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An electronic version of this paper seems to have been lost by the author. What follows is a re-typing of the original paper, by the author, preserving all fonts, styles, and page numbers of the original as faithfully as is possible. Please do not reference the material in this re-typing, without first checking the original paper.

In retyping the following (very minor) errors have been noticed in the original paper, and corrected in this version.

- page 185 : in Example 2.1, in line 7 from the bottom, "$u$" should be "$\mathcal{U}$"
- page 192 : in the proof of Theorem 4.3, in line 5 from the top, "$\{0, 1, 2, 3\}$" should be "$\mathcal{R}$"
- page 192 : in the definition of $p$, $2^{\text{length}(U)+1} - \ldots$ should be $2^{|U|+1} - \ldots$
- page 195 : in line 3, $\ldots p(f(u), u)) \ldots$ should be $\ldots p(f(u), u) \ldots$
- page 196 : "$x \ldots \ll p\ z \ldots$" should be "$\ldots x \ll_p z \ldots$"

# Partial Metric Topology

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ABSTRACT: Metric spaces are inevitably Hausdorff and so cannot, for example, be used to study non-Hausdorff topologies such as those required in the Tarskian approach to programming language semantics. This paper presents a symmetric generalised metric for such topologies, an approach which sheds new light on how metric tools such as Banach's Theorem can be extended to non-Hausdorff topologies.

## INTRODUCTION

In the study of the denotational semantics of programming languages a topological model is constructed for a programming language defined as a system of logic. More often than not this means a $T_0$ model for the lambda calculus in the spirit of Scott [12]. However, the necessity in this approach that all suitable models must be $T_0$ appears to remove any possibility that the theory of metric space (which are of course all $T_2$) can be applied in any way to semantics in Computer Science. Rare exceptions to this rule are the use of quasi-metrics by Smyth in [11] to describe $T_0$ spaces, or the use of a metric super topology for a $T_0$ space by Lawson in [8]. If metrics are to be used at all then the more conventional wisdom in Computer Science would dismiss Scott's $T_0$ approach in favour of a purely metric approach such as that of de Bakker and Zucker [1]. Unfortunately, the latter $T_2$ approach pays the price of losing the notion of partial ordering inherent in $T_0$ spaces, a concept of fundamental importance in any Tarskian approach to fixed point semantics [13]. The distinct advantage of using quasi-metrics is that such generalised metrics can be used to define $T_0$ topologies with partial orderings, and so allow Tarskian semantics. Quasi-metrics are not without their problems though. Being non-symmetric a quasi-metric is arguably an "unnatural" notion of distance. A more important criticism is that the lack of symmetry sheds little light on how to develop tools for reasoning about programs using quasi-metric ideas. The title *Reconciling Domains with Metric Spaces* of Smyth's paper [11] indicates a much desired long term goal allegedly argued for by Dana Scott that partial order semantics should one day have a metric foundation. In *All Topologies come from Generalised Metrics* [5], Kopperman infers that such a foundation might just be possible. This may or may not be of interest to topologists in general as many of the more pleasant $T_2$ properties usually associated with metric spaces may be lost in a process of generalisation. However,

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the point that metrics can, if only in principle, be generalised to explore non-$T_2$-topologies is firmly established. The problem now is that there is little agreement on how this should be done.

In this paper the author's *partial metric* [9] is used to present a unifying framework for all the approaches mentioned above for studying $T_0$ topologies using a distance function.

## BACKGROUND DEFINITIONS AND RESULTS

DEFINITION 2.1: A basis $\mathcal{B}$ for a topology is $\sigma$-disjoint if there exists $B_1$, $B_2$, ... $\subseteq \mathcal{B}$ such that,

$$\mathcal{B} \;=\; \cup\{B_n | n \in \omega\} \text{ and,}$$
$$\forall n \in \omega \; \forall B, B' \in B_n,\; B \cap B' = \phi.$$

DEFINITION 2.2: A *partial ordering* is a binary relation $\ll \subseteq U^2$ such that,

(PO1) $\forall\, x \in U,\; x \ll x$
(PO2) $\forall\, x, y \in U,\; x \ll y \cap y \ll x \Rightarrow x = y$
(PO3) $\forall\, x, y, z \in U,\; x \ll y \cap y \ll z \Rightarrow x \ll z$

Within the field of Computer Science, which originally motivated this work, $\ll$ is used an an *information ordering* in which $x \ll y$ is interpreted as *all the information contained in x is also contained in y.* We now establish the usual relationship in Computer Science between topology and the information ordering. The topology usually placed upon $U$ will at least be $T_0$, and will also be consistent with $\ll$ in the following sense.

