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Abbreviations for the Works

Places, dates and original paginations are given for the first publication of those works that were published in, or close to, Bolzano's lifetime. For the works unpublished until recently (*RZ*, *F* and *F+*) the details are given here of the relevant volume in the *Bernard Bolzano Gesamtausgabe* (*BGA*). For each of the works these were the primary sources for the translation. Most of the works have also had other German editions published and these have always been consulted in the course of preparing the translations. Further details of these editions and other translations of some of the works are to be found in the *Selected Works* on p. 681.

BG Betrachtungen über einige Gegenstände der Elementargeometrie
Considerations on Some Objects of Elementary Geometry
Prague, 1804, X + 63pp.

BD Beyträge zu einer begründeteren Darstellung der Mathematik
Erste Lieferung
Contributions to a Better-Grounded Presentation of Mathematics
First Issue
Prague, 1810, XVI + 152pp.

BL Der binomische Lehrsatz, und als Folgerung aus ihm der polynomische, und die Reihen, die zur Berechnung der Logarithmen und Exponentialgrößen dienen, genauer als bisher erwiesen.
The Binomial Theorem, and as a Consequence from it the Polynomial Theorem, and the Series which serve for the Calculation of Logarithmic and Exponential Quantities, proved more strictly than before.
Prague, 1816, XVI + 144pp.

RB Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege.
Purely Analytic Proof of the Theorem, that between any two Values which give Results of Opposite Sign, there lies at least one real Root of the Equation
Prague, 1817, 60pp.

DP Die drey Probleme der Rectification, der Complanation und der Cubirung, ohne Betrachtung des unendlich Kleinen, ohne die Annahmen des Archimedes, und ohne irgend eine nicht streng erweisliche Voraussetzung gelöst: zugleich als Probe einer gänzlichen Umstaltung der Raumwissenschaft, allen Mathematikern zur Prüfung vorgelegt.

The Three Problems of Rectification, Complanation and Cubature solved without consideration of the infinitely small, without the hypotheses of Archimedes, and without any assumption which is not strictly provable. This is also being presented for the scrutiny of all mathematicians as a sample of a complete reorganisation of the science of space.
Leipzig, 1817, XXIV + 80pp.

RZ Reine Zahlenlehre
Siebenter Abschnitt. Unendliche Größenbegriffe.
Pure Theory of Numbers
Seventh Section: Infinite Quantity Concepts
BGA 2A8 (ed. Jan Berg) Stuttgart, 1976, pp. 100–168.

F Functionenlehre
Theory of Functions
BGA 2A10/1 (ed. Bob van Rootselaar) Stuttgart, 2000, pp. 25–164.

F+ Verbesserungen und Zusätze zur Functionenlehre
Improvements and Additions to the Theory of Functions
BGA 2A10/1 (ed. Bob van Rootselaar) Stuttgart, 2000, pp. 169–190.

PU Paradoxien des Unendlichen
Paradoxes of the Infinite
Leipzig, 1851, 134pp.

Other abbreviations used several times:

OED Oxford English Dictionary Ed. J.A. Simpson and E.S.C. Weiner. 2nd ed. Oxford: Clarendon Press 1989 and OED Online. Oxford University Press. Various dates of access. <http://dictionary.oed.com/>

DSB Dictionary of Scientific Biography Ed. C.C. Gillispie New York: Scribner 1970–

LSJ Greek-English Lexicon Ed. H.G. Liddell, R. Scott, H.S. Jones Oxford: Clarendon Press 1940

All references to German works, whether by Bolzano or by the translator, usually include the original German abbreviations. Some of the most common of these are:

B. = Bd. = Band = volume
S. = Seite = page
Th. = Thl. = Theil (old spelling for Teil) = part
Abth. = Abtheil = section
Aufl. = Auflage = edition

Note that a full point following a numeral indicates an ordinal in German so that 2.B 3.Thl has been rendered Vol. 2 Part 3.

Preface

Bernard Bolzano has not been well-served in the English language. It was almost 150 years after his first publication (a work on geometry) that any substantial appreciation of his mathematical work appeared in English. This was in the first edition of Coolidge's *The Mathematics of Great Amateurs* in 1949. A year later Steele published his translation of the *Paradoxes of the Infinite* with his own, still useful, historical introduction. Over the subsequent half century perhaps about a score of articles or books have been published in English, in all subject areas, that are either about Bolzano's work, or are translations of Bolzano. Some recent works from Paul Rusnock (see *Bibliography*) have gone some way to remedy this neglect in the area of mathematics. The present collection of translations is a contribution with the same purpose.

The main goal of this volume is to present a representative selection of the mathematical work and thought of Bolzano to those who read English much better than they could read the original German sources. It is my hope that the publication of these translations may encourage potential research students, and supervisors, to see that there are numerous significant and interesting research problems, issues, and themes in the work of Bolzano and his contemporaries that would reward further study. Such research would be no small undertaking. Bolzano's thought was all of a piece and to understand his mathematical achievements properly it is necessary to study his work on logic and philosophy, as well as, to some extent, on theology and ethics. Of course, it would also be necessary to acquire the linguistic, historical, and technical skills fit for the purpose. But the period of Bolzano's work is one of the most exciting periods in the history of Europe, from intellectual, political, and cultural points of view. And with over half of the projected 120 volumes of Bolzano's complete works (*BGA*) available, the resources for such research have never been better. The work on mathematics and logic has been particularly well-served through the volumes already published.

It was originally anticipated that each work translated here would be accompanied by some detailed critical commentary on its context and the mathematical achievements it contains. However, to do this properly a substantial proportion of the research described above would have to be completed. In particular, this would involve study of Bolzano's other mathematical and logical writings including his extensive mathematical diaries. To have prolonged the project even further in this way would hardly have been acceptable to any of the parties involved. Though there is, in fact, a certain amount of commentary in footnotes, particularly in *BG* and *RB*; the footnotes are confined, in the main, to translation issues or matters of clarification. Another relevant factor is that in the German *BGA* volumes there is detailed editorial comment on the mathematics as well as textual matters. This is invaluable and must be taken into account in any serious study of Bolzano's mathematics. But, of the nine works translated here, the *BGA* editions have only appeared for *RZ*, *F* and *F+*. In the case of these works, material from some of their most essential editorial footnotes has been included in this volume. The six other works will be published in the *BGA* series in due course. So whatever level of commentary had been given in these translations the coverage across the different works would inevitably be uneven.

