposition to incessant activity in the face of frequent illness did not abate as he grew older. There are, for instance, over 8000 sheets of manuscript in his mathematical diaries in addition to similar diaries in logic, philosophy, and ethics. In 1796, Bolzano entered the Philosophy Faculty in the University of Prague, and for four years he followed courses mainly in philosophy and mathematics. Although he found both subjects rather difficult, he soon discovered in pure mathematics ample scope for the foundational and conceptual investigations, which appealed to him so strongly. In his autobiography Bolzano recollected, 'My special pleasure in mathematics rested therefore particularly on its purely theoretical parts, in other words I prized only that part of mathematics which was at the same time philosophy' (Winter, 1976, p. 64).

In the autumn of 1800, Bolzano began three years of theological study. Although he was basically an orthodox Catholic, he found that his rationalist inclinations did not fit as comfortably as he had hoped with his theological studies. He came to realize that teaching and not ministering defined his true vocation. Educational value no doubt influenced his constant concern for the clarity and correct ordering of concepts in any exposition. While pursuing his theological studies, Bolzano also prepared his doctoral thesis on geometry, which was published in 1804 and is the first translation in this volume. Unable to obtain a mathematics post and torn over his choice of career, Bolzano seemed initially to face an unsure future. However, after deciding in favour of a theological post, events moved swiftly. On 5 April 1804, Bolzano was awarded his doctorate; on 7 April, he was ordained; and on 19 April, he was appointed to the newly formed professorship in religious studies at the University of Prague. Such a post had been created at all universities with a view to curtailing the then current wave of liberalism and free-thinking. In addition to courses of lectures, Bolzano was required to give weekly sermons twice to the students and citizens of Prague. He performed these duties with seriousness and enthusiasm and soon became highly respected and popular in Prague, with over a thousand people regularly attending his sermons. The population of Prague was approximately 80,000 at this time. Despite his successes in the pulpit, Bolzano was never politically suited to such a post, and his appointment was viewed from the start with suspicion by the authorities in Vienna. He would only use the authorized textbook in order to criticize it, and he held distinctly pacifist and egalitarian views. After a long process (which he resisted strongly), Bolzano was dismissed in 1819 for heresy, put under police supervision, and forbidden to publish. This enforced early retirement probably greatly lengthened his life—he suffered from tuberculosis—for he was subsequently able to spend much of his time recuperating, and writing, as a guest on the estate of his friends, Joseph and Anna Hoffmann, at Těchobuz in southern Bohemia. After this crucial turning point in his career, Bolzano began to work on his two major projects: the *WL* on logic, and the *Größenlehre* on mathematics. Although the restrictions on him were gradually lifted after Franz I died in 1835, Bolzano took no further active part in politics, or in the revolution of 1848, the year of his death.

During the 1820's Bolzano worked on *WL*. As we have seen this was regarded both by Bolzano and his readers, for example Husserl, as primarily a work on logic. But from a modern perspective this requires at least as much explanation and empathy as we need in order to understand 'science' in its eighteenth century sense (see the *Note on the Translations*). Logic for Bolzano was a very much broader and richer subject than either the narrow Aristotelian focus on conceptual analysis and on syllogistic kinds of arguments, still popular in the eighteenth century, or the formal mathematical logic that has dominated scientific thinking since the seminal work of Frege at the end of the nineteenth century. We may distinguish two senses of logic in *WL*. There is a very wide sense, what Berg (1973, p. 1) calls 'a kind of metatheory', in which the objects are the various scientific theories themselves. This sense includes a theoretical part that resembles what we might now refer to as a 'philosophy of meaning'. Here Bolzano introduces the notion of a proposition in itself [*Satz an sich*], or the objective content of a proposition,

> ... by *proposition in itself* I mean any assertion that something is or is not the case, regardless whether or not somebody has put it into words, and regardless even whether or not it has been thought. (*WL* §19 as in George 1972, p. 20)

Bolzano distinguishes this abstract notion of proposition from the concrete expression of a proposition in mental or linguistic ways. Not surprisingly this provokes extensive discussion of the distinction between whether *there are* such propositions (which Bolzano defended) and whether such propositions *exist* (which Bolzano disputed). An excellent source for further material on this from Bolzano in translation is Rusnock and George (2004). This wide sense of logic also contains important practical parts, it is what Rusnock (2000, p. 90) calls 'a methodology or theory of science, concerned with the organization and presentation of truths'. It includes heuristics for the discovery of truths and rules for how we should go about composing textbooks.