DEFINITION 2.3: A *weakly order consistent topology* is a weaker version of the *order consistent topology* [2] as used in lattice theory for which in addition suprema of directed sets are their limits. As the work in this paper requires neither directed sets nor lattices we work only with weakly order consistent topologies. An interesting example of a weakly order consistent topology is the topology of all upwardly closed sets,

$$\mathcal{T}[\ll] \;::=\; \{S \subseteq U | \forall x \in S,\; x \ll y \Rightarrow y \in S\}.$$

Thus, for example, for the usual partial ordering $\leq \subseteq (\omega \cup \{\infty\})^2$ on the non-negative integers with infinity,

$$\mathcal{T}[\leq] \;=\; \{\{n, n+1, \ldots, \infty\} | n \in \omega \cup \{\infty\}\}.$$

Each $T_0$ topology is weakly order consistent if and only if it is a topology of upwardly closed sets. Given any $T_0$ topology the information ordering can be recovered using the *specialisation ordering* defined by $x \ll y \Rightarrow x \in \mathrm{cl}(\{y\})$, a topic discussed more fully elsewhere [3]. In Computer Science we are interested in totally ordered sequences $X \in U^\omega$ of the form $X_0 \ll X_1 \ll X_2 \ll \ldots$, called chains, of increasing information, the least upper bound lub$(X)$ of which is intended to capture the notion of the amount of information defined by the chain. To ensure that lub$(X)$ cannot contain more information than can be derived from the members of the chain $X$ we insist that our topologies have the following property.

DEFINITION 2.4: A *Scott-like topology* over a partial ordering $\ll \subseteq U^2$ is a weakly order consistent topology $\mathcal{T}$ over $U$ such that for each chain $X \in U^\omega$, lub$(X)$ exists, and,

$$\forall O \in \mathcal{T},\ \mathrm{lub}(X) \in O \ \Rightarrow \exists k \in \omega \forall n > k,\ X_n \in O.$$

In other words, the least upper bound of a chain must be a limit of that chain. The term *Scott-like topology* introduced here is a weaker version of the term *Scott topology* [2] used in the study of continuous lattices. As the results in this paper do not need the full strength of the Scott topology we work only with the weaker Scott-like topology.

DEFINITION 2.5: A *metric* is a function $d : U^2 \to \mathcal{R}$ such that,

(M1) $\forall\ x, y \in U,\ x = y \ \Leftrightarrow\ d(x, y) = 0$
(M2) $\forall\ x, y \in U,\ d(x, y)\ =\ d(y, x)$
(M3) $\forall\ x, y, z \in U,\ d(x, z)\ \leq\ d(x, y) + d(y, z)$

DEFINITION 2.6: A *quasi-metric* is a function $q : U^2 \to \mathcal{R}$ such that,

(Q1) $\forall\ x, y \in U,\ x = y \ \Leftrightarrow\ q(x, y)\ =\ q(y, x)\ =\ 0$
(Q2) $\forall\ x, y, z \in U,\ q(x, z)\ \leq\ q(x, y) + q(y, z)$

LEMMA 2.1: For each quasi-metric $q : U^2 \to \mathcal{R}$ the relation $\ll_q \subseteq U^2$ defined by, $\forall x, y \in U,\ x \ll_q y \ \Leftrightarrow\ q(x, y) = 0$ is a partial ordering.

LEMMA 2.2: For each quasi-metric $q : U^2 \to \mathcal{R}$ the set of all *open balls* of the form,

$$B^q_\epsilon(x)\ ::=\ \{y \in U | q(x, y) < \epsilon\}$$

for each $x \in U$ and $\epsilon > 0$ is the basis for a weakly consistent topology $\mathcal{T}[q]$ over $\ll_q$.

The following example shows how the well-known Hausdorff metric [7] can be generalised.

EXAMPLE 2.1: For each complete bounded metric $d : U^2 \to \mathcal{R}$ and for each collection of closed sets $\mathcal{U} \subseteq 2^U$ the function $q : \mathcal{U}^2 \to \mathcal{R}$ where,

$$\forall X, Y \in \mathcal{U},\ q(X, Y)\ ::=\ sup\{inf\{d(x, y) | y \in Y\} | x \in X\}$$

is a quasi-metric such that, $\forall X, Y \in \mathcal{U},\ X \ll_q Y \Leftrightarrow X \subseteq Y$.

LEMMA 2.3: For each quasi-metric $q : U^2 \to \mathcal{R}$ the *symmetrisation* function $q^S : U^2 \to \mathcal{R}$ for $q$ where,

$$\forall x, y \in U,\ q^S(x, y)\ ::=\ q(x, y) + q(y, x)$$

is a metric such that $\mathcal{T}[q] \subseteq \mathcal{T}[q^S]$.

LEMMA 2.4: (*The quasi-metric contraction mapping theorem*) For each quasi-metric $q : U^2 \to \mathcal{R}$ such that $q^S$ is complete, and for each function $f : U \to U$ such that,

$$\exists 0 \leq c < 1 \ \forall \ x, y \in U, \ q(f(x), f(y)) \leq c \times q(x, y)$$

called a contraction, there exists a unique $a \in U$ such that $a = f(a)$.

Trivially, each constant function is a contraction, and so the fixed point obtained by the *quasi-metric contraction mapping theorem* is not in general maximal. Thus, as objects with *totally defined* information content will always be maximal this theorem cannot be used within Computer Science to prove that recursive definitions specify such *totally defined objects*. The root of the problem here is that the quasi-metric gives us no way of measuring the definedness of an object, and so no way of discussing total definedness. The next definition attempts to "re-axiomatise" the metric in order to overcome precisely this problem.

