Partial metrizability in value quantales

R. Kopperman*, S. Matthews and H. Pajoohesh

Abstract. Partial metrics are metrics except that the distance from a point to itself need not be 0. These are useful in modelling partially defined information, which often appears in computer science. We generalize this notion to study “partial metrics” whose values lie in a value quantale which may be other than the reals. Then each topology arises from such a generalized metric, and for each continuous poset, there is such a generalized metric whose topology is the Scott topology, and whose dual topology is the lower topology. These are both corollaries to our result that a bitopological space is pairwise completely regular if and only if there is such a generalized metric whose topology is the first topology, and whose dual topology is the second.

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1. Introduction

A partial metric [7] is a generalised metric for modelling partially defined information. For example, if a person is told to visit London in the UK they would instantly ask where in London. A more precise instruction might be to visit London’s Hyde Park, yet again they would ask, where in Hyde Park. A sufficiently precise instruction might be to visit the Prince Albert Memorial, or the Serpentine Gallery in Hyde Park. Let the relation $L_1 \leq L_2$ on such locations be defined by $L_2$ is a place in $L_1$. Then this is a partial ordering. For example,

$$\begin{align*}
\text{London} & \leq \begin{cases} 
\text{Hyde Park} & \leq \begin{cases} 
\text{Serpentine Gallery} \\
\text{Prince Albert Memorial} \\
\text{Trafalgar Square} & \leq \begin{cases} 
\text{Nelson’s Column}
\end{cases}
\end{cases}
\end{cases}
\end{align*}$$

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A partial metric is a means of extending the notion of metric, such as the Euclidean distance between the Serpentine Gallery and Nelson’s Column, to posets. To see how and, most importantly, why this is useful a brief discussion of the programming problem in computer science which led to partial metrics is now given.

A concurrent program is a computer program consisting of two or more processes to be executed in parallel, and also to communicate with each other in some way. A problem arises when two processes are each waiting upon the other for a communication before each itself can communicate. This is a deadly embrace situation known as deadlock where two processes remain alive yet doing nothing as each waits for a communication that can never arrive. In general it is not decidable whether or not an arbitrary concurrent program may deadlock at some time in its execution. However, in a concurrent program for a safety critical application, such as for use in a hospital’s intensive care unit, it is essential that there be some means of proving that deadlock can never occur. The best that can be done is to consider certain concurrent programs where it can be proven that deadlock can never arise. Deadlock is a problem between two processes which are directly or indirectly connected by some path of communication, and so a consideration of such paths provides an analysis of possible deadlock. A cycle is a set of processes $P_1, P_2, \ldots, P_n$ where each $P_i$ communicates by sending messages to $P_{(i+1) \mod n}$, and so receiving messages from $P_{(i-1) \mod n}$. For each $P_i$ let $c_i$ be the largest possible difference at any time between the number of messages sent minus the number received. For example, consider the case for $n = 2$. Suppose $c_1 = 5$, then at any time $P_1$ has sent at least five more messages than it has received. Suppose also that $c_2 = -3$, then $P_2$ has sent at least $-3$ more (i.e. at most 3 fewer) messages than it has received. This cycle of two processes cannot deadlock as at any time there is a net surplus of at least $c_1 + c_2 = 5 + (-3) = 2$ messages being produced by the cycle than being received. If the so-called cycle sum $\sum_{i=1}^{n} c_i > 0$ then this cycle of processes can never deadlock. To prove that a concurrent program will not deadlock it is thus sufficient to prove that each and every cycle sum is positive.

This is the cycle sum theorem [10], later extended to a more sophisticated model of concurrent programming as the cycle contraction mapping theorem [6]. The virtue of these theorems is that they are an extensional treatment of deadlock, one which does not require a detailed understanding of exactly how programs are executed. They prove that a deadlock free computation is necessarily the only possible behaviour for a concurrent program. This is in contrast to the more usual (but more difficult) procedure of constructing the sequence of all intermediate states of an execution, and demonstrating that the limit is a deadlock free computation. The behaviour of a cycle can be studied as the fixed point of a function. The purpose of the cycle sum and cycle contraction mapping theorems is to firstly prove that the fixed point is unique, and secondly that this point is, in a desired sense, totally defined. Being an extensional treatment of deadlock we require not the details of how
programs are executed, only the distinction between the so-called total (the word total is used here in place of complete as in [7]) computations (i.e. the desirable deadlock free executions) and the partial, that is, those initial parts of a total computation. To appreciate the distinction between total and partial objects we return to the analogy of a visitor to London. A reference to London is only partially informative, as it does not refer to a more specific place of interest such as Hyde Park or Trafalgar Square. London is thus a partialization of Hyde Park, Trafalgar Square, etc. where the (extent of) totalness of a place can be measured by the area of ground upon which it stands. London is a partial approximation to Hyde Park as the latter stands upon a smaller space and contained within that of London.

The cycle sum and cycle contraction mapping theorems are in essence inductive results to prove that eventually a total result must follow from the inductive hypothesis that for each step in a computation the next step will steadily increase the (extent of) totalness. To formulate such an hypothesis requires a function to measure the (extent of) totalness of a computation. For example, a tourist might ‘visit’ the partial places London, Trafalgar Square, Nelson’s column. The (extent of) totalness of each partial place, as measured by its area, becomes increasingly precise at each step. In the theorems under discussion the property of deadlock free for programs is partialized precisely to the extent to which it applies to each initial part of a deadlock free computation. This is a downward approach in which a partial object is viewed as a partialized total one. This is in contrast to the established Scott-Strachey least fixed point semantics [9] where the behaviour of a program is viewed as the limit of a chain of partial approximations, an upward approach where a total object (if the notion exists) is a completion of partial objects. While the upward view is necessary to define the semantics for an arbitrary program, the downward view is sufficient to reason about well behaved programs such as those which are deadlock free.

