Partial Metric Spaces

A Fuss about Nothing

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The 2008 Summer Conference
Universitaria de la UNAM
Mexico City
July 29th - August 1st

Acknowledgement

May I thank the organisers for the great privilege of addressing the 2008 Summer Conference in the plenary session on *applied topology*, and for their very generous financial support.

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Abstract

Introduced in 1992, a *partial metric space* is a generalisation of the notion of *metric space* defined in 1906 by Maurice Fréchet such that the distance of a point from itself is not necessarily zero. Motivated by the needs of computer science for non Hausdorff Scott topology, we show that much of the essential structure of metric spaces, such as Banach's contraction mapping theorem, can be generalised to allow for the possibility of non zero self-distances $d(x,x)$. This talk will introduce the essential motivation, theory, and applications for partial metric spaces, leading to the conclusion that the non Hausdorff nature of topology in computer science is calling upon metric topology to reconsider its foundations.

Contents

- A quick review of metric spaces, and of generalised metric spaces.
- What is new about *partial metric spaces*?
- Examples
- A *weighted* contraction mapping theorem
- *Metric space* with a *base point*
- Efficiency oriented languages
- Cost-oriented topology
- Key contributors to *partial metric spaces*
- What's it really all about?

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Metric space

How we model *distance* and its topology

Definition

A **metric space** is a pair $(X,\ d : X \times X \to \Re)$ such that,

$$
\begin{aligned}
d(x,x) &= 0 \\
\text{if} \quad d(x,y) &= 0 \quad \text{then} \quad x = y \\
d(x,y) &= d(y,x) \\
d(x,z) &\leq d(x,y) + d(y,z)
\end{aligned}
$$

Maurice Fréchet, 1906.

Definition

The **open balls** $B_\epsilon(a) = \{x \in A : d(x,a) < \epsilon\}$ are the basis for the usual topology.

Lemma

Each metric open ball topology is **Hausdorff**.

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Generalised metric space

Fréchet's axioms can be relaxed by dropping an axiom

Definition

A **pseudometric space** is a pair $(X, \; d : X \times X \to \Re)$ such that,

$$
\begin{aligned}
d(x,x) &= 0 \\
d(x,y) &= d(y,x) \\
d(x,z) &\leq d(x,y) + d(y,z)
\end{aligned}
$$

Lemma

$d(x,y) = 0$ *is an equivalence relation.*

Lemma

$d'([x],\; [y]) = d(x,y)$ *is a metric over the induced set of equivalence classes.*

Note

Mathematics is, usually, unique up to some equivalence relation.

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Metric spaces without symmetry

Definition

A **quasimetric space** is a pair $(X,\ q : X \times X \to \Re)$ such that,

$$q(x,x) \ = \ 0$$

$$\text{if} \quad q(x,y) = 0 \quad \text{and} \quad q(y,x) \ = \ 0 \quad \text{then} \quad x = y$$

$$q(x,z) \ \leq \ q(x,y) \ + \ q(y,z)$$

Lemma

Let $x \sqsubseteq y$ *iff* $q(x,y) \ = \ 0$. *Then* $(X,\ \sqsubseteq)$ *is a poset.*

Lemma

Let $d(x,y) \ = \ q(x,y) \ + \ q(y,x)$. *Then* d *is a metric (but not non negative).*

Note

Domain theory, a branch of computer science, is unique ***only*** *up to some poset.*

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Poset

Partially ordered set

Definition

A **poset** is a pair $(X, \sqsubseteq \subseteq X \times X)$ such that,

$x \sqsubseteq x$

if $x \sqsubseteq y$ and $y \sqsubseteq x$ then $x = y$

if $x \sqsubseteq y$ and $y \sqsubseteq z$ then $x \sqsubseteq z$

- For today we assume simply the minimal properties that our posets have a least member $\perp \sqsubseteq x$ and are chain-complete.
- Following *domain theory* our research is an asymmetric reconciliation of *poset* theory and T_0 *topology*. However, it is firstly defined *metrically* in a usual symmetric sense.
- The poset $x \sqsubseteq y$ must coincide with the specialisation ordering $x \in cl\{y\}$.

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Non zero self-distance???

