Two Topologies
Are Better Than One

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Abstract

Partially ordered sets and metric spaces are used in studying semantics in Computer Science. Sets with both these structures are hence of particular interest. The partial metric spaces introduced by Matthews are an attempt to bring these ideas together in a single axiomatic framework.

We consider an appropriate context in which to consider these spaces is as a bitopological space, i.e. a space with two (related) topologies. From this starting point, we cover the groundwork for a theory of partial metric spaces by generalising ideas from topology and metric spaces.

For intuition we repeatedly refer to the real line with the usual ordering and metric as a natural example. We also examine in detail some other examples of more relevance to Computer Science.

1 Introduction

In [5] Matthews introduces partial metric spaces and provides clues for developing a general theory. The main contribution of this report is to make precise *how* we generalise metric and topological properties to partial metric spaces. We do this by providing a general framework in which to work, and see how far this process can take us.

We make no claim that this work is near completion. For example, important topics such as function spaces have barely been touched upon. For this reason we avoid any discussion of the applicability of the theory, although [4] does contain some examples, and ideas for the future.

Originality

The originality of this report is fairly hard to judge, since it is linked so closely with the earlier paper [5]. Some results are taken directly from this paper, others are in the paper, but we approach them from a new perspective, and yet others are completely new in their present format. However, many of the later are mainly generalisations from metric spaces, with similar proofs to match. We make these somewhat vague comments more precise by contrasting this report with the earlier paper, although familiarity with [5] is not necessary to read this report.

We make one fundamental change to the partial metric axioms, by allowing the partial metric function to take negative values. This means that we have to recheck even the most fundamental results, and accounts for most of the duplication from [5].

A second difference is in the definition of the open balls. It is arguable, of course, but we feel that the version we adopt here is a more natural generalisation of the metric open balls, since we can choose ε to be arbitrarily small, without the open ball being empty.

Apart from these two changes, the essential difference between the two presentations is in the approach taken. One example will suffice to explain this. In [5] the Cauchy sequences are defined directly in terms of the partial metric. However, we fit the definition into a more general framework, thereby explaining where the definition comes from.

At the heart of our new approach is the role we give the metric induced on our spaces by the partial metric. Rather than considering this as an interesting side effect, we make it central to the theory. This gives us the second topology on our space, and leads to the bitopological approach we adopt to generalisations.

Required Background

Very little Computer Science background is needed for this report, if it can be accepted that partial orders and metric spaces are of relevance to the subject. If not, [6] is recommended for details.

This report is entirely mathematical, and relies on firm foundations in metric spaces (see [8] for example), and to a lesser degree, topological spaces (see [3]). Although partial orders are central to the report, beyond the basic definitions very little background in this area is needed. For this reason we give the relevant information in the appendices (for more details see [1]).

2 Partial Metric Spaces

This section covers much the same ground as [5], with the notable exception of Cauchy sequences, which are considered later. We begin with the axioms, which differ in one regard from those given by Matthews, and will give a number of contrasting examples. The partial metrics on S^{∞} and $S^{\perp}$ were given in [5], but the others are new. Throughout the section we refer back to a natural partial metric on $\Re$ for intuition.

The existence of our first topology is essentially the same as in [5], but we include the proof since our axioms have changed. We then begin to differ in our approach to the subject, in that we regard the order and convergence as derived from the topology, and prove the partial metric definitions of Matthews.

We consider the heart of the subject to be the second topology on our spaces. The existence of this is certainly proved in [5]. However we raise it to a greater degree of prominence for reasons that we explain in the text. We conclude the section with a general discussion of how this bitopological approach should be used in developing the theory.

2.1 Axioms and Basic Results

The axiomatic definition of a partial metric space uses axioms generalised from metric spaces. Our spaces have a natural T_0–topology on them, and every T_0–space has an induced partial order and notion of convergence. In our spaces both these properties can be expressed naturally in terms of the partial metric.

Our main motivating example will be $\Re$, which we will give a partial metric respecting the natural ordering and metric properties. We will also consider other examples, two of which are of particular importance in Computer Science.