The general problem arising from these deadlock studies [10, 6] is how to partialize theories. Given a theory of (now to be known as) total objects how can additional (to be known as) partial objects be introduced and the theory extended yet weakened to apply to them? In each of the above deadlock studies the total objects form a metric space of infinite sequences. For a set \( X \) let, \( d : X^\omega \times X^\omega \to \mathbb{R}^+ \) be the metric such that
\[
d(x, y) = 2^{-\sup\{n \mid \forall m < n, x_m = y_m\}}.
\]
The set of partial objects used in each of the two studies is very different, yet each can be understood as an instance of the same problem of how to partialize the theory of metric spaces. Firstly this involves generalising the notion of a metric. A partial metric (or pmetric) [7] is a function \( p : X \times X \to \mathbb{R}^+ \) satisfying the following conditions.

1) For every \( x, y \in X \), \( p(x, y) \geq p(x, x) \)
2) For every \( x, y \in X \), \( p(x, y) = p(y, x) \)
3) For every \( x, y, z \in X \), \( p(x, z) \leq p(x, y) + p(y, z) - p(y, y) \)
4) For every \( x, y \in X, x = y \iff p(x, y) = p(x, x) = p(y, y) \)
A partial metric is a generalisation of the notion of a metric such that self distance is not necessarily zero. From a partial metric can be defined a metric \( d : X \times X \to \mathbb{R}^+ \), a partial ordering \( \leq \subseteq X \times X \), a weight function \(|\cdot| : X \to \mathbb{R}^+\), and notions of total & partial objects.

\[
d(x,y) = 2 \times p(x,y) - p(x,x) - p(y,y)
\]

\[
x \leq y \Leftrightarrow p(x,x) = p(x,y)
\]

\[
|x| = p(x,x)
\]

\[
x \text{ total} \Leftrightarrow p(x,x) = 0
\]

\[
x \text{ partial} \Leftrightarrow p(x,x) > 0
\]

Note that a pmetric restricted to the total objects is a metric. For all \( x,y \in X \), \( x < y \Rightarrow |x| > |y| \), and so \(|\cdot|\) can be used to measure the (extent of) totalness of each member of \( X \). The Banach contraction mapping theorem can be extended to partial metrics. A contraction is a function \( f : X \to X \) for which there exists a \( 0 < c < 1 \) such that for all \( x,y \in X \), \( p(f(x), f(y)) \leq c \times p(x,y) \). A Cauchy sequence is an \( x \in X^\omega \) such that there exists \( a > 0 \) such that for each \( \epsilon > 0 \) there exists \( k > 0 \) such that for all \( n > k \), \( |p(x_n, a) - p(a,a)| < \epsilon \). p is complete if every Cauchy sequence converges. The partial metric contraction mapping theorem [6] is that each contraction for a complete partial metric has a unique fixed point, and this point is total.

Although originally developed as a partialised theory for extensional reasoning about properties of programs such as deadlock, partial metrics have since been developed in computer science as a theory of partiality for studying continuous lattices, using the induced ordering \( x \leq y \iff p(x,x) = p(x,y) \). A partial metric \( p : X \times X \to \mathbb{R}^+ \) generalises the theory of metric spaces by dropping the requirement that self distance always be zero, and in so doing opens up the study of \( T_0 \) spaces to a symmetric (in contrast to quasimetrics) metric style treatment, which in addition incorporates a weight function \(|\cdot| : X \to \mathbb{R}^+\).

The present paper takes the process of generalisation further by replacing the range \( \mathbb{R}^+ \) of a pmetric by a value quantale.

2. P-metrics and qmetrics in value lattices

**Definition 2.1.** A value lattice is a poset \((V, \leq)\), whose least element is denoted 0 and largest is \( \infty \), such that \((V, \geq)\) is a continuous lattice; together with an associative, commutative operation \(+ : V \times V \to V\) such that 0 is an identity and for each \( R, S \supseteq V \), \((\bigwedge R) + (\bigwedge S) = \bigwedge \{ r + s \mid r \in R, s \in S \}\), where \( \bigwedge \) denotes inf. Here are some simple but useful consequences of this infinite distributive law:

- (s1) For all \( p \in V \), \( p + \infty = \infty \).
- (s2) For all \( p, q, r, s \in V \), \( p \geq q \) and \( r \geq s \) implies \( p + r \geq q + s \).

A value lattice \( V \), is Boolean if for each \( a \in V \), \( a + a = a \).
Like any map preserving arbitrary infima, for each $p$, the map $\cdot + p, \cdot + p(q) = p + q$ has a right adjoint (see [4], chap. 0.3), $\cdot - p$ defined by $\cdot - p(q) = \bigwedge \{ r \in V \mid p + r \geq q \} (= q - p)$. The properties of $-$ (see [2]) include for any $p,q,r \in V$, $S \subseteq V$:

- (d1) $p + r \geq q$ iff $r \geq q - p$.
- (d2) $q - p = 0$ iff $p \geq q$.
- (d3) $p + (q - p) \geq q$.

The reversal of order above – that is, the requirement that $(V, \geq)$, rather than the expected $(V, \leq)$ be a continuous lattice – is due our need to maintain the traditional way of writing axioms for metrics, in order to allow easy comparison between metric spaces and our structures. The reader must be careful when looking at references (except [2]), which use the traditional order. Notice that products of value lattices are again value lattices, and that with the way-above relation denoted $\gg$, that in a product $\prod_i V_i$, $a \gg b$ if and only if each $a_i \gg b_i$ and $\{ i | a_i \neq \infty \}$ is finite. When we write $\ll$, we mean the inverse of $\gg$.

Key examples of value lattices include the extended nonnegative reals, $\mathbb{E} = [0, \infty]$, the unit interval $\mathbb{I} = [0, 1]$, and the two-point set, $\mathbb{B} = \{0, \infty\}$, together with the usual $\leq, +$, except that in $\mathbb{I}$ we truncate addition, via $a + b = \min\{a + u, b, 1\}$, where $+u$ denotes the usual sum. By a qmetric from a set $X$ into a value quantale $V$, we mean a map $q : X \times X \to V$ such that for each $x, y, z \in X$, $q(x, z) \leq q(x, y) + q(y, z)$, and $q(x, x) = 0$.