Thus this volume contains little substantive assessment of Bolzano's mathematical context or his achievements. But the translations contained here, in so far as they are clear, accessible and mathematically faithful to their sources, will, I hope, prove to be a useful step in that direction. They are presented with a view to drawing greater, and wider, attention to the original works so that such an assessment, or at least work towards it, might more likely be made by other people. To help the reader gain some context and orientation on the works there is a general introduction together with short, more technical introductions to each of the three main parts into which the works are grouped: *Geometry and Foundations*, *Early Analysis* and *Later Analysis and the Infinite*.

I have learned a great deal through the preparation of this volume. First, and foremost, the original motivation for the translations—that Bolzano's mathematical work and thinking is still of sufficient interest that it deserves an English version—has been confirmed and reinforced in innumerable ways. Some of Bolzano's mathematics is very good by any standards; some is rather amateurish and long-winded, some is plain eccentric. But in each of the works included here, whether it be to do with notions of proof, concepts of number, function, geometry or infinite collections, his thinking is fundamental, pioneering, original, far-reaching, and fruitful. In each case his key contributions were taken up later by others, usually independently, and of course improved upon, but they all entered into the mainstream of mathematics. I know of no other mathematician, working in isolation, with such a consistent record of independent, far-sighted, and eventually successfull initiatives.

A second lesson for me is that translation is a profoundly interesting process. Translation is often viewed in the English-speaking world as essentially trivial. It is seen as a surface activity, like a change of clothes or a re-wrapping of a parcel: merely a matter of changing the form while preserving the content. Yes, there will be difficulties, so this line of thinking goes, but they are on the 'puzzle' level, a bit of fiddling about and compromise, and a 'good enough' solution will emerge. The translator is seen as subservient to the author and the source text, performing an activity—ideally invisible—of replicating the meaning of the source text in the target language which, if it is English, is tacitly assumed always to be fully fit for the task. Translation is not itself seen as an intellectually interesting, or significant, process. This is not, I think, a caricature, but a widespread attitude, and it must be admitted that translation can be, and often is, done in just such a functional fashion. But such an attitude does little justice to the miracle that is language: the extraordinary richness, colour, mystery, nuance, and unconscious self-expression present in every communication. I wish I had learned earlier than I did of the exalted vision of Walter Benjamin for the translation process (Benjamin, 1999). Then there is also the glorious, detailed, celebration of translation from George Steiner (Steiner, 1998). I would be out of my depth in attempting any summary here of those authors' magisterial works. But I wish to put on record my debt to them for the inspiration and insights that I have gained from these works in particular, among the many others in the rapidly growing field of Translation Studies. It should be noted that Benjamin and Steiner are primarily discussing literary translation. Bolzano's works are neither literary nor technical though they contain technical parts. I have suggested in the *Note on the Translations* that these categories may not be particularly helpful in the context of translation.

Natural languages are not codes. Unlike the situation with a logical calculus, or with formal languages, there is just no possibility of an equivalence of meaning, or equivalence of effect, between natural languages. By their nature they represent vastly complex networks of meanings, associations and usages that are the incommensurable, and constantly changing, products of historical, social, and cultural forces. In practical terms, contemplating a source text, this means there will generally be a huge variety of choices for the exact form of the target text, and a negotiation must ensue between these choices and the source text within a context that includes the purposes of the texts, knowledge of the languages and the subject matter, and the prejudices and inclinations of the translator. The translation process therefore essentially involves an interplay between two kinds of meanings—those that can be accommodated in the source text, and those that can be accommodated in the target language.

A final, and quite unexpected, insight to emerge from this work is that the main strands that inevitably commingle throughout, namely the disciplines of history and of mathematics, and the process of translation, share a common characteristic mode of procedure: namely, what we have just concluded about translation, that at the heart of the process is an interplay between two kinds of meanings.

It is not hard to see the importance of this interplay of kinds of meaning within the discipline of history. To describe the way in which some events, or ideas, were understood at a certain time and place, or how that understanding changed, there are two, closely interacting, stories to tell. There is the view from the time itself, a view seen with the light available at the time, and the sensitivities of the time and the community concerned. But we who are telling the story, trying to take ourselves back into that earlier time, we are ineluctably of our own day, having our own hindsight, education and prejudice; it is a deliberate, imaginative, more or less informed, more or less sympathetic, effort to act as mediator, re-presenting events and ideas of the past in a theatre of the present. The history of mathematics shares completely in this dual aspect of history in general and is perhaps better thought of, at least when the emphasis is on the history, as part of the history of science or, better still, the history of ideas.

Since at least the time of the great Greek contributions to mathematics, there have been those who have explored mathematics for its own sake. Euclid's *Elements* is an outstanding example of assembling knowledge about geometry and arithmetic so as to display and emphasize its deductive structure rather than its practical use. In terms of knowledge about numbers it was *arithmetica* (the science of numbers) rather than *logistica* (or practical calculation). This distinction has been preserved and hallowed ever since. But the vast majority of mathematics throughout history has been motivated by, inspired by, or in the service of, other studies such as astronomy, navigation, surveying, physics, or engineering. This is what came to be called 'mixed mathematics', or by the nineteenth century, 'applied mathematics'. By contrast, the mathematics done for its own sake, became 'pure mathematics', or more commonly in the eighteenth and early nineteenth centuries, 'higher mathematics', or 'general mathematics', which would apply—as a science of quantity—to all quantities in general, with subjects like geometry, arising by specialization to spatial quantities. The abstract entities of pure mathematics, such as numbers, functions, ideal geometric extensions, have a universal, necessary, even—following Kant—an *a priori*, quality. It becomes a challenge for any philosophy of mathematics not only to explain the nature of such entities—seemingly pure and untouchable—but most importantly how they can become embodied and partake in the messy particularity of physical things such as chunks of steel and earth. In other words, how is applied mathematics possible? Here, the interplay of meanings arises in an especially interesting way. The meanings of the very general primitive concepts involved in arithmetic, analysis, geometry, set theory and so on, have typically, since the end of the nineteenth century, been given in a self-contained, implicit fashion through axiom systems. The choice of primitive concepts, and the framing of the axioms have, of course, been guided, or governed, by the mass of informal, intuitive meanings associated with the relevant domain. But when axiomatic theories are 'applied', it is those detached, 'sanitized' meanings that must return to their origins and interact with the meanings arising from particular, experienced, physical things, or patterns of observations. Such issues arising from the existence and success of applied mathematics are closely connected with lively debates in recent decades within the philosophy of mathematics. See, for example, Kitcher (1983) and Corfield (2003).