Partial Metric Axioms

In [5] Matthews introduces the partial metric axioms as a generalisation of the metric axioms. The essential difference being that the distance of a point from itself is not necessarily zero (but always positive). We take this generalisation one step further, and allow the distance function to be negative. We could justify this by saying that very few changes were needed to the proofs in [5] as a result. We will however, give a more convincing argument in section 3.4.

Definition 2.1 *A* **partial metric space** *is a non–empty set S, and a function (the* **pmetric***) $p : S \times S \to \Re$, such that for any $x, y, z \in S$:*

- **P1.** $p(x, y) \geq p(x, x)$.
- **P2.** $x = y$ *if, and only if,* $p(x, x) = p(x, y) = p(y, y)$.
- **P3.** $p(x, y) = p(y, x)$.
- **P4.** $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

We consider the following to be a standard example of a partial metric space. We will consider a number of other examples later.

Example 2.2 *The real line $\Re$ is a set with which we are familiar, and has a natural metric and order associated with it. This space is thus a prime candidate for our first example of a partial metric space. Suppose we define $p : \Re \times \Re \to \Re$ by*

$$p(x, y) = -\min\{x, y\},$$

then the proof that $(\Re, p)$ is a partial metric space is straight–forward. We call p the **usual pmetric** *on $\Re$. Notice that $p'(x, y) = \max\{x, y\}$ is also a pmetric on $\Re$. We will later see why we call p, and not p', the usual pmetric on $\Re$.*

Partial Metric Topology

Our first task is to show that partial metric spaces are T_0–topological spaces. As with metric spaces, we begin with the open balls. For any $x \in S$, $\varepsilon \in \Re$ ($\varepsilon > 0$), we define:

$$B_\varepsilon(x) = \{y \in S \mid p(x,y) < p(x,x) + \varepsilon\}.$$

If there is any possible confusion with the notation, we emphasis that this is a p–open ball, by writing $B_\varepsilon(x;p)$.

Lemma 2.3 *If (S,p) is a partial metric space, then the open balls are a basis for a T_0–topology on S, called the* **pmetric topology***, and denoted by $\mathcal{T}[p]$.*

Proof. Suppose $B_{\varepsilon_x}(x)$ and $B_{\varepsilon_y}(y)$ are open balls in (S,p), and that

$$z \in B_{\varepsilon_x}(x) \cap B_{\varepsilon_y}(y).$$

We define

$$\delta = \min\{p(x,x) + \varepsilon_x - p(x,z), p(y,y) + \varepsilon_y - p(y,z)\} > 0,$$

and show that $B_\delta(z) \subseteq B_{\varepsilon_x}(x) \cap B_{\varepsilon_y}(y)$. Suppose $z' \in B_\delta(z)$, then

$$\begin{array}{rcl} p(x,z') & \leq & p(x,z) + p(z,z') - p(z,z) \\ & < & p(x,z) + \delta \\ & \leq & p(x,x) + \varepsilon_x, \end{array}$$

and $z' \in B_{\varepsilon_x}(x)$. Similarly we prove that $z' \in B_{\varepsilon_y}(y)$. Since $S = \bigcup_{x \in S} B_1(x)$, then the open balls are indeed a basis for a topology on S.
To see that the topology is T_0, we suppose that $x, y \in S$ are distinct points, and that $p(x,x) < p(x,y)$ (wlog). If we let

$$\varepsilon = p(x,y) - p(x,x) > 0,$$

then $x \in B_\varepsilon(x)$, but $y \notin B_\varepsilon(x)$.

$\square$

Example 2.4 *In $\Re$ with the usual pmetric, the open balls are of the form*

$$B_\varepsilon(x) = \{y \in \Re \mid -\min\{x,y\} < -x + \varepsilon\} = (x - \varepsilon, \infty) \subseteq \Re.$$

Partial Order

Once we have a T_0–topology, we immediately have a natural partial order[1] ($\sqsubseteq$) on S, which we call the **induced order**. Unless we specify otherwise, whenever we consider a partial order on S it will be the induced order.

The value of partial metric spaces lies in the fact that the structure of the space can be expressed naturally in terms of the pmetric, thus allowing an analytic approach to many problems. The following lemma gives our first instance of this.