There is a precedent

- In 1942 Karl Menger generalised the concept of *metric space* to that of **statistical metric space** by generalising the notion of *distance* from that of a non negative real number to that of a distribution function.
- In Menger's notation, $F(x; p, q)$ is the probability that the distance of p and q is less than x.
- The relevance to partial metric spaces is that it sets a precedent that the presumed *exactness* of a distance may be questioned.
- Besides that of Karl Menger, are there other generalisations for metric spaces embodying a less than exact notion of *distance*?
- Karl Menger broke the mould of exactness in metric spaces for exact distances. However, for the special case of self-distance in a statistical metric space $F(x; , p, p) = 1$ for any $x > 0$, and so self-distance for Menger is, as for Fréchet, *certainly* zero.

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Partial metric space

A generalised metric space for which self-distance is to be not necessarily zero

Definition

A **partial metric space** is a pair $(X, \; p : X \times X \to \Re)$ such that,

$$
\begin{array}{l}
p(x,x) \; \leq \; p(x,y) \\
\text{if} \;\; p(x,x) \; = \; p(y,y) \; = \; p(x,y) \;\; \text{then} \;\; x = y \\
p(x,y) \;\; = \;\; p(y,x) \\
p(x,z) \;\; \leq \;\; p(x,y) \; + \; p(y,z) \; - \; p(y,y)
\end{array}
$$

Lemma

Each metric space is a partial metric space.

Note

First came non zero self-distance in my thesis, 1985. Then Vickers contributed the triangularity axiom in 1987. The above axioms were presented in the Summer Conference of 1992.

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Properties of partial metric spaces

Lemma

Let $q(x,y) = p(x,y) - p(x,x)$*. Then* q *is a quasimetric.*

Lemma

Let $x \sqsubseteq y$ *if* $p(x,x) = p(x,y)$*. Then* $(X, \sqsubseteq)$ *is a poset.*

Lemma

Let $d(x,y) = 2 \times p(x,y) - p(x,x) - p(y,y)$*. Then* d *is a metric.*

Lemma

The ***open balls*** $B_\epsilon(a) = \{x \in A : p(x,a) < \epsilon\}$ *are the basis for the usual topology. Equivalently, this is the induced quasimetric topology.*

Lemma

Each partial metric topology is T_0*, and is a sub topology of the induced metric topology.*

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What is new about partial metric spaces?

They add *weight* to *metric space*

If, as shown above, each partial metric space can be defined using an equivalent quasimetric, what new construction is added to the theory of *metric spaces*?

Definition

A **weight** is a function $|\cdot| : X \rightarrow \Re$.

Note

Pseudometric adds equivalence relation to metric space, and quasimetric adds poset. But, neither adds weight.

Definition

A **weighted metric space** is a tuple $(X, d, |\cdot|)$ such that,

(X, d) is a metric space

$|\cdot|$ is a weight

$d(x, y) \geq |x| - |y|$

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What is new about partial metric spaces?

They add *weight* to *metric space*

Lemma

For each weighted metric space $(X, d, |\cdot|)$ *let,*

$$p(x,y) \;=\; \frac{d(x,y) \,+\, |x| \,+\, |y|}{2}$$

Then (X, p) *is a partial metric space, and* $p(x,x) = |x|$ *.*

Lemma

For each partial metric space (X, p) *let,*

$$\begin{aligned} d(x,y) &= 2 \times p(x,y) - p(x,x) - p(y,y) \\ |x| &= p(x,x) \end{aligned}$$

Then $(X, d, |\cdot|)$ *is a weighted metric space, and*
$|x| = p(x,x)$ *.*

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What is new about partial metric spaces?

They add *weight* to *quasimetric space*

Definition

A **weighted quasimetric space** is a tuple $(X,\ q,\ |\cdot|)$ such that,

$$(X, q) \quad \text{is a quasimetric space}$$

$$|\cdot| \quad \text{is a weight}$$

$$|x| + q(x, y) \;=\; |y| + q(y, x)$$

Lemma

Not every quasimetric space is weight-able (Matthews, 1992).

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What is new about partial metric spaces?

They add *weight* to *quasimetric space*

Lemma

For each weighted quasimetric space $(X,\ q,\ |\cdot|)$ *let,*

$$p(x,y) \ = \ |x| + q(x,y)$$

Then $(X,\ p)$ *is a partial metric space, and* $p(x,x) = |x|$ *.*

Lemma

For each partial metric space $(X,\ p)$ *let,*

$$\begin{aligned} q(x,y) \ &= \ p(x,y) - p(x,x) \\ |x| \ &= \ p(x,x) \end{aligned}$$

Then $(X,\ q,\ |\cdot|)$ *is a weighted quasimetric space, and* $|x| = p(x,x)$ *.*

How does *weight* relate to *order*?

Weight is consistent with order

However we introduce *weight*, be it into a metric space, into a quasimetric space, or derived from a partial metric space, the induced ordering is consistent with the weight.