The work on these translations over many years has been accompanied, and delayed, by the everyday demands of research and teaching in a leading Department of Computer Science. That research has involved thinking about issues that are fundamental in computing. A particular interest of my own is how we might achieve a much closer integration of human and computer processes than has so far been exhibited by conventional computer systems. Typical questions that arise in pursuing this concern are the following. How is it possible to represent parts of the world? (Humans seem very good at it, but in order to have computers solve problems, or assist in doing so, we have to represent those problems somehow in the computer.) How far can human experience be represented on computers? How is it that we 'make sense' of our experience? How does my 'sense', or view, of some part of the world relate to yours? How does the private become public? A common feature of these questions is that they all have to do with semantics—the formation and communication of meanings. There are two kinds of semantics for a computer program. One is the process in machine memory that the program, and its inputs, evokes. That process may be hugely complex, but it is constrained within the memory and may be studied abstractly and mathematically. This is what the computer scientist usually means by the semantics of a program. The end-user of a program however, is apt to merge the program and its process, and the semantics of the program (and process) is then what the results mean for some application in the world (e.g. the projection of a business budget). The two kinds of semantics are closely related (e.g. via the display or other devices). It is this interplay between meanings of different kinds, that is crucial to the usefulness and the progress of a great deal of computing.[a]

So the fledgling discipline of computer science exhibits a similar interplay of two kinds of meanings in the important area of the semantics of programming. This observation was in fact the origin of the identification of the same commonality between the process of translation, and the disciplines of history and applied mathematics. No doubt there are other disciplines, or practices, that also exhibit this phenomenon. But it articulates, with hindsight, a common theme among my own interests, which was apprehended and felt at an early stage. The fact that this common theme has to do with meaning could hardly be more appropriate for a collection of works by Bolzano. The identification and understanding of the concept of meaning was at the heart of Bolzano's thinking. The attempt at a full-scale philosophical account of meaning was the substance of Bolzano's major logical work, *Theory of Science*. In the words of Coffa '[Bolzano] was engaged in the most far-reaching, and successful effort to date to take semantics out of the swamp into which it had been sinking since the days of Descartes' [Coffa, 1991, p. 23].

It seems to me that the separation of pure and applied mathematics in the later nineteenth century was attended by a kind of discontinuity, a tearing in the fabric of mathematical knowledge. Perhaps it was the degree of abstraction, or the reliance on axiom systems, but after this change there was no longer the semblance of an organic wholeness in knowledge. There was a severance of the connectivity, the mutual exchange and interaction of meanings, between experience and theory, and between sensation and thought. For much of applied mathematics, and engineering, this may not have had an adverse effect. Perhaps the 'experiences' required could usually be correlated with patterns of behaviour, or observation statements, which in turn could be successfully associated with mathematical variables. But the technology of computing now allows for, and requires, a deeper level of engagement between a 'formal' machine and experience. This

[a] I am indebted to the work of Brian Cantwell Smith for this analysis of the semantics of a program, for further details see [Smith, 1996, p. 32 ff.].

is an engagement at a cognitive level, more primitive than the recognition and use of standardized behaviours, one in which observations and their interpretation are context dependent, open and negotiable. It is the very nature of modern formal systems that make them, despite any amount of ingenious elaboration, unsuitable to act as foundation, or 'grounding', for the ways in which we are now using, and thinking about, computing. That nature, I believe, can be traced back in many ways to the nineteenth century. In order to develop a broader theory of computation, one that would include the existing theory but better support the ways computers are being used, and the closer integration of human and computing processes, we need, arguably, a breadth of outlook that would embrace both the formal and the informal. We need to restore the connectivity and wholeness that, I am suggesting, was broken during the nineteenth century. Of course, any such restoration, or enrichment of the formal sciences, is now a twenty-first century matter. Indeed a broader theory of computation, possibly based on just such an enrichment, is one of the current goals of the Empirical Modelling research group at the University of Warwick with which I have been actively involved for the past ten years.

History can, however, give us insight that would be foolish to ignore, into the weaknesses, or the fault-lines, the stresses and strains, the needs and motives, which attended the ways that pure mathematics and logic developed in their most formative decades. It would be a major historical, and philosophical, project to examine the suggestion that this development involved some kind of lasting discontinuity, or qualitative change, in the nature of mathematical knowledge. Bolzano's work is not a bad starting place for this study. He was thoroughly imbued with the integrated thinking about mathematics of the eighteenth century, yet he was also, ironically, to have some of the ideas that would contribute, later in the nineteenth century, to the very separation I have decried. But Bolzano's work would only be a starting point. The centre of gravity of this project would doubtless lie later in the nineteenth century.

I have always looked back with gratitude to the late G. T. Kneebone (of the former Bedford College of London University) as the one who first suggested Bolzano to me as 'an interesting person . . . only mentioned in footnotes' in relation to logic and the foundations of mathematics. I am very pleased to thank Clive Kilmister, formerly of King's College, London, who encouraged me in this area of study, Dan Isaacson of Oxford who has enthusiastically supported the work from the earliest days, and Graham Flegg, a founding member of the Open University, who very effectively managed my PhD thesis on Bolzano's early mathematical works (Russ, 1980*a*). That thesis had a long appendix consisting of (rather crude) translations of the five early works. It was the seed from which this volume has grown.