## THE PARTIAL METRIC

DEFINITION 3.1: A *partial metric or pmetric* [9] (pronounced "p-metric") is a function $p : U^2 \to \mathcal{R}$ such that,

(P1) $\forall \ x, y \in U, \ x = y \ \Leftrightarrow \ p(x, x) = p(x, y) = p(y, y)$
(P2) $\forall \ x, y \in U, \ p(x, x) \leq p(x, y)$
(P3) $\forall \ x, y \in U, \ p(x, y) \ = \ p(y, x)$
(P4) $\forall \ x, y, z \in U, \ p(x, z) \ \leq \ p(x, y) + p(y, z) - p(y, y)$

The pmetric axioms P1 thru P4 are intended to be a minimal generalisation of the metric axioms M1 thru M3 such that each object does not necessarily have to have zero distance from itself. In this generalisation we manage to preserve the symmetry axiom M2 to get P3, but have to "massage" the transitivity axiom M3 to produce the generalisation P4 (originally suggested to the author in [16]). Consequently a metric is precisely a pmetric $p : U^2 \to \mathcal{R}$ such that,

$$\forall \ x \in U, \ p(x, x) = 0.$$

"Half" of the metric axiom M1 is preserved as,

$$\forall \ x, y \in U, \ p(x, y) = 0 \Rightarrow x = y$$

However, the converse implication does not generally hold. $p(x, x)$, referred to as the *size* or *weight* of $x$, is a feature used to describe the amount of information contained in $x$. The smaller $p(x, x)$ the more defined $x$ is, being totally defined if $p(x, x) = 0$.

EXAMPLE 3.1: In Computer Science a *flat domain* is a partial ordering of the form $\ll \subseteq \ (S \cup \{\perp\})^2$ consisting of a set $S$ of *totally defined* objects together with the special *undefined* object $\perp \notin S$ (pronounced "bottom"), and ordering defined by,

$$\forall \ x, y \in S \cup \{\perp\}, \ x \ll y \Leftrightarrow x = \perp \vee x = y \in S.$$

Such a domain can be defined by the *flat pmetric* $p : (S \cup \{\perp\})^2 \to \{0, 1\}$ where,

$$\forall\ x, y \in S \cup \{\perp\},\ p(x,y) = 0\ \Leftrightarrow\ x = y \in S$$

Note how the condition $p(x,x) = 0$ precisely captures the flat domain notion $x \in S$ of total definedness.

DEFINITION 3.2: An *open ball* for a pmetric $p : U^2 \to \mathcal{R}$ is a set of the form,

$$B^p_\epsilon(x)\ ::=\ \{y \in U | p(x,y) < \epsilon\}$$

for each $\epsilon > 0$ and $x \in U$.

Note that, unlike their metric counterparts, some pmetric open balls may be empty. For example, if $p(x,x) > 0$ then $B^p_{p(x,x)}(x) = \phi$.

THEOREM 3.1: The set of all *open balls* of a pmetric $p : U^2 \to \mathcal{R}$ is the basis of a topology $\mathcal{T}[p]$ over $U$.

*Proof:* As, $U = \cup_{x \in U} B^p_{p(x,x)+1}(x)$ and, for any balls $B^p_\epsilon(x)$ and $B^p_\delta(y)$,

$$B^p_\epsilon(x) \cap B^p_\delta(y) = \cup\{B^p_\eta(z) | z \in B^p_\epsilon(x) \cap B^p_\delta(y)\}$$

where, $\eta ::= p(z,z) + min\{\epsilon - p(x,z), \delta - p(y,z)\}$. □

THEOREM 3.2: For each pmetric $p$, open ball $B^p_\epsilon(a)$, and $x \in B^p_\epsilon(a)$, there exists $\delta > 0$ such that $x \in B^p_\delta(x) \subseteq B^p_\epsilon(a)$.

*Proof:*

Suppose $x \in B^p_\epsilon(a)$.
Then $p(x,a) < \epsilon$.
Let $\delta ::= \epsilon - p(x,a) + p(x,x)$.
Then $\delta > 0$ as $\epsilon > p(x,a)$.
Also, $p(x,x) < \delta$ as $\epsilon > p(x,a)$.
Thus $x \in B^p_\delta(x)$.

Suppose now that $y \in B^p_\delta(x)$.
$\therefore\ p(y,x) < \delta$.
$\therefore\ p(y,x) < \epsilon - p(x,a) + p(x,x)$.
$\therefore\ p(y,x) + p(x,a) - p(x,x) < \epsilon$.
$\therefore\ p(y,a) < \epsilon$ (by P4).
$\therefore\ y \in B^p_\epsilon(a)$.
Thus $B^p_\delta(x) \subseteq B^p_\epsilon(a)$. □

Using the last result it can be shown that each sequence $X \in U^\omega$ converges to an object $a \in U$ if and only if,

$$\lim_{n \to \infty} p(X_n, a) = p(a,a).$$

THEOREM 3.3: Each pmetric topology is $T_0$.

*Proof:* Suppose $p : U^2 \to \mathcal{R}$ is a pmetric, and suppose $x \neq y \in U$, then, from P1 & P2 (wlog) $p(x,x) < p(x,y)$, and so,

$$x \in B^p_\epsilon(x)\ \wedge\ y \notin B^p_\epsilon(x),$$

where, $\epsilon\ ::= (p(x,x) + p(x,y))/2$. □

So far we have shown that a partial metric $p$ can quantify the amount of information in an object $x$ using the numerical measure $p(x,x)$, and also that $p$ has an open ball topology. This would not be of much use in Computer Science without a partial ordering.

