Introducing *Complete Points* to Topology

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Dagstuhl, August 20-25th, 2006

Outline

Abstract

Today we consider a largely unknown notion of *complete point* introduced by W.W. Wadge, which although closely related to notions in domain theory, is not fully meaningful therein. We show how this notion relates to work in bitopology, and in fuzzy set theory.

Complete points

... have no holes or gaps

In reasoning about concurrent programs Bill Wadge proposed,

A complete point is one that has, in an information content sense, "no holes or gaps", one that "cannot be further completed", and "should extend to a much wider context [than analysing dataflow deadlock]".

This notion of *complete point* was used to define the correctness notion of *absence of deadlock* in a dataflow program.

Complete points versus *partial points*

How do complete points relate to domain theory?

The less/more complete is a data point, the more/less partial is it. Thus there is, so to speak, an invariant relationship between the two conceptions.

Thus, we wish, a partial point to be thought of as an *incomplete point*.

Thus a complete point serves as a semantic correctness value of what a program **should ultimately do** when executed, whereas domain theory models what a program **does do** when executed.

Use of the word *complete*

Is this not confusing?

At first sight the use of the word *complete* for points seems to clash with the notion of a set being *complete* in some sense, or the *completion* of a set.

Actually there is no conflict here, as what we are doing is *completing a point* to complement other existing notions of *completing a set*.

To avoid confusion we will use the terms *point-complete* when discussing a point, and *set-complete* when discussing a set.

Use of the word *complete*

Is this not confusing?

The difficulty here is, in classical set theory, thinking of a point as being anything other than a point no more, no less.

Domain theory suceeds admirably in avoiding this problem by using non Hausdorff topology to describe each point we wish to interpret as being *partial*.

But, as we shall argue, there are situations where such T_0-separation is not concrete enough to capture a notion of *point-complete*.

Point-complete point versus *partial point*

How do complete points relate to domain theory?

Observation 1 :

There is no obvious way (in general) to *complete* (so to speak) domain theory in order to establish a theory of complete points.

Conversely, it seems most unlikely that one could determine domain theory from a theory of complete points.

Thus, our challenge is to understand how partial and complete points can work together, and what this means for point set topology.

Point-complete points versus *partial points*

How do point-complete points relate to domain theory?

Observation 2 :
A complete point is maximal (in the domain theory sense), but, a maximal point is not necessarily complete.

But, we won't be able to see this in domain theory, until we introduce a quantiy for each point, such as by using partial metrics.

The distinction between *maximal* and *point-complete* would be meaningful in quantitative domain theory.

Point-incomplete descriptions for domains

A rationale

If a partial point is to be identified with an incomplete point, then the domain of all points has to be identified with a *point-incompleton* (so to speak) of the structure of complete points.

Example : the formal ball model is a *point-incompletion* of a metric space.

The challenge then is how to use domain theory to establish point-incomplete mathematics.

The good news, for me, is that there is now a growing interest in *domain representation*.

P-incomplete descriptions for domains

A **domain representation** of a space is a domain whose maximal points are (isomorphic to) that space.

However, as there may (in general) be maximal points which are not point-complete, domain representation is not really what we want in order to study the relationship between partial points and complete points.

We need a framework in which each point can be determined as being either *partial* (equivalently *point-incomplete*) or *point-complete*.

So far we have *partial metrics*, but, as will be argued later, it is possible to go further.

Partial metrics

as introduced in 1992

A **partial metric space** is a pair $(X, p : X \times X \to [0, \infty))$ such that,

$$
\begin{array}{lcll}
p(x,y) & = & p(y,x) & \textit{symmetry} \\
p(x,x) & \leq & p(x,y) & \textit{small self-distances} \\
p(x,z) & \leq & p(x,y) + p(y,z) - p(y,y) & \textit{sharpened triangularity} \\
p(x,x) & = & p(x,y) = p(y,y) \Rightarrow x = y & \textit{separateness}
\end{array}
$$

Partial metrics firstly introduce non zero self-distance, and secondly, sharpen triangularity.

Example 1

for computation over a metric space

Let $X = \{0, 1, \ldots\} \rightarrow S$ be the set of all infinite sequences over some set S. They have an obvious metric $d(x, y) = 2^{-sup\{n | \forall i < n . x_i = y_i\}}$.