Over the years in which these translations have been prepared I have worked closely with two particular colleagues, Meurig Beynon and Martin Campbell-Kelly, who, in their very different ways, have been an indirect, but profound, influence to broaden and deepen my thinking about the nature of mathematics,

computing, and history. That has helped to shape and improve this work and I am grateful to them both. It is a personal sadness that John Fauvel, a great friend and collaborator, who did so much to support the completion of this volume by way of encouragement, vision, and advice has not lived to see its completion. His faith in his friends was a powerful source of energy for many people and many projects. It is a particular pleasure to record the debt I owe to David Fowler for his wonderful warmth and wisdom, his inspiring care and enthusiasm for the history of mathematics and for his unremitting prodding, teasing, and scholarly advice to me, which is now finally bearing fruit. Again, alas, the pleasure is overshadowed, at the time of writing, by sorrow at the loss of David after a long illness.

Both in the early stages and the more recent stages of preparation I have been very fortunate to have been able to draw on the expertise, and vast knowledge of Bolzano's mathematics, possessed by both Jan Berg and Bob van Rootselaar—through their published editorial commentary in the *Gesamtausgabe* volumes and through personal contact. Their unfailing detailed advice and support has been truly invaluable; I only regret that time has not allowed the inclusion of more of their technical insights from the *BGA* volumes. It has also been an enormous benefit in the final stages of preparation of the translations to be able to incorporate a substantial number of detailed and careful corrections, revisions, and suggestions from Annette Imhausen and Birte Feix. It was a great pleasure to be able to visit the Bolzano-Winter Archive in Salzburg in July 2003 and benefit from the resources there and the deep knowledge, and wise advice, of Edgar Morscher. Also in the later stages of this work it has been invaluable to benefit as I have from the encouragement, advice, and expert knowledge of both Peter Simons and Paul Rusnock. I am also glad to acknowledge the role of a former research student, David Clark. His thesis, Clark (2003), concerning the meaning of computing and the relationship of language to programming, is strongly connected with some of the issues touched upon earlier in this *Preface*. Our discussions were undoubtedly helpful to my own thinking about issues important for Bolzano's work and for computing. A great deal of the complicated typesetting of these works is due to Howard Goodman to whom I remain grateful for introducing me to the workings of TeX, the world of font families, and the subtle aesthetics and benefits of good typesetting. I am pleased also to thank Xiaoran Mo and Vincent Ng for help with some of the typesetting and for re-drawing all Bolzano's original figures so they could appear in convenient positions within the text in *BG* and *DP* for the first time. I am grateful to Ashley Chaloner for a postscript program that allows for the flexible display of the Bolzano function illustrated on p. 352.

A great number of other people have helped very significantly in the preparation of these translations. Help in terms of moral support is as valuable in a major project as technical advice. I have enjoyed both kinds of support for the project as a whole, or for specific parts of it, from the following: Joanna Brook, Tony Crilly, William Ewald, Jaroslav Folta, Ivor Grattan-Guinness, Jeremy Gray, the late Detlef Laugwitz, Dunja Mahmoud-Sharif, David Miller, Peter Neumann, Graham Nudd, Karen Parshall, Hans Röck, Jeff Smith, Jackie Stedall, the late Frank Smithies,

Julia Tompson, Claudia Wegener and Amanda E. M. Wright. My apologies, and thanks, go to any others whose names have inadvertently been omitted here and who have contributed in one way or another to this volume.

I am grateful to the University of Warwick, and in particular the Department of Computer Science, for their patient and generous support of this work, over the years, in innumerable ways. It is also a pleasure to acknowledge the support of a History of Science Grant from the Royal Society in 1992 which helped to fund visits to libraries and collections in both Prague and Vienna. I am pleased to acknowledge assistance from the Austrian National Library in Vienna for supplying digitized images of the title pages for *BG*, *BD*, *BL*, *RB*, and *PU* as well as the three images of manuscript pages from *RZ* and *F*.

Finally, I am very pleased to express my great thanks to Oxford University Press. All long relationships have times of strain, when character is tested, and patience and faith are stretched. I was gratefully amazed at the patience and understanding of Elizabeth Johnston during a long period when circumstances prevented my making progress with the work. I count myself fortunate that the new, vigorous management style of Alison Jones came at a time when I was, finally, able to complete the work; it was just what was needed. I am also very grateful to Anita Petrie who has helped calmly, and professionally, to smooth the way through a complex production process.

In spite of so much assistance I am all too conscious that the work will inevitably still contain errors, omissions, and defects of many kinds for which I alone am responsible. I should be very grateful to receive details of these as they are identified by readers. Corrections, comments, and suggestions for improvement may be sent to me at sbr@dcs.warwick.ac.uk. I anticipate maintaining a website with the original German texts translated here and with all known corrections as these are discovered. This website will be at http://www.dcs.warwick.ac.uk/bolzano/

The referencing system adopted in the volume is very simple. All references to works by Bolzano are made by a group of one or more upper-case characters and identified in the *Selected Works* section. The translated works have various textual forms further identified by a number. For example, the version of *PU* edited by van Rootselaar is referenced as *PU*(5). The selection of Bolzano's work is a very small fraction of what is available, being only what is needed for this volume. There is extensive bibliographic detail in the sources mentioned at the beginning of the *Selected Works*. All references to works by other authors are in a standard Harvard Style and included in the *Bibliography*.

Note on the Texts

The German source texts of the nine works translated in this volume are very varied in their origin and status. The first five works were published between 1804 and 1817, each by different publishers in Prague or Leipzig, and no manuscript copies are known to remain. None of these works has yet appeared in the *BGA*. But important traces and precursors of the ideas for these works are to be found in the manuscript material of the mathematical diaries appearing in the *BGA* Series II B. These diaries have already been published for material written up to the year 1820.