**Definition 2.2.** A $V$-pseudometric space is a pair $(X, p)$, consisting of a set $X$ and a function $p : X \times X \to V$ satisfying the following conditions:

- P1) For every $x, y \in X$, $p(x, y) \geq p(x, x)$,
- P2) For every $x, y \in X$, $p(x, y) = p(y, x)$,
- P3) For every $x, y, z \in X$, $p(x, y) \leq p(x, y) + p(y, z) - p(y, y))$.

Its associated qmetric is $q_p : X \times X \to V$, defined by $q_p(x, y) = p(x, y) - p(y, y)$. The dual of any qmetric is the qmetric defined by $q^*(x, y) = q(y, x)$ (so $q^*_p(x, y) = p(x, y) - p(x, x)$).

A $V$-metric is a $V$-pseudometric which also satisfies:

- P4) For every $x, y \in X$, $x = y$ iff $p(x, y) = p(x, x) = p(y, y)$.

**Definition 2.3.** Given a $V$-qmetric $q : X \times X \to V$, the ball (or closed ball) about $x \in X$ of radius $r \in V$ for is the set $N_r(x) = \{ y \in X \mid q(x, y) \leq r \}$; also, $N^*_r(x) = \{ y \in X \mid q^*(x, y) \leq r \}$, and the open ball about $x$ of radius $r$ is $B_r(x) = \{ y \mid q(x, y) \ll r \}$.

A subset $U$ of $X$ is open in $X$ if for each $x \in U$ there is an $r \gg 0$ such that $N_r(x) \subseteq U$. We write $\tau_q$, for the collection of all open subsets of a $V$-qmetric space $(X, q)$, $\tau_p = \tau_{q_p}$, and $\tau_{p^*} = \tau_{(q^*)_p}$, for a $V$-pseudometric space.

**Theorem 2.4.** Let $(V, \geq, +)$ be a value lattice.

(a) If $p : X \times X \to V$ is a $V$-pseudometric, then $q_p$ is a $V$-qmetric.
(b) For each $V$-metric, $\tau_q$ is a topology. For each $x \in X$, $\{B_r(x)|r \gg 0\}$ and $\{N_r(x)|r \gg 0\}$, are neighborhood bases for $\tau_q$ at $x$. For each $x \in X, r \in V, N^*_r(x)$ is a closed set in $\tau_q$. Also, if $r \gg 0$ then $B_r(x)$ is an open set in $\tau_q$, so in particular, the set of all open balls is a base for the topology $\tau_q$.

Proof. (a) Certainly, if $\tau$ is a topology then the set of all open balls is a base for the topology $\tau_q$.

Thus for every $b \in X$, $a \in B_r(x)$.

Now by the above for every $x,y,z \in X$, $q(x,z) = q(x,y) + q(y,z)$. The others use facts about the continuous lattice ($\mathcal{V}$, $\geq$).

By Theorem 2.4, any $V$-metric and $q(x,y) = p(x,y) - p(y,y)$, then $q(x,y) = 0$. Next, we show that if $V$ is a value lattice and $a,b,c \in V$ then $(a+b) - c \leq (a-c)+b$. By definition of $\geq$, $a \leq (a-c)+b$.

Thus for every $b \in V, a+b \leq (a-c)+c+b$. Therefore $(a+b) - c \leq (a-c)+b$.

Now by the above for every $x,y,z \in X$, $q(x,z) = p(x,z) - p(z,z) \leq (p(z,y) + (p(x,y) - p(y,y))) - p(z,z) \leq (p(x,y) - p(y,y)) + (p(y,z) - p(z,z)) = q(x,y) + q(y,z)$.

(b) These proofs are left to the reader. That $\tau_q$ is a topology, is straightforward (or see [5]). The others use facts about the continuous lattice ($\mathcal{V}, \geq$), which generalize those about $I$:

$$a \ll r, s \text{ if } r \wedge s \gg a \text{ and } r, s \gg a \text{ if } r \wedge s \geq a.$$  

By Scott continuity of $+$, if $q(x,y) \ll r$ then $\bigwedge \{ q(x,y) + s | s \gg 0 \} = q(x,y) + \bigwedge \{ s | s \gg 0 \} = q(x,y) \ll r$ so for some $s \gg 0$, $q(x,y) + s \ll r$, thus if $z \in N_q(x)$ then $z \in B_r(x)$, so $B_r(x)$ is open.

Theorem 2.5. In any $V$-metric space, $x \in \overline{cl}(y)$ if and only if $q(x,y) = 0$. In any $V$-pseudo metric space, $x \in \overline{cl}(y)$ if and only if $p(x,y) = p(y,y)$.

Proof. By Theorem 2.4, $N_0^*(y)$ is closed, so $\overline{cl}(y) \subseteq N_0^*(y)$. But if $x \notin \overline{cl}(y)$ then for some open $T, x \in T$ and $y \notin T$. By Theorem 2.4, $N_0^*(x) \subseteq T$ for some $\epsilon \gg 0$, thus in particular, $q(x,y) \ll \epsilon$, so $q(x,y) \neq 0$. This shows the reverse inclusion, $N_0^*(y) \subseteq \overline{cl}(y)$. The assertion about $V$-pseudo metric spaces is seen by noting that by the above, $x \in \overline{cl}(y)$ if and only if $q_p(x,y) = 0$, and certainly this happens if and only if $p(y,y) \geq p(x,y)$, so they are equal by P1.

Corollary 2.6. For a $V$-metric, $\tau_q$ is $T_0$ if and only if for each $x,y \in X$, $q(x,y) = q(y,x) = 0 \Rightarrow x = y$. A $V$-pseudo metric $p$ is a $V$-metric if and only if $\tau_p$ is $T_0$.

Proof. It is known that a topology is $T_0$ if and only if, for each $x,y \in X$, $x \in \overline{cl}(y) \& y \in \overline{cl}(x) \Rightarrow x = y$. Thus by Theorem 2.5, $\tau_q$ is $T_0$ if and only if $q(x,y) = q(y,x) = 0 \Rightarrow x = y$, and so $\tau_p$ is $T_0$ if and only if $p(x,y) = p(y,y) = p(x,x) \Rightarrow x = y$.