Problem : To compute a sequence in order we have to introduce the finite sequences as *parts* of the *whole*. However, self-distance is no longer zero for each finite sequence, although the other metric axioms all hold.

Solution : *partialise* (so to speak) the metric axioms as necessary to accommodate the *parts* of each $x \in X$.

Example 2

for computation over a metric space

Let $(-\infty, \infty)$ be the real line with the usual (Euclidean) metric, and let X be the set of all closed intervals.

Let $p : X \times X \to [0, \infty)$ be defined by $p([a, b], [c, d]) \ = \ max\{b, d\} - min\{a, c\}$. Then p is a partial metric.

The computation of a real number such as π can be modelled by a chain such as,

$$[0, 10] \ \supseteq \ [3.1, 3.2] \ \supseteq \ [3.14, 3.15] \ \supseteq \ \ldots \ \supseteq \ [\pi, \pi]$$

of *partial real numbers* where, $[a, b] \supseteq [c, d]$ iff $p([a, b], [a, b]) \ = \ p([a, b], [c, d])$.

Example 3

to show that a maximal point is not necessarily complete

Let $X = \{a, b_0, b_1, \ldots, c_0, c_1, \ldots\}$
Let $p : X \times X \to [0, \infty)$ be the partial metric such that,

$$
\begin{aligned}
p(b_n, c_n) &= p(b_n, b_n) = p(b_n, a) = 1 + 2^{-n} \\
p(c_n, c_n) &= 0 \\
p(a, a) &= 1
\end{aligned}
$$

Let $x \sqsubseteq y$ iff $p(x, x) = p(x, y)$.

Then $(X, \sqsubseteq)$ is a domain, and a is maximal but not complete (as $p(a, a) > 0$).

Equivalences

Relating partial metrics to other distance functions

And so, a partial metric, if describing a domain, has an in-built means, namely *non zero self-distance*, of distinguishing between maximal and point-complete points.

We wish now to bring to the fore non zero self-distance, by showing how it relates to other distance functions.

Equivalences

Relating partial metrics to other distance functions

From weighted metrics to partial metrics :
Suppose $D = (X, d, |\cdot| : X \to [0, \infty))$ is a *weighted metric space*, i.e. d is a metric and $|\cdot| : X \to [0, \infty)$ is such that $d(x, y) \geq |x| - |y|$. Then $p_D(x, y) = \frac{d(x,y)+|x|+|y|}{2}$ is a partial metric.
Now we can introduce the partial ordering $x \sqsubseteq y$ iff $|x| = d(x, y) + |y|$.
From partial metrics to weighted metrics :
Let $d_p(x, y) = 2 \times p(x, y) - p(x, x) - p(y, y)$, and $|x| = p(x, x)$. Then $(X, d_p, |\cdot|)$ is a weighted metric space.
Thus in their generalisation partial metrics retain the notion of metric, and enhance each point with a *weight*.

Equivalences

Relating partial metrics to other distance functions

From weighted quasi-metrics to partial metrics :
Suppose $Q = (X, q, |\cdot|)$ is a *weighted quasi-metric space*, i.e. q is a quasi-metric, and $q(x, y) + |x| \ = \ q(y, x) + |y|$. Then $p_Q(x, y) \ = \ q(x, y) + |x|$ is a partial metric.

From partial metrics to weighted quasi-metrics :
Let $q_p(x, y) \ = \ p(x, y) - p(x, x)$ and $|x| \ = \ p(x, x)$. Then $(X, q, |\cdot|)$ is a weighted quasi-metric space.

Equivalences

Relating partial metrics to other distance functions

A **based metric space** is a triple $(X, d, \phi \in X)$ such that (X, d) is a metric space.

From based metrics to topped partial metrics
Let $p_d^\phi(x, y) \;=\; d(x, y) + d(x, \phi) + d(y, \phi)$. Then p_d^ϕ is a partial metric, and $\top_{p_d^\phi} \;=\; \phi$.

From topped partial metrics to based metrics
Let $d_p(x, y) \;=\; d(x, \top_p)$. Then $(X, d_p, \top_p)$ is a based metric space.

Thus, in effect, a based metric space is a metric space with a *view* of that space from the *base point*.

Equivalences

Relating partial metrics to other distance functions

Each of these equivalences embodies a metric space, together with an elaboration in the form of a *weight* for each point.