The next two works translated in this volume, *RZ* and *F*, date from the 1830s and until their appearance in the *BGA* (as volumes 2A8 in 1976, and 2A10/1 in 2000, respectively) they had only been published in partial forms. The works were part of Bolzano's mathematical legacy bequeathed to his former student Robert Zimmermann who took up a post as professor of philosophy at Vienna in 1850. The manuscripts came with Zimmermann and remain to this day in the Austrian National Library. They were 'discovered' (among very many other mathematical manuscripts) about 1920 by M. Jašek who realized their significance and began to publish material (mainly in Czech) about them. For further bibliographic details see Jarník (1981). The discovery led to a version of *F* being published by the Royal Bohemian Society for Sciences under the editorship of K. Rychlík in 1930. There are three versions of the manuscript for *RZ* and two for *F*, some are in Bolzano's own hand, others are in the hand of a copyist with corrections and additions by Bolzano. It should be noted that what is referred to here as *RZ* is only the final section of the whole work published in *BGA* 2A8. It is this same section that was given a partial publication in Rychlík (1962). Bolzano's handwriting can be notoriously hard to decipher and Rychlík included only the more easily readable parts and omitted parts he thought were not relevant. In fact some of the parts he omitted were significant improvements to Bolzano's theory. And he included material crossed out by Bolzano. The present author claims no proficiency in reading Bolzano's handwriting and has relied entirely on the expert editors of the *BGA* volumes. The 1930 edition of *F* is, according to van Rootselaar the editor of the *BGA* edition, 'still a valid edition and has always been taken into account in preparation of the present edition' [*BGA* 2A10/1, *Editionsbericht*]. However, it did not include the important improvements and corrections, found later within another separate manuscript and reported on in van Rootselaar (1964). These are included in *BGA* 2A10/1 and in the present translation as *F+*.

In writing his manuscripts Bolzano frequently referred to previous paragraphs by means of the symbol § without any specific numbers. In many cases the *BGA* editors have identified the appropriate paragraph number but in many other cases, especially in *F* and *F+*, the specific paragraph could not clearly be identified so only the section symbol § appears.

The final work *PU* was published posthumously in 1851 following Bolzano's request to his friend Příhonský to act as editor in its publication. Until recently it was believed that only substantial parts of the manuscripts that Příhonský used are extant, but not the final version used by the publishers. However, in the summer of 2003 Edgar Morscher discovered the printer's copy among the papers of Příhonský in Bautzen.[b] For further details of *PU* see Berg (1962, p. 25).

For the early mathematical works there is a facsimile edition *EMW* but both this and the original copies of these five works are now quite rare. Uniquely among these works *RB* also appeared in the Proceedings of the Royal Bohemian Society of Sciences and so enjoyed a significant European circulation. Schubring (1993) is a useful report on a number of reviews of several of the early works, especially in Germany, showing there was significant distribution to some important centres. With the exception of *BL* they have all had later editions as indicated in the *Selected Works*. Each of the first editions of these works has a significant number of errors, either deriving from Bolzano's manuscripts or from transcription errors by the printers. In each case the subsequent editions have corrected some first edition errors but introduced further errors themselves. The work *BL* had its own list of misprints included at the end but there are misprints and numerous omissions even in this list.

The original publications of these early works were made in the German Fraktur font. The convention for giving emphasis in this font was to adopt an extra 'spacing' of the characters of a word. This has been reproduced in these translations by the use of a slanting font although this has in many cases led to a more frequent use of emphasis than would now be usual, and in some cases hardly seems appropriate. It may, of course, have occurred sometimes in error, or merely to help the printer in justifying a line. We have sought to retain it in all cases.

For the works *RZ*, *F* and *F+* the emphasis is given by italics in the *BGA* volumes which corresponds to underlining in the manuscript, and such cases have been rendered with a slanting font in the translations. There is also excellent commentary by way of footnotes in those volumes on variants and corrections in the different manuscript versions. *PU* is the first publication of Bolzano's work appearing in a Roman font and having the diagrams *in situ* within the text.

Since the present volume is primarily for those who cannot easily read the original German it is not appropriate to give detailed accounts of the variations and errors in those texts. After taking account of later editors' comments and corrections such errors have mostly been corrected silently in making

[b] Personal communication, February 2004.

the translations. Occasionally attention is drawn to uncertain, or particularly significant, cases. Fuller details of all known variations and corrections do appear, however, in the electronic version of the German texts on the associated website (see p. xix).

Where footnotes by the editor of *BGA* 2A8, Jan Berg, have been translated and used here in *RZ* this has been indicated by '(JB)' following the footnote. Bob van Rootselaar, the editor of *BGA* 2A10/1, has authorized a general use of his footnotes in the translations *F* and *F*+. For all these works *RZ*, *F* and *F*+ it is the *BGA* editions that have been the source, not the original manuscripts transcribed in those editions. Thus when there appear to be errors in the source this may be due to original error in the manuscript or to transcription error. When categorical statements are made about errors these have been checked with the editor concerned.

Note on the Translations

The first five of these translations began life as the appendix to an unpublished PhD thesis (Russ 1980*a*). A version of *RB* very similar to the thesis version was published as Russ (1980*b*). Revised versions of *BD* and *RB*, and the *Preface* of *BG*, appeared in Ewald (1996). In the main these revisions are the translations by the present writer in their versions of the time (1994). They have significant differences from the earlier thesis versions, which in some important cases are due to improvements made by William Ewald. The versions of all five early works appearing in the present volume are so different from the thesis versions, and from those in the Ewald collection, that it is hard to find a single sentence in them in common with the earlier versions. This is partly because of errors or omissions being corrected, but it is also witness to the flexibility and richness of language, the wider experience of the translator and the changing context in which the translations are presented. A text that deals significantly with concepts and meanings that are open to interpretation, rather than being tightly constrained or closed, thereby has a complexity and dynamic, a purposefulness, and yet an autonomy, that fully merits the metaphor of having a 'life' of its own. The present versions not only reflect better knowledge of the source and target languages but also of the purposes and contexts, both cultural and technical, in which the texts were written and in which they are now being re-presented.