Lemma 2.7. $p(x,y) = \max(x,y)$ is an $I$-metric on $I$ and is also a $B$-metric on $B$. Also, $\tau_\sigma$ is the Scott (or upper) topology, $\sigma = \{(x,1)|x \in (0,1)\} \cup \{\emptyset, I\}$, and $\tau_{\sigma^*}$ is the lower topology, $\omega = \{[0,x]|x \in (0,1)\} \cup \{\emptyset, I\}$.

Proof. Since for every $x, y \in I$, $\max(x,y) = \max(x,y) \geq \max(x,x)$, $p$ satisfies P1 and P2; also P4 is clear. For P3, if $y = \max(x,y,z)$ then $p(x,z) \leq y + p(y,z) - p(y,y) = p(x,y) + p(y,z) - p(y,y)$; if $z = \max(x,y,z)$ then $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.
\( p(x, y) + z - p(y, y) = p(x, y) + p(y, z) - p(y, y) \) and the case \( x = \max(x, y, z) \) is similar. Also if \( p(x, x) = p(y, y) \), then \( x = y \). Thus \( p \) is a \( \mathbb{I} \)-pmetric on \( \mathbb{I} \).

Now we show that \( \tau_p = \sigma \) and \( \tau_{p^*} = \omega \): Let \( A \in \tau_p \). Then for each \( x \in A \) there exists \( r > 0 \) (notice that here \( r \geq 0 \) if and only if \( r > 0 \)), such that \( N_r(x) \subseteq A \). Thus \( \{ y | x - r \leq y \} = N_r(x) \subseteq A \), hence \( \uparrow (x-r) \subseteq A \). Also, if \( \bigvee D \in A \) and \( D \) is directed, then there is some \( r > 0 \) such that \( N_r(\bigvee D) \subseteq A \), and by properties of \( \bigvee \), for some \( d \in D \) we have \( \bigvee D - r \leq d \), showing \( A \in \sigma \). If \( x \in A \in \sigma \), then \( \uparrow x \subseteq A \). But \( x = \bigvee(x-1/n) \in A \), \( n \in \mathbb{N} \) and \( \{x-1/n|n \in \mathbb{N}\} \) is a directed set, thus there is \( m \in \mathbb{N} \) such that \( x-1/m \in A \).

So \( \uparrow(x-1/m) \subseteq A \). Therefore \( N_{1/m}(x) \subseteq A \) and hence \( A \in \tau_p \). Now let \( A \in \tau_{p^*} \); then for every \( x \in A \) there is \( r > 0 \) such that \( N_r^*(x) \subseteq A \). Thus \( \{ y | y < x + r/2 \} \subseteq \{ y | y - x \leq r \} = N_r^*(x) \subseteq A \). So \( (X-\uparrow(x+r/2)) \subseteq A \). Therefore \( A \in \omega \). For the reverse inclusion, assume \( A \in \omega \). Then for every \( x \in A \) there is a \( a \in X \) such that \( x \in (X-\uparrow a) \subseteq A \). Thus \( x < a \) and hence \( r = (a-x)/2 \geq 0 \) and \( x \in N_{r^*}(x) \subseteq A \).

For the assertions about \( \mathbb{B} \), since \( \mathbb{B} \subseteq \mathbb{I} \), \( p \) is a \( \mathbb{B} \)-pmetric on \( \mathbb{B} \). Note that \( N_0(\infty) = \{ \infty \} \) and \( N_0(0) = N_{\infty}(x) = \mathbb{B} \) for \( x \in \mathbb{B} \), so \( \tau_p = \sigma \), and a similar proof shows \( \tau_{p^*} = \omega \). \( \square \)

**Lemma 2.8.** If \( f : X \rightarrow Y \) and \( p_f \) is a \( V \)-pseudopmetric on \( Y \), then \( p_f(x, y) = p(f(x), f(y)) \) defines a \( V \)-pseudopmetric on \( X \). Also, \( f \) is continuous from \((X, \tau)\) to \((Y, \tau_p)\), if and only if \( \tau_{p_f} \subseteq \tau \).

**Proof.** Since \( p \) is a \( V \)-pseudopmetric on \( Y \), \( p_f \) is on \( X \). We distinguish open \( q_{p_f} \)-balls in \( Y \) from open \( q_{p_f} \)-balls in \( X \), denoting the former by \( B_f^V(x) \), the latter by \( B_f^X(x) \). By definition of \( p_f \), \( B_f^X(x) = \{ y | p(f(x), f(y)) \leq p(f(y), f(y)) \leq r \} = f^{-1}(B_f^V(f(x))) \). Of course, \( f \) is continuous if and only if the inverse image of each set in the base of open \( q_{p_f} \)-balls is open, that is, if and only if each open \( q_{p_f} \)-ball is open; the latter occurs if and only if \( \tau_{p_f} \subseteq \tau \). \( \square \)

Recall that a bitopological space \((X, \tau, \tau^*)\) is completely regular if whenever \( x \in T \in \tau \) then there is a pairwise continuous \( f : (X, \tau, \tau^*) \rightarrow (\mathbb{I}, \sigma, \omega) \), such that \( f(x) = 1 \) and \( f \) is 0 off \( T \); it is zero-dimensional if whenever \( x \in T \in \tau \) then there is a pairwise continuous \( f : (X, \tau, \tau^*) \rightarrow (\mathbb{B}, \sigma, \omega) \) such that \( f(x) = \infty \) and \( f \) is 0 off \( T \). A bitopological space \((X, \tau, \tau^*)\) is said to have a property pairwise, if both \((X, \tau, \tau^*)\) and its dual, \((X, \tau^*, \tau)\) have the property.