Lemma

If $x \sqsubseteq y$ *then* $|x| \geq |y|$

- Good! But, what does it mean? First we ask this question of *domain theory*, with a view later of asking *general topology*.
- *Domain theory* is a model of *computation* as **increasing** information.
- In contrast, *weight* is a function **decreasing** to 0 .
- As domain theory models the information that has been computed, then weight must model how much information has yet to be computed.

An example of a partial metric space

To describe the notion of *flat domain* in the theory of metric spaces

Definition

A **flat domain** is a poset $(X \cup \{\perp\},\ \sqsubseteq)$ such that,

$\perp \notin X$

if $x \sqsubset y$ then $x = \perp$

Example

For each set X and $\perp \notin X$ let,

$$p(x,y) = \begin{cases} 0 & \textit{if } x = y \in X \\ 1 & \textit{otherwise} \end{cases}$$

Then p is a partial metric, and $(X \cup \{\perp\},\ \sqsubseteq)$ is a flat domain.

$\perp$ (pronounced *bottom*) represents *no output so far*, while x is *the output* if it ever comes. That is, at each moment in time **all** or **nothing** has been output.

An example of a partial metric space

To describe the notion of *flat domain* in the theory of metric spaces

"The answer to the Great Question . . . Of Life, the Universe and Everything is ..." Forty-two . . . " said Deep Thought with infinite majesty and calm.
"Forty two!" yelled Loonquawl.
"Is that all you've got to show for seven and a half million years' work?"
"I checked it very thoroughly," said the computer, "and that quite definitely is the answer. I think the problem, to be quite honest with you, is that you've never actually known what the question is."

$\perp \quad \sqsubset$ 

From *The Hitchhiker's Guide to the Galaxy* (1979) (Douglas Adams, 1952-2001).

Another example of a partial metric space

The first real domain

Let ω be the set of all **natural numbers**,

$$\omega \;=\; \{1,\, 2,\, 3,\, ,\; \ldots\; \}$$

Let $P\omega$ denote the set of all subsets of ω .
$P\omega$ is historically important in domain theory, as it was the first real domain, that is, the model defined by Dana Scott (1969) for the λ-*calculus* (see Stoy 1977).
Let,

$$p(x,\, y) \;=\; 1 - \sum_{n \in x \cap y} 2^{-n} \quad \textit{for any } x, y \;\in\; P\omega$$

Then p is a partial metric, with the (usual) subset ordering, $x \sqsubseteq y$ iff $n \in x \Rightarrow n \in y$. Also, $\perp \;=\; \{\,\}$ and $\top \;=\; \omega$.

Another example of a partial metric space

The *interval domain*

Example

For all closed intervals on the real line let,

$$p([a,\ b],\ [c,\ d]) \ = \ max\{b,\ d\} \ - \ min\{a,\ c\}$$

Then p is a partial metric, $|[a,b]| = b - a$, and $[a,b] \sqsubseteq [c,d]$ iff $[c,d] \subseteq [a,b]$.

Note

To ensure that the intervals form a domain, a little more work is required. For example, as each domain has to have $\perp$ *we might only consider the domain of closed sub intervals of* $[0,1]$ *.*

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What do we know so far?

- The notion of *metric space* can be generalised to meaningfully introduce *non zero self-distance*.
- Equivalently, *metric space* can be generalised to introduce *weight*.
- Equivalently, *quasimetric space* can be generalised to introduce *weight*.
- Equivalently, each *partial metric space* is a *metric space*, a *weight*, and a *poset* as a single formulation.
- Partial metric spaces are consistent with *domain theory*, the so-called *Scott-Strachey* order-theoretic topological model for a logic of computer programs (1969).
- Domain theory (1969) is founded upon only *poset* and *topology*, reconciled to quasimetrics by Smyth (1988). Thus domain theory precedes partial metric spaces (1992). So, why bother?