We can take our definition of *complete point* as being one whose weight (=self-distance) is zero.

Michael Bukatin has recently noted that this is analogous to a *sheaf* where a topology is elaborated with algebraic information. But, in our situation, not one but two topologies (see below) are elaborated by a partial metric.

Equivalences

Relating partial metrics to other distance functions

We have *complete* versions of *Cauchy sequence*, *convergent sequence*, and Banach's *contraction mapping theorem* for partial metrics into $[0, \infty)$.

Notions of *set-completion* which may or may not combine quantale-valued metric and order have been considered, and we expect to publish results soon.

Abstracting from partial metrics

Partial metrics demonstrate that, if only in the case of metric spaces, a notion of complete point can be introduced.

As each metric space (X, d) gives rise to a topological space (X, τ_d), to what kind of topological structure can it be said that a partial metric gives rise?

A clue is that there is not one, but two topologies. τ_p has the basis of *open balls* $\{y | p(x, y) < \epsilon\}$, and the metric topology from the metric $d_p(x, y) = 2 \times p(x, y) - p(x, x) - p(y, y)$.

Thus we look at bitopological spaces.

Bitopology

A bitopological space is a tuple $(X, \tau \subseteq 2^X, \sigma \subseteq 2^X)$.

Now we have to decide what should be the relationship between the two topologies.

$\tau \subseteq \sigma$, where σ is to be understood as the topology of (classical) points, and τ is to be understood as what may be known about those points.

Thus (X, τ) is a theory of knowledge for (X, σ).

Bitopology

We can now consider a bitopological definition for *complete point*, based upon the simple idea that a point is complete precisely when our knowledge of it is it.

For (X, τ, σ), $x \in X$ is *point-complete* if
$\forall O \in \sigma \,.\, x \in O \;\Rightarrow\; O \in \tau$.

For a partial metric space (X, p), if $x \in X$ has self-distance zero (i.e. point-complete) then it will belong to each τ_p ball centred on it, as it does to every τ_{d_p} ball centred on it.

Thus an *incomplete* or *fuzzy* point is one where knowledge of that point is less than what is the point.

Category theory

How to define categories of partial metrics?

Non expansive maps seem to be the most natural choice for functions over partial metric spaces, perhaps with the additional restriction that self-distance is preserved.

What is the partial metric distance between two functions? Still an open question as partial-metric categories do not seem to be Cartesian-closed friendly.

Kim Wagner has solved simple domain equations such as $D = A + (D \to D)$ in enriched categories, which as such can be interpreted either metrically or as usual domain theory. We are now trying to introduce non zero self-distance into Kim's work by formulating a theory of *partially enriched categories*.

Fuzzy set theory

Two completely different communities having the same mathematics

Just a few weeks ago Michael Bukatin spotted that Ulrich Hohle has given a characterisation of topology over fuzzy sets, termed *many valued topology*. It seems that both he and us have in the same time frame been using the same axioms without knowing it.

Whereas our slogan is *non zero self-distance partialises mathematics*, Hohle's is *"equality implies existence"*,

$$E(x, y) \leq \begin{cases} E(x, x) \\ E(y, y) \end{cases}$$

This is synonymous with *small self-distances* for partial metrics.

Dichotomous topology

Lawrence Michael Brown

Brown uses (X, τ, σ) where τ is a topology, and σ are the closed sets from a second, possibly different topology.

Motivated by fuzzy sets, Brown clearly shows that there may be more than one way to consider multiple topological structures to capture fuziness without changing classical set theory.

What we know so far about partial metrics

Computation makes sense of introducing non zero self-distance into mathematics, but it is additional to domain theory.

Topologically speaking, we consider bitopological structures of the form $(X, \tau \subseteq 2^X, \tau' \subseteq 2^X)$. But such dances of 'tango topology' are cumbersome, more of a cry for help than a final *homogemnous* solution.

There is now enough evidence to suggest that the notion of *point* in point set topology can be, and needs to be generalised to a *incomplete-point*. Ultimately what we need is nothing less than a *incomplete-point set topology* to partialise point set topology.

More on partial metrics

Resources for finding out more about partial metrics

Our web site at `www.dcs.warwick.ac.uk/pmetric` introduces and contains research material on partial metrics.

... And finally

Thank you for listening.

Any questions?