DEFINITION 3.3: For each pmetric $p : U^2 \to \mathcal{R}$, $\ll_p \subseteq U^2$ is the binary relation such that,

$$\forall x, y \in U,\ x \ll_p y \Leftrightarrow p(x,x) = p(x,y).$$

THEOREM 3.4: For each pmetric $p$, $\ll_p$ is a partial ordering.

*Proof:* We prove P01 thru PO3.

(PO1) $\forall x \in U,\ x \ll_p x$ as $p(x,x) = p(x,x)$.

(PO2) $\forall x, y \in U,\ x \ll_p y\ \wedge\ y \ll_p x$
$\Rightarrow\ p(x,x) = p(x,y) = p(y,y)$ (by P3)
$\Rightarrow\ x = y$ (by P1).

(PO3) $\forall x, y, z \in U,\ x \ll_p y \wedge y \ll_p z$
$\Rightarrow\ p(x,x) = p(x,y) \wedge p(y,y) = p(y,z)$.
But by P4, $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$
$\therefore\ p(x,z) \le p(x,x)$
$\therefore\ p(x,z) = p(x,x)$ (by P2)
$\therefore\ x \ll_p z$. $\Box$

EXAMPLE 3.2: The concept of a *vague real number* might be constructed as a nonempty closed interval on the real line. The function $p : \{[a,b] | a \le b\}^2 \to \mathcal{R}$ over all such intervals where,

$$\forall [a,b], [c,d],\ p([a,b],[c,d])\ ::=\ max\{b,d\} - min\{a,c\}$$

is a pmetric such that $[a,b] \ll_p [c,d] \Leftrightarrow [c,d] \subseteq [a,b]$, read as $[c,d]$ is a more precise version of $[a,b]$. Also we can use $p([a,b],[c,d])$ to measure the degree of vagueness of a vague number $[a,b]$.

EXAMPLE 3.3: Gilles Kahn's model of parallel computation [4] consists of a set of computing processes sending unending streams of information from one process to another. Such streams can easily be modelled using the well-known *Baire metric* $d : (S^\omega)^2 \to \mathcal{R}$ of $\omega$-sequences over a set $S$ defined by,

$$\forall x, y \in S^\omega,\ d(x,y)\ = 2^{-\sup\{i | i \in \omega \wedge \forall j < i, x_j = y_j\}}$$

However, such networks of processes must have partially defined streams of information as well as the totally defined infinite streams if a least fixed point semantics [13] is to be possible, and so we need to add the set $S^*$ of all finite sequences. The desired *initial segment* partial ordering on $S^* \cup S^\omega$ is,

$$\forall x, y \in S^* \cup S^\omega,\ x \ll y \Leftrightarrow \text{length}(x) \le \text{length}(y)\text{and},\ \forall i < \text{length}(x), x_i = y_i.$$

The Baire metric can be extended to the set $S^* \cup S^\omega$ of all finite and infinite sequences over $S$ by the *Baire pmetric* $p : (S^* \cup S^\omega)^2 \rightarrow \mathcal{R}$ where,

$$\forall x, y \in S^* \cup S^\omega,\ p(x, y) = 2^{-sup\{i | i \leq \text{length}(x) \wedge i \leq \text{length}(y) \wedge \forall j < i, x_j = y_j\}}.$$

Note that for each $x$, $p(x, y) = 2^{-\text{length}(x)}$, and thus the condition $p(x, x) = 0$ (as in Example 3.1) can be used to distinguish between the totally defined objects which comprise the Baire space and the remaining partial objects.

If $x \ll y$ for an information ordering $\ll$ then $y$ must have at least as much information as $x$. To see that $\ll_p$ does indeed have this property the following result can be deduced from axioms P1 & P2.

$$\forall x, y \in U,\ x \ll_p y \ \Rightarrow\ p(x, x) \geq p(y, y)$$

From this we can conclude that totally defined objects are indeed maximal in the pmetric framework, and also derive an interesting result for chains. If $X \in U^\omega$ is a chain converging to $a \in U$, and if in addition,

$$\lim_{n \rightarrow \infty} p(X_n, X_m) \ = \ p(a, a)$$

then the least upper bound of $X$ must exist, and this will be $a$.

THEOREM 3.5: For each pmetric $p$, $\mathcal{T}[p] \subseteq \mathcal{T}[\ll_p]$, that is, $\mathcal{T}[p]$ is a weakly order consistent topology over $\ll_p$.

*Proof:* It is sufficient to show that,

$$\forall x \in U\ \forall \epsilon > 0.\ B^p_\epsilon(x) \ = \ \cup \{\{z | y \ll_p z\} | y \in B^p_\epsilon(x)\}.$$

Suppose $x, y, z \in U$ and $\epsilon > 0$ are such that $y \in B^p_\epsilon(x)$ and $y \ll_p z$. Then,

$$\begin{aligned} p(x, z) &\leq p(x, y) + p(y, z) - p(y, y) \qquad \text{(by P4)} \\ &= \ p(x, y) \quad \text{as} \quad y \ll_p z \\ &< \ \epsilon \quad \text{as} \quad y \in B^p_\epsilon(x). \end{aligned}$$