Lemma 2.5 *Suppose (S, p) is a partial metric space then*

$$x \sqsubseteq y \textit{ if, and only if, } p(x, x) = p(x, y).$$

Proof. Suppose first that $x \sqsubseteq y$. Then, for all $\varepsilon > 0$, $y \in B_\varepsilon(x)$ and we have,

$$p(x, x) \leq p(x, y) < p(x, x) + \varepsilon.$$

Since this is true for all $\varepsilon > 0$, then $p(x, y) = p(x, x)$.
Conversely, suppose that $p(x, x) = p(x, y)$, and $x \in U$ with $U \in \mathcal{T}[p]$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. However,

$$p(x, y) = p(x, x) < p(x, x) + \varepsilon,$$

implies that $y \in B_\varepsilon(x)$, and so $y \in U$. Since this holds for any $U \in \mathcal{T}[p]$, we see that $x \sqsubseteq y$.

□

One topic of interest, is when we can put a pmetric on a partially ordered set that captures the order. We introduce some terminology now, and will return to this subject later.

Definition 2.6 *If $(S, \leq)$ is a partially ordered set, then a pmetric p on S is* **satisfactory** *if p induces the ordering $\leq$ on S.*

Example 2.7 *The usual pmetric on $(\Re, \leq)$ is satisfactory since*

$$x \sqsubseteq y \iff -x = -\min\{x, y\} \iff x \leq y.$$

The pmetric $p'(x, y) = \max\{x, y\}$ is not satisfactory, and in fact induces the dual ordering.

[1]The specialisation order, see the appendices for details.

Examples of Partial Metric Spaces

Let us consider some other examples of partial metric spaces. We omit the proofs since they are straightforward.

Example 2.8 *Let $P\omega$ denote the power set of the natural numbers, with the subset ordering. This set plays an important role in Computer Science, since Scott showed that it can be considered as a semantic model of the λ–calculus (see [7] for example).*

If we define $p : P\omega \times P\omega \to \Re$ by,

$$p(x, y) = 1 - \sum_{n \in x \cap y} 2^{-n}, \quad \textit{for any } x, y \in P\omega,$$

then p is a satisfactory pmetric on $P\omega$.

Example 2.9 *A second example of interest to Computer Science, is the set S^∞ of finite and infinite sequences over some set S, with the prefix ordering, as discussed in [4, 6].*

If we denote the length of a sequence $x \in S^\infty$ by $l(x)$, then the function $p : S^\infty \times S^\infty \to \Re$ defined by,

$$p(x, y) = 2^{-\sup\{i \in \mathbf{N} | i \leq \min\{l(x), l(y)\}, \forall j < i, x_j = y_j\}}, \quad \textit{for any } x, y \in S^\infty,$$

is a satisfactory pmetric on S^∞, called the **Baire pmetric**. *Intuitively, the value of the supremum is the first instance where the sequences differ, taking care if one sequence is shorter than the other.*

Example 2.10 *In common with other theories, it is always useful to have some examples which satisfy the axioms, if not the motivating intuition. These become a source of possible counter–examples when developing the theory.*

For us, metric spaces can play this role since they must have a flat induced order. We mention specifically the function $p^f : \Re \times \Re \to \Re$ defined by

$$p^f(x, y) = \tfrac{1}{2}|x - y|, \textit{ for all } x, y \in \Re,$$

which we call the **flat pmetric** *on $\Re$.*

Example 2.11 *As a final example we consider the flat domain $(S^\perp, \leq)$ over a set S, where $S^\perp = S \cup \{\perp\}$ and $x \leq y$ if, and only if, $x = \perp$ or $x = y \in S$. If we define $p : S \times S \to \Re$ to be the discrete metric, and extend this to $S^\perp$ by defining $p(x, \perp) = p(\perp, x) = 1$, for all $x \in S^\perp$, then we have a satisfactory pmetric on $S^\perp$.*

Convergence

As with all topological spaces, we have a notion of convergence in partial metric spaces. In common with metric spaces, this can be expressed naturally in terms of the pmetric.