**Theorem 2.9.** If \((X, \tau, \tau^*)\) is completely regular then there is a value lattice \( V \) and a \( V \)-pseudopmetric such that \( \tau = \tau_p \) and \( \tau^* \supseteq \tau_{p^*} \). Further, if \((X, \tau, \tau^*)\) is pairwise completely regular then there is a value lattice \( V \) and a \( V \)-pseudopmetric such that \( \tau = \tau_p \) and \( \tau^* = \tau_{p^*} \). The analogous result holds for zero-dimensionality in place of complete regularity, with Boolean value lattices.

Conversely, if there is a value lattice and a \( V \)-pseudopmetric such that \( \tau = \tau_p \) and \( \tau^* \supseteq \tau_{p^*} \), then \((X, \tau, \tau^*)\) is completely regular, and converses also hold in the other three cases.

Throughout the above, \( p \) is a \( V \)-pmetric if and only if \( \tau \) is \( T_0 \).
Proof. Let $PC(X,I)$ be the collection of all pairwise continuous functions from $(X,\tau,\tau^*)$ to $(\mathbb{I},\sigma,\omega)$, and define $A = \mathbb{I}^{PC(X,I)}$. With the pointwise order, $A$ is a value lattice and for $\phi,\psi \in A$, $\phi \gg \psi$ if and only if $\phi(f) > \psi(f)$, for every $f \in PC(X,\mathbb{I})$ and $\{f|\phi(f) \neq 1\}$ is finite. Now define $p : X \times X \to A$ such that $p(x,y)(f) = \max\{f(x),f(y)\}$ for every $f \in PC(X,\mathbb{I})$. A coordinatewise proof then shows that $p$ is an $A$-pseudometric on $X$. Now we show that $\tau = \tau_p$. Let $T \in \tau$. If $x \in T$ then there is a pairwise continuous function $g : (X,\tau,\tau^*) \to (\mathbb{I},\sigma,\omega)$ such that $g(x) = 1$ and $g$ is 0 off $T$. Now take $r \in A$ such that $r(g) = 1/2$ and $r(f) = 1$ for $f \neq g$. Then $N_r(x) = \{y|g(y) \geq 1/2\} \subseteq T$. Thus $\tau \subseteq \tau_p$. Now assume that $T \in \tau_p$. For each $x \in T$, there exists $r \gg 0$ such that $N_r(x) \subseteq T$. Let $I = \{f \in PC(X,\mathbb{I})|r(f) \neq 1\}$. Then $N_r(x) = \bigcap\{f_i(x) - f_i(y) \leq r(f_i)\}$, where $i \in I$. Hence $x \in \bigcap\{y|f_i(x) - f_i(y) < r(f_i)/2\} \subseteq T$, where $i \in I$. Therefore $x \in \bigcap\{y|y \in f_1^{-1}((f_1(x) - r(f_1)/2,1])\}$. Since every $f_i$ is continuous, $\bigcap\{y|y \in f_i^{-1}((f_1(x) - r(f_1)/2,1])\} \in \tau$ and since $I$ is finite, $\bigcap\{y|y \in f_i^{-1}((f_1(x) - r(f_1)/2,1])\} \in \tau$. Thus $T \in \tau$ and by the above we now have $\tau = \tau_p$. Similarly $\tau_{p^*} \subseteq \tau^*$. For zero-dimensionality replace $A$ by $\mathbb{B}^{PC(X,\mathbb{I})}$ and proceed as in the completely regular case.

In the “pairwise” cases, the above $p$ satisfies $\tau = \tau_q$ and $\tau_{p^*} = \tau^*$. For the converses, consider the relation $\preceq_p$ on subsets of $X$, defined by $S \preceq_p T \iff (\exists r \gg 0)(N_r(S) \subseteq T)$, where $N_r(S)$ is defined to be $\bigcup_{x \in S} N_r(x)$. Then $\preceq_p$ can easily be seen to satisfy the properties (a1)-(a3) of an auxiliary relation given below (where (a3) results from the interpolation property of the way-below relation $\gg$ in the continuous lattice $(\mathbb{V},\geq)$). Further, it is clear that $R,S \preceq_p T \Rightarrow R \cup S \preceq_p T$, $R \preceq_p S,T \Rightarrow R \preceq_p S \cap T$, $\varnothing \preceq_p \varnothing$ and $X \preceq_p X$. These are the defining properties of a quasiproximity (see [3]). Each quasiproximity $\preceq$ has a dual, $\preceq^*$, defined by $S \preceq^* T \iff X \setminus T \preceq X \setminus S$, and gives rise to a topology, $\tau_q = \{T|x \in T \Rightarrow \{x\} \preceq T\}$. Certainly, $\tau_p = \tau_q$, and $\tau_{p^*} = \tau_{q^*}$. In the reference just mentioned, Urysohn’s lemma is shown for quasiproximities; thus, if $S \preceq T$ then there is a pairwise continuous function $f : (X,\tau_q,\tau_{q^*}) \to (\{0,1\},\sigma,\omega)$ such that $f[S] = \{1\}$, $f[X \setminus T] = \{0\}$, and as a result, if $x \in T \in \tau_p$ then, letting $S = \{x\}$, there is a pairwise continuous function $f : (X,\tau_p,\tau_{p^*}) \to (\{0,1\},\sigma,\omega)$ such that $f(x) = 1$ and $f[X \setminus T] = \{0\}$; the same then holds for $(X,\tau_{p^*},\tau_p)$ using $\preceq^*$. This yields the results for complete regularity and pairwise complete regularity, and those involving 0-dimensionality are simpler, since in this case, for each $x \in X$, $r \gg 0$, the function defined by $f(y) = \begin{cases} 1 & y \in N_r(x) \\ 0 & y \notin N_r(x) \end{cases}$ is pairwise continuous from $(X,\tau_p,\tau_{p^*})$ to $(\{0,1\},\sigma,\omega)$.

The last statement is immediate from Corollary 2.6.

If $\leq$ is a transitive, reflexive relation on $X (= a$ pre-order) then the Alexandroff topology is $\alpha(\leq) = \{T \subseteq X|x \in T \& x \leq y \Rightarrow y \in T\}$.

**Theorem 2.10.** (a) For any topology $\tau$, $(X,\tau,\alpha(\geq))$ is pairwise 0-dimensional. Thus there is a Boolean value lattice $\mathbb{V}$ and a $\mathbb{V}$-pseudometric such that $\tau = \tau_p$. 