By way of contrast it is with the narrow sense of Bolzano's logic that we are likely to feel more at home today. Here he treats relations of deducibility [*Ableitbarkeit*] and of ground-consequence [*Abfolge*]. The former is actually what we would now call logical consequence. It is defined in terms of Bolzano's 'logic of variation'. Any part of a proposition which is not itself a proposition is called an idea in itself. When an idea is considered 'variable' by Bolzano this means we should consider the class of propositions that arise by successive substitution for that idea from among a class of ideas of the same kind. This is something short of the notion of a propositional function but it is a simple, original, and powerful notion. It allows Bolzano to define such concepts as compatibility, deducibility, validity and analyticity. For example,

> Propositions $M, N, O, \ldots$ are *deducible* from propositions $A, B, C, D, \ldots$ with respect to variable parts $i, j, \ldots$, if every class of ideas whose substitution for $i, j, \ldots$ makes all of $A, B, C, D, \ldots$ true also makes all of $M, N, O, \ldots$ true. (*WL* §155 as in George 1972, p. 209)

Bolzano's exposition at *WL* §155 is very detailed and thorough, extending to thirty-six numbered paragraphs and two explanatory notes. Several commentators have pointed out the similarity between Bolzano's definition of deducibility here and that of logical consequence given in Tarski (1983) originally appearing almost exactly one hundred years after the publication of *WL*. For example, see Etchemendy (1990). It should be noted that what we (and Rusnock and George) have rendered as 'deducibility' [*Ableitbarkeit*] is translated in Berg's works (and others) by 'derivability'. We take up the meaning of the ground-consequence relationship in the context of the early work *BD* where it undoubtedly had its roots (see pp. 18–20). What we have called the narrow sense of logic for Bolzano is almost entirely contained in the major section of *WL* entitled *Theory of Elements* that is about the nature and properties of ideas and propositions in themselves. The remaining parts of *WL* are concerned with logic in the wide sense.

This has been the merest sketch of Bolzano's logic and the contents of his *WL*. Space does not allow for further elaboration here of the remarkable and original treatment of logic in that work. Nor is it necessary because his logic is already much better served in English translation than his mathematics. The interested reader is referred to the editions of *WL* already mentioned, in addition to the detailed commentaries in Berg (1962) and Rusnock (2000, Ch. 4). On the logic of variation see also the chapters by Siebel and by Morscher in Künne (1997). For readers of French there is also an excellent further resource in the extensive and detailed study of both Bolzano's logic and his mathematics in Sebestik (1992).

The distinguished Bolzano scholar, Eduard Winter in *BGA* E1, has emphasized that for Bolzano logic and religion were inextricably related. On the one hand religion was the starting point for his logic. And at the same time Bolzano said himself that logic was the key to understanding his writings. The most fundamental role of logic becomes for him a moral, or ethical matter,

> The division of the totality of truths in disciplines and their presentation in individual treatises should be undertaken throughout in accordance with the laws of morality, and as a consequence also so that the greatest possible good (the greatest possible promotion of the general well-being) is thereby produced. (*WL* §395 as in Rusnock 2000, p. 91)

Bolzano understood religion in a rather special way. The supernatural was somewhat secondary and its agency, for example in miracles, was often to be understood metaphorically. Religion was primarily the wisdom by which people can live together more tolerantly. It is in this vision of logic and religion serving one and the same end that we might hope to find resolution of the enigma referred to earlier. Rusnock has described the position of logic for Bolzano as follows:

> The development of Bolzano's logic is thus guided by two strong principles: a commitment to make logic serve human ends, and an insistence on rigour in his characteristic sense. Neither asserts complete dominance: his logic might therefore be described as formalism with a human face. (Rusnock 2000, p. 92)

The provisional conclusion we draw here on the extraordinary blending in his life of energy and piety, of philosophy and mathematics, and of religion and logic, is that Bolzano was following in, and living out with great dedication, the historic and admirable tradition of Bohemian reform. He grew up at a time when this tradition was coming to public awareness in a new way through a revival of education in the Czech language in the schools and a vigorous movement at every level to recover the values and identity of the Czech peoples. Bolzano was a reformer in every sphere in which he worked. He sought social and political reform. He called for reform in the Church and in theology. He heralded and worked for reform in education at all levels in Bohemia. But principally he saw himself as having the vision and gifts to reform knowledge on a grand scale. Especially that fundamental preliminary to knowledge which was logic, and especially that part of knowledge for which he had obvious talent which was mathematics. This was his mission and his best way of promoting the common well-being.