Thus, $z \in B^p_\epsilon(x)$. $\square$

Thus $\mathcal{T}[p]$ is a Scott-like topology over $\ll_p$ if each chain $X$ has a least upper bound and if,

$$\lim_{n \rightarrow \infty} p(X_n, X_m) \ = \ p(\text{lub}(X),\ \text{lub}(X))$$

THEOREM 3.6: For each pmetric $p : U^2 \rightarrow \mathcal{R}$, $\mathcal{T}[p] = \mathcal{T}[\ll_p]$, if and only if,

$$\forall x \in U, \exists \epsilon > 0,\ B^p_\epsilon(x) \ = \ \{y | x \ll_p y\}$$

*Proof:* Suppose first that, $\forall x \in U \exists \epsilon > 0$, $B^p_\epsilon(x) = \{y | x \ll_p y\}$. Then, $\forall O \in \mathcal{T}[\ll_p]$,

$$O \ = \ \cup_{x \in O} \{y | x \ll_p y\} \ = \ \cup_{x \in O} B^p_\epsilon(x) \in \mathcal{T}[p]$$

$$\therefore\ \mathcal{T}[\ll_p] \ \subseteq \ \mathcal{T}[p]$$

$$\therefore\ \mathcal{T}[p] \ = \ \mathcal{T}[\ll_p] \qquad \text{(by Theorem 3.5)}$$

Suppose now that, $\mathcal{T}[p] = \mathcal{T}[\ll_p]$. Then, $\forall x \in U, \{y | x \ll_p y\} \in \mathcal{T}[p]$. Thus by Theorem 3.2,

$$\forall x \in U\ \exists \epsilon > 0,\ x \in B^p_\epsilon(x) \subseteq \{y | x \ll_p\}.$$

But, if $x \in B^p_\epsilon(x)$ then $\{y | x \ll_p y\} \subseteq B^p_\epsilon(x)$. Thus,

$$\forall x \in U\ \exists \epsilon > 0,\ B^p_\epsilon(x) \ = \ \{y | x \ll_p y\}. \ \Box$$

Having now established the relationship of the open ball topology $\mathcal{T}[p]$ to both the upward closure $\mathcal{T}[\ll_p]$ and the weakly order consistent topology we now move on to consider the relationship of the partial metric to the quasi-metric.

## PARTIAL AND QUASI-METRICS

THEOREM 4.1: For each pmetric $p : U^2 \to \mathcal{R}$ the function $q : U^2 \to \mathcal{R}$ where,

$$\forall x \in U,\ q(x, y) = \ p(x, y) - p(x, x)$$

is a quasi-metric such that $\mathcal{T}[p] = \mathcal{T}[q]$ and $\ll_p = \ll_q$.

*Proof:* We show first $q$ is a quasi-metric by proving Q1 and Q2.

(Q1$\Rightarrow$) $\forall x, y \in U,\ x = y \Rightarrow q(x, y) = 0$ (by definition of $q$).

(Q1$\Leftarrow$) $\forall x, y \in U,\ q(x, y) = q(y, x) = 0$
$\Rightarrow\ p(x, y) - p(x, x) = p(y, x) - p(y, y) = 0$
$\Rightarrow\ p(x, x) = p(x, y) = p(y, y)$ (by P3)
$\Rightarrow\ x = y$ (by P1).

(Q2) $\forall x, y, z \in U,\ p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$
$\Rightarrow\ q(x, z) \leq q(x, y) + q(y, z)$.

Thus $q$ is a quasi-metric.

Now, $\mathcal{T}[p] = \mathcal{T}[q]$ as,

$$\forall x \in U,\ \forall \epsilon > p(x, x),\ B^p_\epsilon(x) \ = \ B^q_{\epsilon - p(x,x)}(x)$$

$$\forall x \in U\ \forall 0 < \epsilon \leq p(x, x),\ B^p_\epsilon(x) \ = \ \phi$$

$$\forall x \in U\ \forall \epsilon > 0,\ B^q_\epsilon(x) \ = \ B^p_{\epsilon + p(x,x)}(x).$$

Finally, $\ll_p = \ll_q$ as,

$$\forall x, y \in U,\ p(x, x) = p(x, y) \ \Leftrightarrow\ q(x, y) = 0. \ \Box$$

DEFINITION 4.1: A *weighted quasi-metric* over a set $U$ is a pair $\langle q, |\ | \rangle$ consisting of a quasi-metric $q : U^2 \to \mathcal{R}$ and a *weight function* $|\ | : U \to \mathcal{R}$ specified by,

$$(WQ) \quad \forall x, y \in U,\ q(x, y) + |x| \ = \ q(y, x) + |y|.$$

A quasi-metric $q$ is *weightable* if there exists a function $|\ | : U \to \mathcal{R}$ such that $\langle q, |\ | \rangle$ is a weighted quasi-metric.

THEOREM 4.2: For each weighted quasi-metric $\langle q, |\ |\rangle$ over a set $U$ the function $p : U^2 \rightarrow \mathcal{R}$ where,

$$\forall x, y \in U,\ p(x,y)\ ::=\ q(x,y) + |x|$$

is a pmetric such that $\mathcal{T}[p] = \mathcal{T}[q]$ and $\ll_p = \ll_q$.

*Proof:* We show first that $p$ is a pmetric by proving P1 thru P4.

(P1$\Rightarrow$) Trivial.