(b) For each continuous bounded dcpo, there is a value lattice \( V \) and a \( \forall \)-metric such that its Scott topology, \( \sigma \), is \( \tau_p \) and its lower topology, \( \omega \), is \( \tau_p^\ast \). If the dcpo is algebraic as well, then \( V \) can be assumed Boolean.

Proof. (a) Notice that \( U \in \alpha(\geq_x) \) if and only if \( x \in U \) and \( y \in \text{cl}(\{x\}) \) imply that \( y \in U \). Now consider \( x \in T \in \tau \) then define \( f : (X, \tau, \alpha(\geq_x)) \to (\mathcal{B}, \sigma, \omega) \) such that \( f = \infty \) on \( T \) and \( f = 0 \) on \( X \setminus T \). Since \( T \in \tau \) implies \( X \setminus T \in \alpha(\geq_x) \), \( f \) is pairwise continuous. Now let \( x \in U \in \alpha(\geq_x) \). Since \( (X \setminus \text{cl}(\{x\})) \in \tau \), \( \text{cl}(\{x\}) \in \alpha(\geq_x) \). Define \( f : (X, \alpha(\geq_x), \tau) \to (\mathcal{B}, \sigma, \omega) \) by \( f = \infty \) on \( \text{cl}(\{x\}) \) and \( f = 0 \) on \( X \setminus \text{cl}(\{x\}) \). By the construction, \( f \) is pairwise continuous.

(b) For each continuous bounded dcpo, \( (P, \sigma, \omega) \) is pairwise completely regular and \( \sigma \) is \( T_0 \) (see [3]); additionally if \( P \) is algebraic, then \( (P, \sigma, \omega) \) is pairwise 0-dimensional, using the fact that for compact \( x \), \( \uparrow x = \uparrow\uparrow x \), so a base for the open sets in \( \sigma \) is a subbase for the closed sets in \( \omega \). \( \square \)

3. Value quantales

First, we recall the definition and a few basic properties of value quantales from [2]. Assume \( V \) is a complete lattice. Then \( V \) is completely distributive if for any family \( \{x_{i,j} \mid j \in J, k \in K_j \} \) of elements of \( V \),

\[
\bigwedge_j \bigvee_{k \in K_j} x_{j,k} = \bigvee_{f \in M} \bigwedge_{j \in J} x_{j,f(j)},
\]

where \( M = \prod_{j \in J} K_j \).

Assume \( V \) is a complete lattice and \( p, q \in V \). Then \( q \) is well above \( p \), denoted by \( q \succ p \), iff for any subset \( S \subseteq V \), if \( p \geq \bigwedge S \), then for some \( r \in S \), \( q \geq r \).

Raney ([8]) has shown that a complete lattice is completely distributive if and only if each \( p = \bigwedge \{ q \mid q \succ p \} \). This is the criterion we shall use below.

The “way-above” relation \( \succ \) for a continuous lattice \((L, \geq)\) and \( \succ \) on a completely distributive lattice \((V, \geq)\) differ in that for the former (unlike the latter), the set \( S \) must be directed (by \( \geq \)). Like \( \succ \), \( \succ \) satisfies most of the axioms for an auxiliary relation, \( \succ \) on a poset \((P, \geq)\): if \( p, q, r, s \in P \) and \( S \subseteq P \),

- (a1) \( q \succ p \) implies \( q \geq p \);
- (a2) \( s \geq q \), \( q \succ p \) and \( p \geq r \) implies \( s \succ r \); and
- (a3) (Interpolation Property) If \( q \succ p \), then for some \( r \), \( q \succ r \) and \( r \succ p \).

Property (a3) is more special, and holds for \( \succ \) in any continuous lattice \((L, \geq)\) ([4]), and for \( \succ \) in any completely distributive lattice \((V, \geq)\) ([8]). The definitions of completely distributive lattice (for \( \succ \)) and continuous lattice (for \( \succ \)) amount to the statement that the relation is approximating:

- (a4) Each \( p \) is the inf, \( \bigwedge \{ r \mid r \succ p \} \).

For an arbitrary set in a completely distributive lattice, \( q \succ \bigwedge S \) iff for some \( r \in S \), \( q \geq r \), and for directed set \( D \) in a continuous lattice, that \( q \succ \bigwedge D \) iff for some \( r \in D \), \( q \geq r \).

However, many auxiliary relations, like \( \succ \), are subdirecting : for each \( x \in X \), \( \{ y \mid y \succ x \} \) is directed by \( \geq \). This need not hold for \( \succ \); for example, in \( \mathbb{B}^F \).
For some \( r > 0 \) if and only if, there is at most one \( f \) such that \( r(f) < \infty \), and this collection is clearly not directed. But we need this property for 0:

A value distributive lattice is a completely distributive lattice \( V \) satisfying the following two conditions:

- (v1) \( \infty \neq 0 \).
- (v2) If \( p > 0 \) and \( q > 0 \), then \( p \wedge q > 0 \).

A value quantale \( V \) \( = < V, \leq, + > \) consists of a value distributive lattice \( V, \leq \) and an operation + on \( V \) satisfying definition 1.

We now describe a special case of the construction of value quantale from [2]. Assume \((L, \geq)\) is a continuous lattice in which \( \bot \neq \top \), and let \( A = A(L) = \{ x \in L \mid x \gg \bot \} \), where \( \gg \) denotes the way-above relation on \( L \). Then \( \top \in A \) and if \( a, b \in A \), then \( a \wedge b \in A \). By a round upper set in \( L \), we mean a nonempty \( I \subseteq A \) for which:

- (r1) \( j \geq k \in I \Rightarrow j \in I \), and
- (r2) \( \forall j \in I \downarrow j \cap I \neq \emptyset \).