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A *weighted* contraction mapping theorem

To model the *Cycle Sum Test*

- On one occasion the *order* and *topology* of domain theory did not have a counterpart for *weight*.
- In fact, the need to find a proof for Wadge's *Cycle Sum Test* led to the necessity for a notion of *non zero self-distance* (Matthews, 1985) to work with domain theory and metric spaces.
- *Partial metric space* (Matthews, 1992) is thus the eventual formalisation of Wadge's intuition (1981),

  > *"A complete object (in a domain of data objects) is, roughly speaking, one which has no holes or gaps in it, one which cannot be further completed."*

- That is x is **complete** if $p(x,x) = 0$, otherwise **partial.**

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A *weighted* contraction mapping theorem

To model the *Cycle Sum Test*

- The early conception of a **correct** computer program was one that always **terminated**.
- By the 1970s computer scientists had many programs (such as for computing the value of π) which did not terminate, but should be nonetheless *correct*.
- Wadge intuited that if at each stage in the execution of a program progress was made, then, at the end of time, be it finite or infinite, the result must be *correct*.
- But! *correctness* as *termination* did not allow for the possibility of a correct program (such as for computing π) taking infinitely long.
- Domain theory (of the 1970s) could model infinitely long programs, but, had no machinery for identifying those to be treated as being *correct*.

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A *weighted* contraction mapping theorem

To model the *Cycle Sum Test*

- A program passes the Wadge *Cycle Sum Test* if each possible *cycle* in the execution necessarily results in a **net increase** in the amount of data produced by that cycle.
- The Cycle Sum Test was hard to prove in the very machine oriented world of computer science.
- Once the metrical abstraction of the *partial metric* was established, all the messy machine detail could be scrapped, and the Test reduced to the obvious *weighted* generalisation of Banach's contraction mapping theorem (1922).

Theorem

Each contraction mapping in a complete partial metric space has a unique fixed point, and this point is complete (Matthews, 1995).

Based metric space

A way to *view* a metric space

Definition

A **based metric space** is a tuple $(X, d, \phi \in X)$ such that (X, d) is a metric space.

Lemma

For each based metric space (X, d, ϕ) *let,*

$$p(x, y) = \frac{d(x, y) + d(x, \phi) + d(y, \phi)}{2}$$

Then (X, p) *is a partial metric space,* $x \sqsubseteq \phi$, $|\phi| = 0$, *and* $|x| = d(x, \phi)$ *for each* $x \in X$.

This suggests that the asymmetry and weight found in partial metric spaces are not actually far removed from the original mathematics of metric spaces, and not dependent upon domain theory.

Based metric space

Another way to *view* a metric space

The base point could be $\perp$ (to suit domain theory) or $\top$ to suit metric topology.

Lemma

For each based metric space (X, d, ϕ) *(and constant c) let,*

$$p(x,y) = c + \frac{d(x, y) - d(x, \phi) - d(y, \phi)}{2}$$

Then (X, p) *is a partial metric space,* $\phi \sqsubseteq x$ *,* $|\phi| = c$*, and* $|x| = c - d(x, \phi)$ *for each* $x \in X$ *.*

- Thus *partial metric space* with $\top$ or $\perp$ is equivalent to *metric space with base point*.
- The problem is, deciding whether the base point should be $\top$, $\perp$, or conceivably something else?

Properties of partial metric spaces

- Once the taboo against non zero self-distance has been broken, other questions soon arise.
- For example, (Heckmann 1999) demonstrated that the so-called *small self-distances* axiom $p(x,x) \leq p(x,y)$ can be dropped as follows.
- Define a partial metric to be **weak** if it does not have to satisfy the small self-distances axiom.

Lemma

Let $p'(x,y) = max\{p(x,x),\, p(x,y),\, p(y,y)\}$ *for a weak partial metric* p *. Then* p' *is a partial metric, and has the same topology as* p *.*

- This example reinforces the conception of a partial metric space being a threefold combination of *metric*, *poset*, and *weight*, but, none getting lost in the mix.

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Two topologies are better than one

- For each partial metric p, and $a > 0$, and any b, let $p'(x,y) = a \times p(x,y) + b$. Then p' is a partial metric, having the same poset, and same topology as p.
- In contrast to the usual convention for metric space, distance could be negative. That is, if p is a partial metric, then, $p'(x,y) = p(x,y) - c$ is equivalent.
- Let $p^*(x,y) = p(x,y) - p(x,x) - p(y,y)$ be the **dual** partial metric of p. Then $\sqsubseteq_{p^*} = \sqsupseteq_p$.
- Let $p(x,y) = max\{x, y\}$ over the real line. Then p is a partial metric, with the usual ordering $\sqsubseteq = \leq$.
- Each partial metric p gives rise to a bitopological space, $(X, \tau[p], \tau[d])$ where $\tau[p] \subseteq \tau[d]$.

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Efficiency oriented languages

A new application for partial metric spaces?

- Michel Schellekens has advocated **complexity spaces**, and **efficiency oriented programming languages**.
- This is evidence that the **algorithms and complexity** genre of computer science can be unified with that of **denotational semantics**, which is founded upon **domain theory**.
- However, there is little historical, or natural affinity between the two sub disciplines to call upon.
- Partial metric spaces do suggest that the quantitative notion of **weight** can be introduced to the qualitative notion of **topology**.