(P1$\Leftarrow$) $\forall x, y \in U,\ p(x,x) = p(x,y) = p(y,y)$
$\Rightarrow\ |x|\ =\ q(x,y) + |x|\ =\ |y|$ (by Q1)
$\Rightarrow\ |x|\ =\ q(x,y) + |x|\ =\ q(y,x) + |x|\ =\ |y|$ (by WQ)
$\Rightarrow\ q(x,y)\ =\ q(y,x)\ =\ 0$
$\Rightarrow\ x = y$ (by Q1).

(P2) $\forall x, y \in U,\ 0 \leq q(x,y)$
$\therefore\ \forall x, y \in U,\ |x| \leq q(x,y) + |x|$
$\therefore\ \forall x, y \in U,\ p(x,x) \leq p(x,y)$ (by Q1).

(P3) $\forall x, y \in U,\ q(x,y) + |x|\ =\ q(y,x) + |y|$ (by WQ)
$\therefore\ \forall x, y \in U,\ p(x,y)\ =\ p(y,x)$.

(P4) $\forall x, y, z \in U,\ q(x,z) \leq q(x,y) + q(y,z)$ (by Q2)
$\therefore\ \forall x, y, z \in U,\ q(x,z) + |x| \leq (q(x,y) + |x|) + (q(y,z) + |y|) - |y|$
$\therefore\ \forall x, y, z \in U,\ p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$ (by Q1).

Thus $p$ is a pmetric.
Now, $\mathcal{T}[p] = \mathcal{T}[q]$ as,

$$\forall x \in U\ \forall \epsilon > x,\ B^p_\epsilon(x)\ =\ B^q_{\epsilon - |x|}(x)$$
$$\forall x \in U\ \forall 0 < \epsilon < |x|,\ B^p_\epsilon(x)\ =\ \phi$$
$$\forall x \in U\ \forall \epsilon > 0,\ B^q_\epsilon(x)\ =\ B^p_{\epsilon + |x|}(x).$$

Finally, $\ll_p = \ll_q$ as,

$$\forall x, y \in U,\ p(x,x) = p(x,y)\ \Leftrightarrow\ q(x,y) = 0.\ \ \Box$$

Theorems 4.1 and 4.2 have established an algebraic equivalence between the partial metric and a class of quasi-metrics, raising the question of whether every quasi-metric topology is also a partial metric topology.

THEOREM 4.3: Not every quasi-metric is weightable.

*Proof:* Let $q : \{a, b, c\}^2 \rightarrow \{0, 1, 2, 3\}$ be the unique quasi-metric such that,

$$\begin{array}{ll} q(a,b)=0 & q(b,a)=2 \\ q(a,c)=1 & q(c,a)=1 \\ q(b,c)=3 & q(c,b)=0 \end{array}$$

Suppose by way of contradiction that there exists a weight function $|\ |:\{a,b,c\}\rightarrow\mathcal{R}$ for $q$, then,

$$\begin{aligned} &|b|+q(b{,}c) \\ &\qquad=(|b|+q(b,a))+1 \\ &\qquad=(|a|+q(a,b))+1 \qquad \text{(by WQ)} \\ &\qquad=|a|+q(a,c) \\ &\qquad=|c|+q(c,a) \qquad \text{(by WQ)} \\ &\qquad=(|c|+q(c,b))+1 \\ &\qquad=(|b|+q(b,c))+1 \qquad \text{(by WQ).} \end{aligned}$$

Unfortunately, Theorem 4.1 does not answer the question of whether or not every quasi-metric topology can be defined using a partial metric. Theorem 4.3 only shows that the method of defining a pmetric for a quasi-metric topology using a weight function will not always work. The next result does answer the question for finite quasi-metric topologies.

THEOREM 4.4: For each quasi-metric $q:U^2\rightarrow\mathcal{R}$ over a finite set $U$ there exists a pmetric $p:U^2\rightarrow\mathcal{R}$ such that $\mathcal{T}[p]=\mathcal{T}[q]$ and $\ll_p=\ll_q$.

*Proof:* Let $\ll_q^*\subseteq 2^U$ be the set of all chains in $\ll_q$. Let $\$:U\rightarrow\omega$ be the function where,

$$\forall x\in U,\ \$(x)::=2^{max\{\mathrm{length}(c)|c\in\ll_q^*\wedge\mathrm{lub}(c)\ll_q x\}}$$

Then it can be shown that the function $p:U^2\rightarrow\omega$ where,

$\forall x,y\in U,\ p(x,y)\ ::=$

$$2^{|U|+1}-\max\{\sum_{x\in c}\$(x)|c\in\ll_q^*\wedge\mathrm{lub}(c)\ll_q x\wedge\mathrm{lub}(c)\ll_q y\}$$

is a pmetric such that $\mathcal{T}[p]=\mathcal{T}[q]$ and $\ll_p=\ll_q$ . □

An important implication of Theorem 4.4 is that any finite partial ordering can be defined by a partial metric.