(Note in particular, that we do not require that \( I \) be directed by \( \geq \).) Let \( R = (R[L], \geq) \) denote the poset of round upper sets in \( L \), with reverse set inclusion. Since \( R \) is an inf-closed subset of \((2^A, \geq)\)(\( \cong (\mathbb{B}^A, \geq) \)), \( R \) is a complete lattice with \( \bigwedge S = \bigcup S \cup \{ \top \} \).

For \( a \in L \), define \( \theta(a) = \{ x \in A \mid x \gg a \} \). Then \( \theta(a) \in R \) and for \( a \in A \), and \( I \in R \), notice that if \( a \in I \Rightarrow \theta(a) \gg I \), since if \( a \in I \geq \bigwedge S \) then \( a \in I \subseteq \bigwedge S \) so for some \( J \in R, a \in J \), showing \( \theta(a) \subseteq J \), thus \( \theta(a) \geq J \). Thus \( I = \bigcup \{ \theta(a) \mid a \in I \} \subseteq \bigcup \{ J \gg I \} \subseteq \bigcup \{ J \leq I \} = I \), so in particular, each \( I = \bigwedge \{ J \gg I \} \), thus \( R \) is completely distributive by Raney’s result.

Also note that if \( J \gg I \) then for some \( a \in I \), \( J \geq \theta(a) \), that is, \( J \subseteq \theta(a) \); this with the previous paragraph shows \( J \gg I \Leftrightarrow (\exists a \in I)J \subseteq \theta(a) \). Here are some other properties of \( \theta \) that we need later:

- (\( \theta_1 \)) \( \theta \) preserves direct inf: For let \( D \) be directed by \( \geq \). Then for \( x \in A \), \( x \in \theta(\bigwedge D) \Leftrightarrow x \gg \bigwedge D \Leftrightarrow \) for some \( y \in L \), \( x \gg y \gg \bigwedge D \Leftrightarrow \) for some \( z \in L \) there is a \( d \in D \) such that \( x \gg z \gg d \Leftrightarrow \) there is a \( d \in D \) such that \( x \gg d \Leftrightarrow \) \( x \in \bigcup_{d \in D} \theta(d) = \bigwedge \theta(D) \). Hence \( \bigwedge \theta(D) = \theta(\bigwedge D) \) as required.

- (\( \theta_2 \)) For each \( a, b \in L \), \( a \gg b \Leftrightarrow \theta(a) \geq \theta(b) \): If \( a \gg b \) then \( b = \bigwedge \{ a, b \} \) a directed set, so by \( \theta_1 \), \( \theta(b) = \bigwedge \{ \theta(a), \theta(b) \} \leq \theta(a) \). Conversely, if \( \theta(b) \geq \theta(a) \), then \( \{ x \gg a \} \subseteq \{ x \gg b \} \), so \( \bigwedge \{ x \gg a \} \geq \bigwedge \{ x \gg b \} \). But since \( L \) is a continuous lattice, \( a = \bigwedge \{ x \gg a \} \) and \( b = \bigwedge \{ x \gg b \} \). Hence \( a \gg b \).

In \( R \), clearly the smallest element, called 0, is \( \top \) and the largest, \( \infty \), is \( \{ \top \} \).

The two differ, since \( \bot \neq \top \), so by (a4), for some \( a \neq \top \), \( a \gg \bot \); thus \( a \in A \) so \( 0 \neq \{ \top \} = \infty \).

Note also that if \( I, J \gg \bot \) then for some \( a, b \in A \) we have \( I \subseteq \theta(a) \), \( J \subseteq \theta(b) \), so \( I, J \subseteq \theta(a \wedge b) \), and \( a \wedge b \in A \), showing that \( I, J \gg \theta(a \wedge b) > 0 \). Thus \( \{ I \gg 0 \} \) is \( \geq \)-directed, so that \( R \) is a value distributive lattice in the terminology of [2].

Now suppose that \( * : L \times L \to L \) is a binary operation on \( L \) such that \((L, *, \bot, \bot)\) is a commutative monoid and for any \( a \in L \), the function \( a * \_ \) is a value quantale preserving infs and the way above relation; that is, any indexed family \( \{ b_i \}_{i \in I} \)
Proof. By Theorem 2.9, for $a \triangleright \triangleright b$, and whenever $b \triangleright \triangleright b'$, then $a \star b \triangleright \triangleright a \star b'$. Then $\star : A \times A \rightarrow A$.

For $I, J \in \mathcal{R}$, define $I + J = \{a|\exists x \in I, y \in J|a \geq x \times y\}$. Clearly, $+$ is associative, commutative and monotone. Also 0 is a unit for +, because of the following: If $a \in I + A$ then for some $x \in I$ and $y \in A$, $a \geq x \times y$. Thus $a \geq x \times y \geq x \times \perp = x$, and since $I$ is an upper set, $a \in I$. For the reverse set inclusion, if $a \in I$ then by (r2) there is $m \in I$ such that $a \triangleright \triangleright m$. Thus $a \triangleright \triangleright m = m \star \perp = \bigwedge_{k \geq 1} k \triangleright \triangleright m \star k$. Thus there is $k_0 \triangleright \triangleright \perp$ such that $a \geq m \star k_0$. Hence $a \in I + A$. Thus $I + A = I$ for every $I$.

Note also that for $S \subseteq \mathcal{R}$ and $I \in \mathcal{R}$, $I + \bigwedge S = I + \bigcup_{\{\top\}} \{a|\exists x \in I, y \in S \cup \{\top\}|a \geq x \times y\} = \bigcup_{J \in S \cup \{\top\}} \{a|\exists x \in I, y \in J|a \geq x \times y\} = \bigcup_{J \in S \cup \{\top\}} (I + J) = \bigwedge_{J \in S} (I + J)$, so $(\mathcal{R}, +)$ is a value quantale. Further:

$(\theta_2)$ Each $\theta(a + b) = \theta(a) + \theta(b)$. For if $t \in \theta(a) + \theta(b)$ then $\exists x, y \in \theta(a), y \in \theta(b)$ such that $t \geq x \times y$. Thus $t \geq x \times y \geq a \star b$, hence $t \in \theta(a \star b)$. But if $t \in \theta(a \star b)$ then $t \triangleright \triangleright a \star b = \bigwedge_{x \triangleright \triangleright \top \times y \triangleright \triangleright \top} \{x \triangleright \triangleright y\}$. Hence by definition of $\triangleright \triangleright$ there are $w \triangleright \triangleright a$ and $v \triangleright \triangleright b$ such that $t \geq w \times v$. Therefore $t \in \theta(a) + \theta(b)$. As a result of this and $\theta_2$, we also have:

$(\theta_3)$ Each $\theta(a \star b) = \theta(a) \star \theta(b)$. For if $t \in \theta(a \star b)$ then $\exists x, y \in \theta(a), y \in \theta(b)$ such that $t \geq x \times y$. Thus $t \geq x \times y \geq a \star b$, hence $t \in \theta(a \star b)$. But if $t \in \theta(a \star b)$ then $t \triangleright \triangleright a \star b = \bigwedge_{x \triangleright \triangleright \top \times y \triangleright \triangleright \top} \{x \triangleright \triangleright y\}$. Thus by definition of $\triangleright \triangleright$ there are $w \triangleright \triangleright a$ and $v \triangleright \triangleright b$ such that $t \geq w \times v$. Therefore $t \in \theta(a) \star \theta(b)$.

We have two special cases in mind: Assume $K$ is a nonempty set and let $\mathbb{F}$ be $(I, \leq, +)$ (with truncated addition $+$ as introduced preceding Definition 1) or $(\mathbb{B}, \leq, +)$. Let $K$ be any nonempty set. Then as a product of continuous lattices, $L = \mathbb{F}^K$ is also a continuous lattice with the pointwise order, and in it, $a \triangleright \triangleright \perp$. If and only if for each $i \in K$, $a(i) \triangleright \triangleright \perp_i$ and for all but a finite number of $i$, $a(i) = \top_i$. Thus in particular, for $\mathbb{F} = \mathbb{B}$, $A = \{r \in \mathbb{B}^K|r^{-1}[\{\top\}]$ is cofinite}, and for $\mathbb{F} = \mathbb{I}$, $A = \{r \in [0, 1]^K|r^{-1}[\{1\}]$ is cofinite}.

Let $\star$ be pointwise addition; certainly $\star$ is in both cases an associative, commutative operation preserving $\leq$ and $\triangleright \triangleright$, for which 0 is the unit, since these all hold coordinatewise. Thus $\star$ obeys the assumptions made of it, so $(\mathcal{R}[\mathbb{F}^K], +)$ is a value quantale. Following [2], we call the $\mathcal{R}[\mathbb{F}^K]$ the value quantales of subsets, and denote them by $\Gamma(K)$, and we call the $\mathcal{R}[\mathbb{B}^K]$ the value quantales of fuzzy subsets, and denote them by $\Lambda(K)$.

One special property of $\Gamma(K)$ worth noticing is that (r2) is trivial, since for $r \in A$, it is easy to see that $r \triangleright \triangleright r$; for $\Lambda(K)$ it is worth noticing for (r2) that for $r, s \in A$, $r \triangleright \triangleright s$ if and only if $r(i) > s(i)$ whenever $r(i) \neq 1$.

**Theorem 3.1.** In theorem 2.9 and its corollaries, “value lattice” can be improved to “value quantale”.

**Proof.** By Theorem 2.9, for $K = PC(X, \mathbb{F})$ there is a $K$-pseudometric $p$ such that $\tau = \tau_p$ and $\tau^* \supseteq \tau_p^*$. Let $\theta : K \rightarrow \mathcal{R}[\mathbb{F}^K]$ be as defined above. In addition to the properties already established, notice that $\theta(0) = 0$ and $\theta(\infty) = \infty$. We finish the proof by showing that for the $\mathcal{R}[\mathbb{F}^K]$-pseudometric $d = \theta \circ p$, $\tau_d = \tau_p$ and $\tau_d^* = \tau_p^*$. 
For suppose $T \in \tau_p$; if $x \in T$ then for some $r \gg 0$, $N_r(x) \subseteq T$. But $r \in A(F^K)$ so $\theta(r) \succeq A(F^K)$, the 0 of $\mathcal{R}[F^K]$. Further, if $y \in N_0(x)$, we have $q_d(x, y) \leq \theta(r)$ so $\theta(p(x, y)) \succeq \theta(p(y, y)) \leq \theta(r)$, thus by ($\theta_4$), $p(x, y) - p(y, y) = q_d(x, y) \leq r$, showing $y \in T$. This shows $N_0(x) \subseteq T$, so $T \in \tau_d$.

Conversely, suppose $T \in \tau_d$; if $x \in T$ then for some $s \gg 0$, $s \in \mathcal{R}[F^K]$, $N_s(x) \subseteq T$. By the beginning of the discussion of $\theta$ there is an $r \in K$, $r \gg 0$, so that $\theta(r) \leq s$. If $q_p(x, y) \leq r$ then $p(x, y) - p(y, y) \leq r$, thus $\theta(p(x, y)) - \theta(p(y, y)) \leq \theta(r) \leq s$, so $y \in T$. This shows $N_s(x) \subseteq T$, so $T \in \tau_p$.

The above show that $\tau_p = \tau_d$, and a similar proof shows that $\tau_p^* = \tau_d^*$. This completes the proof that throughout Theorem 2.9, the value lattice $F^K$ and pseudometric $p$ can be replaced by the value quantale $\mathcal{R}[F^K]$ and pseudopmetric $d$. □

References


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Department of Mathematics, City College, City University of New York, New York, NY 10031, USA.
Department of Computer Science, University of Birmingham, UK.

S. G. MATTHEWS
sgm@dcs.warwick.ac.uk
Department of Computer Science, University of Warwick Coventry, CV4 7AL, UK.

H. PAJOOHESH
hpajoohesh@yahoo.com
Department of Mathematics Shahid Beheshti University Teheran 19839, IRAN.
BCRI and CEOL, University College Cork, (Unit 2200, Cork Airport Business Park, Kinsale Road, Co. Cork), Ireland
Department of Computer Science, University of Birmingham, UK.