Lemma 2.12 *Suppose* (x_n) *is a sequence in a partial metric space* (S,p) *and* $a \in S$*, then*

$$x_n \to a \text{ if, and only if, } \lim_{n\to\infty} p(x_n,a) = p(a,a).$$

Proof. It is clear that $\lim_{n\to\infty} p(x_n,a) = p(a,a)$ if, and only if, for any $\varepsilon > 0$, there exists $N \geq 1$ such that $x_n \in B_\varepsilon(a)$, for any $n \geq N$. Since the open balls are a basis for $\mathcal{T}[p]$, the result follows.

□

Suppose (x_n) is a sequence in a partial metric space (S,p), and we define $\mathcal{L}(x_n)$ to be the set of limit points of (x_n). It is almost immediate that if $a \in \mathcal{L}(x_n)$ and $a' \sqsubseteq a$, then $a' \in \mathcal{L}(x_n)$, since any open set containing a' must contain a, and hence eventually contain (x_n). We will be looking at $\mathcal{L}(x_n)$ again in section 3.1.

Example 2.13 *In* $\Re$ *with the usual pmetric, the sequence* $(1/n)$ *has* $\mathcal{L}(1/n) = (-\infty, 0]$.

2.2 A Second Topology

Before giving the second topology on a partial metric space (S,p), we motivate the need for it. We have already seen how the induced convergence and partial order can be expressed naturally in terms of the pmetric. It would seem that the next logical step is to consider other possible topological properties of $\mathcal{T}[p]$ in terms of the pmetric.

Let us see where this takes us. In metric spaces, dense sets are used to approximate points in the space, and in generalising to partial metric spaces, we would like to preserve this meaning. Suppose $b \in \Re$, and we have the usual pmetric on $(-\infty, b] \subseteq \Re$, then the set $\{b\}$ is dense in $\mathcal{T}[p]$. But how can b "approximate" every point in $(-\infty, b]$ in any intuitive sense of the word?

The problem is that in considering the pmetric topology, we are really only investigating the order properties on S. There is however, a metric structure on the space as well, not captured by $\mathcal{T}[p]$. So, we are not looking at our spaces in the correct context, and it is not surprising that we are only getting part of the picture. We see how we can view partial metric spaces as bitopological[2] spaces, and consider the implications of this in developing a theory of partial metric spaces.

Induced Metric Topology

If we look again at the definition of the open balls, we see that in a partial metric space (S,p), the actual values of $p(x,y)$ and $p(x,x)$ (for some $x, y \in S$) are of less significance than the difference $p(x,y) - p(x,x) \geq 0$. This simple observation will lead us to our second topology on S.

[2] A bitopological space is just a set with two topologies, see [2] for example.

Lemma 2.14 *Suppose (S, p) is a partial metric space, and we define a function $d : S \times S \to [0, \infty)$ by*

$$d(x, y) = 2p(x, y) - p(x, x) - p(y, y), \text{ for any } x, y \in S.$$

Then (S, d) is a metric space, and if $\mathcal{T}[d]$ is the metric topology on S, we have $\mathcal{T}[p] \subseteq \mathcal{T}[d]$.

Proof. We can immediately prove from the definitions that d is a metric,

$$d(x, y) = 0 \iff p(x, x) = p(x, y) = p(y, y) \iff x = y,$$

$$\begin{aligned} \text{and} \quad d(x, y) &= 2p(x, y) - p(x, x) - p(y, y) \\ &\leq 2p(x, z) + 2p(z, y) - 2p(z, z) - p(y, y) - p(x, x) \\ &= d(x, z) + d(z, y). \end{aligned}$$

To see that $\mathcal{T}[p] \subseteq \mathcal{T}[d]$, we note that,

$$p(x, y) - p(x, x) \leq d(x, y), \quad \text{for any } x, y \in S,$$

which implies that, for any $\varepsilon > 0$, $B_\varepsilon(x; d) \subseteq B_\varepsilon(x; p)$.

□

We call d the **induced metric** on (S, p). Unless we specify otherwise, whenever we consider a metric on S, it will be the induced metric. Now we have a metric on our spaces, we similarly extend definition 2.6. We will then see some examples of induced metrics, and in particular how different partial metric spaces can have the same induced metric space.