The works *BL* and *DP* are published here in English in their entirety for the first time. There have been short extracts of the former work that appear in Rusnock (2000, pp. 64–69) and of the latter work contained in Johnson (1977). The major works *RZ* and *F* went unrecognized and unpublished in any form before the twentieth century; they have also not appeared in English before apart from some short extracts of each of them to be found in Rusnock (2000) and some extracts of *F* in Jarník (1981).

The translation of *PU* in Steele (1950) was the first of any of Bolzano's works to be available in English. It contains an historical introduction and a detailed summary of the contents of the sections laid out in parallel to the similar summary given by the posthumous editor, Příhonský, but reflecting 'a more modern analysis'. Steele has in many ways taken greater liberties with both text and mathematics than I have done and yet his translation still reads for the most part like an older style of English than its date would suggest. For both these reasons, as well as the intrinsic merit of the work, its influence in the nineteenth century, and the difficulty of obtaining Steele's version, it has seemed appropriate to include a new translation in this volume.

It is likely that a translation is generally read by people who are not in a position to evaluate the quality of the translation in its relation to the original source text. Nevertheless, it may not be without interest, and relevance, to the reader to know something of the translator's perception of both the process and the product.

The explicit principle governing these translations, for much of their life, has been to preserve everything to do with Bolzano's mathematics as faithfully and accurately as possible. This has led to the retention of his original notations and layout (unless these were obviously a restriction or convention of the time, such as the placing of collected diagrams on a foldout page). It has also led to the attempt to reflect closely the original terminology and thereby it has occasioned a somewhat literal rendering of the text, sometimes at the expense of a more natural English. The latter is probably inevitable, and even desirable, if a priority is to be made of the thought over the language in which it is expressed in so far as this possible. But the basic principle of 'preserving the mathematics', while perhaps sounding innocent enough is, I now believe, naïve and flawed. The mathematics cannot, in general, be sharply separated from the insights and the attitudes to concepts and proofs, or beliefs about the status of mathematical objects, or even the motives for developing new theories. And even to the extent that the mathematics *can* sometimes be considered apart from these informal surroundings, the 'meanings' of either the mathematics, or of its informal framework, cannot be 'preserved'. We experience thought as almost inseparable from language; it is commonplace to find we do not know our thoughts until they are articulated, by ourselves or others. Thus it is not to be expected that we can translate Bolzano's language into the form he would have used if he had been expressing his thoughts in English. They would, in English, have been different thoughts. It is, in general, just not possible to separate content sharply from language. The challenge, therefore, is to translate both language and thought together. It is perhaps more useful to think in terms of transformation. The work of translation becomes that of transforming Bolzano's German thoughts into English thoughts in a way that respects their meanings—bringing them as close together as language and our understanding make possible.

It is a pleasing, albeit somewhat misleading, pair of images that Benjamin conjures in *The Task of the Translator*: 'While content and language form a certain unity in the original, like a fruit and its skin, the language of the translation envelops its content like a royal robe with ample folds' (Benjamin 1999, p. 76). But, of course, this is not always so. There is often not such unity in an original text and to be sure about the 'folds' of the translation assumes a direct access to the content of the original text. It is all too easy, in the first instance, to 'read into' the translated text a content that again fits skin-tight to the target language. Nevertheless Benjamin's imagery makes a serious point vividly: the same content will not 'fit' its expression in different languages equally well. The translator is thus not subservient to the preservation of an author's content but may at times need to re-create ideas afresh in the target language. It is this exalted, creative vision of translation that Benjamin extols in his essay. He suggests that translation is not so much a transmission as a re-creation. It is only information, he says, that can be transmitted and that is the inessential part of a text: attending primarily to the transmission function of a translation 'is the hallmark of bad translations'. With that, all the translations of this volume might seem to be condemned; it has been my chief aim to convey Bolzano's way of thinking about mathematical domains, about proofs and concepts, his particular insights into, among other things, numbers, functions and multitudes. Is this 'information'? For Benjamin the essential substance of a text is what it contains in addition to information, 'the unfathomable, the mysterious, the "poetic" '. A consequence of this, again calling for the re-creative function, is that a translation can be seen as part of the 'afterlife' of the original. The initial shock at Benjamin's apparent disparagement of the information content of a text is partly relieved by the fact that he explicitly refers to 'literary' texts.

It still appears to be common in writings on translation to make a broad distinction between 'literary' and 'non-literary' texts, or between 'literary' and 'technical' texts. This is in spite of some extensive studies of text types (such as in Reiss 2000). Although the meaning of 'literary' here is obviously wide, such binary divisions are clearly inadequate and unsuited for their purpose. The *OED* suggests that a primary meaning of 'literary' is 'that kind of written composition which has value on account of its qualities of form'. But on such a criterion a great deal of good mathematics would be literary. The widely admired qualities in mathematical writing of succinctness and clear structure, of economy and precision, and of appropriate notations are pre-eminently qualities of form. Such a classification is presumably not intended. A good deal of philosophy is surely both literary and technical and most texts are neither. As a counterpart to the wide spectrum of text types is the huge range of contents, or meanings, that lie in between Benjamin's 'information' and 'the unfathomable'. In this space lie assumptions, motives, contexts, viewpoints, interpretations and the use of metaphor, among many other components essential to texts of all sorts. All Bolzano's texts in this volume reside in large part in this space. This is for the simple reason that he is, in each of his works, exploring uncharted territory—he is doing original, radical, conceptual analysis of the abstract objects of mathematics, of meanings, of logical relationships, and of the nature of the infinite. Perhaps the spectrum of texts might be characterized with respect to translation in terms of their degree of openness to imaginative and varied interpretation. Thus literary or poetic texts would be deemed highly 'open', while more factual texts such as instruction manuals with their schematic diagrams would be relatively 'closed'. The place of scientific, or technical, texts would depend very largely, I suggest, on the place of those texts within the discipline at the time of their composition. The more they are presenting material that is original, fundamental, and tentative, the more open they are in the sense employed here. It is worth reflecting on the fact that the idea of a 'technical term' with which we are so familiar today, would mean little more at the time Bolzano was writing than the much weaker idea of a 'term of art'. Many of the very words Bolzano was using for major philosophical ideas had only been introduced as German (rather than Latin) philosophical terms less than a century before he began writing. For example, Christian Wolff introduced early in the eighteenth century *Vorstellung* for 'representation' (*repres-entatio*) in general, but it shows how fluid matters were that for representations of *things* Wolff introduced *Begriffe* (concepts), while decades later Kant, for the same purpose was using *Erkenntnisse* (cognitions). To the extent that the translations of this volume engage with, and re-create the exploratory philosophical aspects of Bolzano's work they are open to interpretation, and call for interaction with the reader. In this respect they enter, I hope, at least part of the way into Benjamin's vision for translation.