### Additional Results on Weighted Quasi-metrics

Since the Queen's College Summer School, Hans-Peter Kunzi [6] has shown that every $\sigma$-disjoint topology can be defined by a weighted quasi-metric. This result is an important result only in as much as it helps to clarify the class of quasi-metric topologies which are also pmetric. Unfortunately Kunzi's construction is probably not of much use though to computer scientists who wish to use the notion of weight as a tool for reasoning about the total definedness of programs. In Kunzi's construction only a topology with a single maximal object (and so for example not the Baire pmetric space) can have an object of size 0. Kunzi has also helped further clarify the relationship between quasi-metric and pmetric topologies [6] by showing that the *Sorgenfrey line* with base $\{[a,b]|a \leq b \in \mathcal{R}\}$ of semi-open intervals is an example of a quasi-metric topology which is not a pmetric topology.

## METRICS AND PARTIAL METRICS

A $T_0$ space can also be studied by providing it with a metric space "refinement" super topology in the spirit of Lawson [8]. For us such a refinement can be provided by a *weighted metric*.

DEFINITION 5.1: A *weighted metric* over a set $U$ is a pair $\langle d, |\ |\rangle$ consisting of a metric $d : U^2 \to \mathcal{R}$ and a *weight function* $|\ | : U \to \mathcal{R}$ specified by,

$$(WM) \quad \forall x, y \in U,\ d(x,y) \ \geq\ |x| - |y|$$

A metric $d$ is *weightable* if there exists a weight function $|\ |$ such that $\langle d, |\ |\rangle$ is a *weighted metric*.

The next two results show the algebraic equivalence between the *partial metric* and the *weighted metric*.

THEOREM 5.1: For each pmetric $p : U^2 \to \mathcal{R}$ the pair $\langle p^m : U^2 \to \mathcal{R}, || : U \to \mathcal{R}\rangle$ where,

$$\forall x, y \in U,\ p^m(x,y) \ ::=\ 2 \times p(x,y) - p(x,x) - p(y,y)$$
$$\forall x \in U,\ |x| \ ::=\ p(x,x)$$

is a weighted metric such that $\mathcal{T}[p] \subseteq \mathcal{T}[p^m]$, and,

$$\forall x, y \in U,\ p(x,y) \ =\ (p^m(x,y) + |x| + |y|)/2.$$

*Proof:* By Theorem 4.1, the function $q : U^2 \to \mathcal{R}$ where,

$$\forall x, y \in U,\ q(x,y) \ ::=\ p(x,y) - p(x,x)$$

is a quasi-metric such that $\mathcal{T}[p] = \mathcal{T}[q]$. Thus by Lemma 2.3 $p^m$ is a metric such that,

$$\forall x, y \in U,\ p^m(x,y) \ =\ q(x,y) + q(y,x).$$

Thus, $\mathcal{T}[p] \subseteq \mathcal{T}[p^m]$.

Finally, WM holds as by P2 $\forall x, y \in U,\ p^m(x,y) \ \geq\ |x| - |y|.\Box$

THEOREM 5.2: For each weighted metric $\langle d, |\ |\rangle$ over a set $U$ the function $p : U^2 \to \mathcal{R}$ where,

$$\forall x, y \in U,\ p(x,y) \ ::=\ (|x| + |y| + d(x,y))/2$$

is a pmetric such that $d = p^m$ and $\forall x \in U,\ |x| = p(x,x)$.

*Proof:* As
M1 $\Rightarrow$ P1 and
WM $\Rightarrow$ P2 and
M2 $\Rightarrow$ P3 and,
M1 & M3 $\Rightarrow$ P4. $\square$

We now turn to constructing a contraction mapping theorem for partial metrics. Unlike the quasi-metric version in Lemma 2.4 the *partial metric contraction mapping theorem* below can be used to prove that a recursive definition over a pmetric topology defines a total object. To do this we first need to generalise the notions of *Cauchy sequence* and *complete metric* to partial metric topology.

DEFINITION 5.2 For each pmetric $p : U^2 \to \mathcal{R}$, and for each $X \in U^\omega$, $X$ is a *Cauchy sequence* if there exists

$$\lim_{n,m\to\infty} p(X_n, X_m).$$

A sequence is Cauchy in the pmetric sense precisely when it is Cauchy, in the metric sense of the word, with respect to $p^m$. A nice example of a Cauchy sequence is the *chain*, as each chain is Cauchy. Thus the *chain*, so important in Tarskian fixed point semantics, can be considered in the pmetric framework as being merely a particular form of Cauchy sequence.

DEFINITION 5.3 A pmetric $p : U^2 \to \mathcal{R}$ is *complete* if every Cauchy sequence $x \in U^\omega$ converges to an object of size

$$\lim_{n,m\to\infty} p(X_n, X_m).$$

As with the pmetric definition of Cauchy sequence, $p$ is complete precisely when $p^m$ is complete in the metric sense of the word. Thus for a complete pmetric each chain must have a least upper bound, this also being a limit. In other words $\mathcal{T}[p]$ is a *Scott-like topology for each complete pmetric p.* Thus if the Scott topology [2] is relevant to programming language semantics, as undoubtedly it is, then it is reasonable to claim that so is the *complete partial metric space* introduced in this paper.

THEOREM 5.3: (*The partial metric contraction mapping theorem*) For each complete pmetric $p : U^2 \to \mathcal{R}$, and for each function $f : U \to U$ such that,

$$\exists\ 0 \le c < 1\ \ \forall x, y \in U,\ \ p(f(x), f(y)) \le c \times p(x, y)$$

called a *contraction*, firstly there exists a unique $a \in U$ such that $a = f(a)$, and secondly $p(a, a) = 0$.