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Cost-oriented topology

There is no such thing as a free lunch

- A typical algorithm to generate the sequence of all prime numbers would take longer and longer to produce each number.
- In domain theory we have order theory and topology to model the primes as follows.

$$\bot \sqsubseteq \langle 2 \rangle \sqsubseteq \langle 2,\ 3 \rangle \sqsubseteq \langle 2,\ 3,\ 5 \rangle \sqsubseteq \langle 2,\ 3,\ 5,\ 7 \rangle \sqsubseteq \ \ldots$$

- But, there is no means here to model the **complexity** of the algorithm which would inform us that it takes longer and longer to produce each prime.

Cost-oriented topology

There is no such thing as a free lunch

- Wadge envisaged the idea of a **hiaton**, a *pause object*.
- For example, the following sequence includes both the necessary domain theory for expressing prime numbers, and the pauses.

$$\langle *,\ 2,\ 3,\ *,\ 5,\ *,\ 7,\ *,\ *,\ *,\ 11,\ \ldots \rangle$$

- At present Wadge's **hiaton** remains the most intuitive argument for motivating computer science research into cost-oriented topology, while Schellekens *complexity spaces* is perhaps the most substantive theory available.
- What we really need is for applied topology to break free of computer science, and to take on the challenge of defining a new sub discipline of **cost-oriented topology**.

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Key contributors to partial metric spaces

See `partialmetric.org` for links to publications

Michael Bukatin – quantitative domains, *relaxed* metrics, relations with *fuzzy* sets and Höhle's *many valued topology*.

Reinhold Heckmann – *weak* partial metric drops the *small self-distance* axiom $p(x,x) \leq p(x,y)$.

Ralph Kopperman – all topologies come from generalised metrics, bi-topology, partial metrizability (into value quantales).

Hans-Peter Künzi – asymmetric topology, quasi metrics, quasi uniformities.

Steve Matthews – *partial metric* (X,p), contraction mapping theorem, data flow, *based metric* (X,d,ϕ).

Simon J. O'Neil – negative distance $p(x,y) < 0$, dual partial metric $p^*(x,y) = p(x,y) - p(x,x) - p(y,y)$.

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Key contributors to partial metric spaces

See `partialmetric.org` for links to publications

Homeira Pajoohesh – lattices, partial metrizability (into value quantales).
Michel P. Schellekens – characterising partial metrizability, semivaluations, quantitative domains, *efficiency oriented languages*.
Mike Smyth – constructive maximal point space and partial metrizability.
Steve Vickers – $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$ axiom, *topology via logic*.
Bill Wadge – *Lucid*, *complete object*, *cycle sum test*.
Pawel Waszkiewicz – quantitative domains.

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What's it really all about?

Topology, *Nothing*, *View*, and *Cost*

Nothing defined,

$$x + 0 \; = \; 0 + x \; = \; x$$

Nothing partially known,

$$x + \bot \; = \; \bot + x \; = \; \bot \sqsubseteq x$$

Distance defined,

$$d(x, y)$$

Distance partially known,

$$p(x, y)$$

The topology of *nothing*,

$$\bot \sqsubseteq x \sqsubseteq y \in O \in \tau \; \subseteq \; 2^X$$

A fuss about *nothing*

A background tutorial for partial metric spaces

Speaker : Steve Matthews

The plenary session in applied topology has introduced the essential concepts and results of *partial metric spaces*. In so doing there was not time to describe the background in computer science that actually gave rise to the conception of non zero self-distance in metric spaces. It is thus instructive to give a tutorial upon how concerns in programming language design of the 1970s came to be related to metric spaces, and from there, how metric topology is returning full circle to influence, what is now known in computer science as, **discrete mathematics**. In short, this tutorial is intended to be an inspiring example of how an infinitary concept such as *metric space* from continuous mathematics can be re-discovered to simplify the *finitary* structure of contemporary computer science. For applied topology there is a useful, liberating lesson here that *finitary* concepts are not trivial, but naturally arise in a modern context as *partial* approximations to simplify their infinitary counterparts.

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Reference for tutorial

Download via `partialmetric.org`

- **Lucid, a Nonprocedural Language with Iteration**. E.A. Ashcroft and W.W. Wadge. Communications of the Association for Computing Machinery. Vol. 20, Issue 7, pp. 519-526. July 1977. ISSN:0001-0782.