I wish I had known earlier about the range of approaches and valuable insights already gained in the burgeoning and important subject of Translation Studies. A useful short introduction to the central issues and history of the subject may be found in Bassnett (2002). One of the major strands of theoretical thought in recent decades, which embraces both linguistic and cultural perspectives on translation, is the so-called 'functionalist approach'. The main argument of functionalist approaches is the apparently innocuous idea 'that texts are produced and received with a specific purpose, or function, in mind' Schäffner (2001, p. 14). That this might offer a governing principle for translation can be traced back to the *skopos* theory put forward by Vermeer in 1978 in German but with an English exposition in Vermeer (1996). Schäffner goes on to suggest that since the purpose of the target text may be different from the purpose of the source text, arguments about literal versus free translations, and similar contrasts, become superfluous: the style of translation should match the purposes of the texts. In this context the traditional dimensions of faithfulness and freedom in translation clearly become less significant.

The issue of purpose has been a useful consideration for the present volume. In the case of these texts of Bolzano there were undoubtedly multiple purposes. Some of them, such as making his ideas known to mathematicians of the time, and of gaining feedback on the value of the approach he was adopting, are clearly ones which cannot pertain to this translation. The time has gone. One purpose of these translations is that English-reading scholars of the early twenty-first century might appreciate in detail the context, insight, novelty and substance of his ideas and contributions to mathematics. A practical consequence of this purpose includes the demands mentioned above of preserving Bolzano's notations and paying close attention to his terminology. So although the principle enunciated above, of 'preserving the mathematics', may be flawed theoretically the outcome in practice has, I hope, not suffered unduly.

Translation has played an important, but neglected, role in the long histories of several scientific subjects. For an example of a rare study of translation in science see Montgomery (2000). Now that systematic and rigorous studies are being made to understand the way translation processes operate it will be important that historical scientific works also become part of Translation Studies, and that the theory and practice of translation maintain close connection over the whole spectrum of text types. It is possible that some standard and long-accepted translations could usefully both be informing current translation studies and themselves be reconsidered in the light of such studies.

In many translations of scientific and philosophical texts, including some works of, or on, Bolzano (e.g. George 1972, and Berg 1962), it has been common to give lists of the principal 'equivalences' between key German and English words. Although such lists may sometimes have their place, for example with certain technical terms, it has not seemed appropriate here. It will be clear from the whole tenor of this *Note* that I am sceptical about any attempts to mechanise the translation process. I hesitate to proclaim the consistency that the idea of equivalence suggests, and in any case I am unconvinced of the wisdom of aiming at a strict consistency. It has, however, often been convenient to give the original German word or phrase in square brackets following its translation. For example, it may be useful for the reader to know that it is the same German word (*Grund*) that has been variously rendered 'basis', 'foundation', 'ground' or 'reason' (among others). And conversely, that several German verbs *beweisen*, *erweisen*, *nachweisen*, *dartun* have, on occasion, all been rendered by the appropriate form of 'prove'. While *beweisen* has almost always been translated 'prove', the other terms mentioned have also given rise to suitable forms of 'establish', 'demonstrate' or 'show'.

Bolzano uses the verbs *bezeichnen* and *bedeuten* very frequently and as an experiment in consistency they were, for some time during revision of these translations, systematically rendered 'designate', or 'denote', respectively in such contexts as 'Let *x* designate (denote) a whole number'. It now seems unlikely there is any systematic distinction of meaning intended by the choice of these terms, but this explains the frequency of occurrences of the slightly cumbersome 'designate'.

The important term *Wissenschaft* is used on several occasions in *BG* and *BD*. It means a body of knowledge, especially knowledge rendered coherent or systematic by its subject matter, or the way in which it was acquired. There is no equivalent modern English word. In spite of the large range of alien connotations it is usually rendered by 'science' and after some experiment with 'subject' and 'discipline' I have fallen in with the conventional term. But the reader should strive to think only of the eighteenth-century meaning of 'science'. Dr Johnson's *A Dictionary of the English Language* (4th edition, 1773) gives as the first meaning for the entry *science* simply 'knowledge'. The *OED* offers 'knowledge acquired by study' and 'a particular branch of knowledge', each illustrated by quotations up to the early nineteenth century. Fichte (twenty years Bolzano's senior) used *Wissenschaftslehre* to mean an overall system of thought, and indeed one of his titles (in translation by Daniel Breazeale) runs *Concerning the Concept of the Wissenschaftslehre or of So-Called Philosophy* (in the collection Fichte (1988)). Breazeale decides not to translate *Wissenschaftslehre* at all in his own work while remarking that "Science of Knowledge' which has long been the accepted English translation of *Wissenschaftslehre*, is simply wrong' (Fichte 1994, p.xxxi). Bolzano's major philosophical work, also entitled *Wissenschaftslehre*, has been translated in each of the editions George (1972), and Berg (1973), with the title *Theory of Science*. In the opening section Bolzano explains that *Wissenschaft* has no generally accepted meaning (this is in 1837) and he declares his own meaning: a collection of truths of a certain kind, provided enough of them are known to deserve to be set forth in a textbook. It is clear from the title page of the work that he conflates his *Wissenschaftslehre* with his own broad understanding of logic. For further details the interested reader should consult the editors' introductions in the two translations just mentioned. Returning to the early works translated in this volume, the adjective *wissenschaftlich* has been translated, uncomfortably, but now for obvious reasons, as 'scientific'.