*Proof:* Suppose $u \in U$, then,

$$\begin{aligned}
&\forall\ n, k \in \omega,\ p(f^{n+k+1}(u), f^n(u)) \\
&\quad \le p(f^{n+k+1}(u), f^{n+k}(u)) + p(f^{n+k}(u), f^n(u)) - p(f^{n+k}(u), f^{n+k}(u)) \\
&\quad \le c^{n+k} \times p(f(u), u)\ +\ p(f^{n+k}(u), f^n(u)).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\forall\ n, k \in \omega,\ \ p(f^{n+k+1}(u), f^n(u)) \\
&\quad \leq (c^{n+k} + \ldots + c^n) \times p(f(u), u) + p(f^n(u), f^n(u)) \\
&\quad \leq c^n \times ((1 - c^{k+1})\ /\ (1 - c)) \times p(f(u), u) + c^n \times p(u, u) \\
&\quad \leq c^n \times ((p(f(u), u)\ /\ (1 - c)) + p(u, u)).
\end{aligned}$$

Thus as, $\forall n \in \omega,\ p(f^n(u), f^n(u)) \leq c^n \times p(u, u)$ we see that $\langle f^n(u) | n \in \omega \rangle$ is a Cauchy sequence such that,

$$\lim_{n,m \to \infty} p(f^n(u), f^m(u)) = 0.$$

Thus as $p$ is complete we can choose $a \in U$ such that $X$ converges to $a$ and $p(a, a) = 0$. Thus,

$$\lim_{n \to \infty} p(f^n(u), a) = 0.$$

But, $p(f(a), a) = 0$ as,

$$\begin{aligned}
&\forall\ n \in \omega,\ \ p(f(a), a) \\
&\quad \leq p(f(a), f^{n+1}(u)) + p(f^{n+1}(u), a) - p(f^{n+1}(u), f^{n+1}(u)) \\
&\quad \leq c \times p(a, f^n(u)) + p(f^{n+1}(u), a).
\end{aligned}$$

Thus $a = f(a)$, and $p(a, a) = 0$ by P1 & P2.

Suppose $b \in U$ is such that $b = f(b)$, then,

$$p(a, b) \ = \ p(f(a), f(b)) \leq c \times p(a, b).$$

Thus, as $c < 1$, $p(a, b) = 0$, and so $a = b$. Thus the fixed point of $f$ is unique.□

## CONCLUSIONS AND FURTHER WORK

The principal conclusion from the research in this paper is that generalised metric topology has a largely unexplored potential in the field of non-Hausdorff partial order topology. This conclusion is justified for the following reasons. Firstly, this research both supports the quasi-metric approach used by Smyth in [11] and the metric refinement approach used by Lawson in [8] to model such topologies. Secondly, we have shown that such work can be conducted using a symmetric distance function. Thirdly, we have shown that such work can be conducted within metric topology itself by adding the notion of a *weight* to points in a metric space. Finally, we have shown that *Banach's contraction mapping theorem* can be generalised to many $T_0$ topologies for applications in program verification. The author's intuition in this work is that the relationship,

$$pmetric \ \equiv \ metric \ + \ size$$

is a way of turning the analytic notion of a metric into a more logic based construct.

The *cycle sum test* of Wadge [14] for proving Kahn's data flow networks [4] free of deadlock was the original motivation provided by Computer Science for this work. Consisting of solely infinite computations Kahn's data flow model needs an alternative to the now obsolete notion of program correctness based upon terminating programs. Wadge argued that there should be a suitable generalisation of termination based upon the following intuition.

> "A complete object (in a domain of data objects) is, roughly speaking, one which has no holes or gaps in it, one which cannot be further completed."

Inspired by Wadge's intuition, the partial metric framework is now proposed by the author as a theory of *complete and partial* objects. By formulating Kahn's domain of message passing streams for parallel computing as the *Baire pmetric space* we can both formulate and prove the cycle sum test. In [10] the author's *cycle contraction mapping theorem* is introduced for extending the cycle sum test to lazy data flow networks as used by the lazy LUCID [15] data flow programming language.

The primary motivation for this work has been to develop metric based tools for program verification in which the notion of the *size* of an object in a domain plays a pivotal role in quantifying the extent of its definedness. This is the amount of information currently known about an object in a computation. To successfully verify programs using the pmetric approach will require further work on how "useful" pmetrics can be defined. For example, the following construction can be used to ensure that all maximal objects in a pmetric space have size 0. For each pmetric $p : U^2 \rightarrow \mathcal{R}$ for which the set $V \subseteq U$ of all maximal objects is such that $\forall x \in U \; \exists \; z \in V, \; x \ll_p z$ let $pp : U^2 \rightarrow \mathcal{R}$ be such that,

$$\forall \; x, y \in U, \; pp(x, y) = p(x, y) - inf\{p(z, z) | z \in V \wedge (x \ll_p z \vee y \ll_p z)\}$$

Then $pp$ is a pmetric such that $\ll_{pp} = \ll_p$ and all maximal objects have size 0.

As noted in [1] there are considerable rewards to be gained from a theory of computation based upon the concept of distance instead of the more traditional partial ordering approach. The latter is much more suited to sequential computing, but the former appears to be more suitable for the future needs of parallel computing.

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