The range, references and connotations of *Größe* in German are different from those of 'quantity' in English which is nevertheless often the best translation. Each term has a complex pattern of usages and meanings overlapping in English with those of 'magnitude' and 'size'. When those latter terms are used in these translations they are always translating *Größe*, so we shall not usually indicate the German. But just as 'quantity' in early nineteenth century English was sometimes synonymous with 'number', so *Größe* was sometimes close to, but not the same as, *Zahl*. 'Number' in this volume is usually translating *Zahl* or *Anzahl*, so the occasions where the source has been *Größe* are indicated with square brackets or a footnote. (For example, both devices are used for this purpose on p. 87.) Quantity is the more general term embracing number, space, time, and multitude. So the work *RZ* (*Pure Theory of Numbers*) is a part of the overall unfinished enterprise *Größenlehre* (*Theory of Quantity*). Thus the late alteration by Bolzano of *Zahlen* into *Größen* at the opening of *RZ* on p. 357 is of interest and some surprise.

For a long time I have been convinced that 'set' is not the appropriate translation of Bolzano's use of *Menge* although it has been rendered this way in all previous translations of his mathematical work (including my own). In Bolzano's time, and still today, it is a word with a very wide everyday usage roughly meaning 'a lot of', or 'a number of'. It has also, since Cantor in 1895, been appropriated by mathematicians to take on a well-known technical meaning later enshrined in various axiomatisations, such as that of Zermelo-Fraenkel. Philip Jourdain, in translating Cantor's defining work, initially used 'aggregate' for *Menge*, while sometimes also needing to use 'number' (e.g. in the title of Cantor (1955)). But 'set' quickly became the standard English mathematical term for *Menge*, whether it be the axiomatized concept or Cantor's informal 'gathering into a whole of definite and distinct objects of intuition or thought'. For Bolzano working eighty years earlier, for example in *RB* with reference to collections of terms in a series, it would be misleading to use the term 'set' for his use of *Menge*. Later in *WL* §84 and *PU* §§3–8 where Bolzano is making more careful distinctions among collections, even if we are generous over his ambiguous use of 'part', his definition of *Menge* fails to correspond to the informal notion introduced by Cantor. In a thorough recent examination made by Simons (1997) of Bolzano's distinctions among various concepts of collection the proposal is made to translate his *Menge* with 'multitude'. This has been adopted in these translations with, I hope, quite satisfactory results. There are still some occurrences where 'number' must be used. For example, in *RB*§§1–3 are some good examples of *Menge* being used interchangeably with *Anzahl* in precisely the same context.

The term *Vielheit* is even more vexed than *Menge* and has been variously translated by others in the past as 'multiplicity', 'set', 'multitude', 'plurality', and 'manifold'. From what has already been said, plurality and multiplicity are the most obvious candidates. It occurs about 25 times in *PU*. Literally *Vielheit* means a many-ness and Bolzano describes it in *WL* §86, as well as *PU* §9, as a collection, the parts of which are 'units of kind A'. Many things of a specific kind suggest a grammatical plural although the grammatical connotations are not particularly suited to the context of use in *PU*. It is adopted in Simons (1997) with qualification as 'concrete plurality'; we have used simply 'plurality'.

I am grateful to Edgar Morscher for pointing out the difference between *gegenstandslos*, a fairly recent word meaning to become void, or irrelevant, and *gegenstandlos*, an older word—the one Bolzano uses—to mean, of an idea, that it is without an associated object (or, it is as it has usually been translated, 'empty'). The latter word is used in contrast to *gegenständlich* meaning, of an idea, that it does have objects associated with it. Bolzano often cites the cases of 0 or $\sqrt{-1}$ as number ideas that are empty (e.g. *PU* §34). See also the footnote on p. 594.

The translator of a text must also act to some extent as an editor. There have been, in this volume of translations numerous decisions of whether mathematical material should be displayed or left in-line with consequent problems of small font size and 'gappiness' in the lines of text. The manuscript works of *RZ*, *F* and *F+* have closely preserved many of the human inconsistencies of the original manuscripts, for example, the number of dots used in a sequence or equation to indicate subsequent continuation. An editor faces an almost irrestible urge to 'tidy up' and to render 'consistent' the variations introduced by normal human production. We have not always managed to resist these urges. Continuation dots in the published early works, as in the later works rendered faithful to the manuscripts, have been reproduced as the conventional three dots (whether or not the original had one or two or more dots). The conventional German practice of continuation dots in arithmetic expressions being 'on the line' has been replaced by the English form of dots centred vertically at the operator level. The few cases of equation labels on the left-hand side of an equation have been replaced consistently by right-hand side labels. The breaking of very long expressions from one line to the next has not always followed the original form. Apart from these matters we have intended to follow Bolzano's notations exactly with only two exceptions: the matter of decimal commas on p. 268, and the subscripts inside an omega symbol on p. 574.

Some might regard it as undue pedantry to imitate such matters as Bolzano's astrological symbols labelling equations (e.g. in *BL*), or his wavering inconsistently in *F* between centred and right-hand superscripts. But it is hard to exaggerate the significance of writing and notation for our thinking. Bolzano knew this and took seriously the choice of good mathematical notation (see *BD* II §6 on p. 106). Usually he (or his printer) maintained a style of superscripting fairly consistently within one work. It seems surprising therefore, and should not be suppressed, that he was using, to any extent, superscripts so easily confused (we might suppose) with powers. It is hard enough to take ourselves back into the thinking of earlier times, surely it would be folly to deliberately erase anything that might possibly serve, alone or cumulatively, as a clue to the thinking we are working to recover?