Definition 2.15 *If $(S, \leq, d)$ is a partially ordered metric space, then a pmetric p on S is* **satisfactory** *if p induces the ordering $\leq$ and the metric d on S.*

Example 2.16 *If we apply this discussion to $\Re$ with the usual pmetric p, we find that*

$$\begin{aligned} d(x, y) &= 2p(x, y) - p(x, x) - p(y, y) \\ &= x + y - 2\min\{x, y\} \\ &= |x - y|. \end{aligned}$$

So the usual pmetric is satisfactory on $\Re$ with the usual order and metric. The pmetric $p'(x, y) = \max\{x, y\}$ on $\Re$ induces the dual ordering and usual metric, and the flat pmetric $p^f(x, y) = \frac{1}{2}|x - y|$ induces the flat order and usual metric.

Example 2.17 *As another example we consider the space $P\omega$ with the pmetric defined in (2.8). This induces the metric,*

$$d(x, y) = \sum_{n \in x \triangle y} 2^{-n}, \quad \text{for any } x, y \in S,$$

where $x \triangle y = (x \setminus y) \cup (y \setminus x)$.

Towards A General Theory

We are now in a position to regard a partial metric space (S, p) not as a set S with a distance function p, but as a bitopological space $(S, \mathcal{T}[p], \mathcal{T}[d])$. It is then a simple step to say that we regard a topological property as holding on (S, p) if it holds for both the topologies, and that a metric property holds on (S, p) if it holds on (S, d).

The fact that $\mathcal{T}[p] \subseteq \mathcal{T}[d]$, merely simplifies the issue in most cases. Indeed we find that the work we have to do in developing a theory of partial metric spaces is significantly reduced, since many results come directly from the induced metric space.

The difficulty though, is in accepting that bitopology is the correct context in which to view partial metric spaces. This is not something that we can prove in anyway. However, throughout the next section we use the bitopological approach to generalise many ideas to partial metric spaces. In each case we give examples as to why we consider the resulting definitions to be intuitively correct.

More generally, what we are seeking is a theory of partial metric spaces with as much of the flavour of metric spaces as possible, while respecting the additional structure, namely the partial order. The fact that we are interested in spaces with two structures, is perhaps the most convincing case for a two topology approach.

3 Bitopological Properties

Perhaps the only way to see if the bitopological approach is the correct one, is to see how far it will take us, if it agrees with our intuition, and if it proves useful in applications. In this section we pursue the first, and find (unsurprisingly) many interesting connections with metric spaces. By considering examples we hopefully show how the intuition agrees with the theory we are developing. As for applications, these will come only once the theory is sufficiently well developed.

We take four of the most fundamental ideas from topology and metric spaces; separability, continuity, completeness and compactness, and generalise to partial metric spaces. Of these only completeness has been considered before by Matthews, in [5]. An immediate benefit of our general approach is that we can cover a lot of ground fairly quickly. Although we continue to use the natural pmetric on $\Re$ as a motivating example, in this section we also consider the spaces $P\omega$ and S^∞ in some detail.

3.1 Separability

Separable spaces in topology have countably dense subsets, and since we used these as motivation for the bitopological approach in the last section, it seems appropriate to look at these first. We begin by reconsidering convergence, and then move on to dense sets, before starting our investigation into the properties of $P\omega$ and S^∞.

Proper Convergence

We call convergence in the bitoplogical sense proper convergence to distinguish it from our existing definition. It turns out to be useful to have both notions of convergence available to us. We give the definition (which uses the fact that $\mathcal{T}[p] \subseteq \mathcal{T}[d]$), and then see how we can express proper convergence naturally in terms of the pmetric.

Definition 3.1 *If (x_n) is a sequence in a partial metric space (S, p), then $c \in S$ is a* **proper limit** *of (x_n), written $x_n \to c$ (properly), if $x_n \to c$ in (S, d). If a sequence has a proper limit then we say that the sequence is* **properly convergent**.

Lemma 3.2 *Suppose (x_n) is a sequence in a partial metric space (S, p) and $c \in S$, then*

$$x_n \to c \text{ (properly) if, and only if, } \lim_{n \to \infty} p(x_n, c) = \lim_{n \to \infty} p(x_n, x_n) = p(c, c).$$

Proof.

$$\begin{aligned} x_n \to c \text{ (properly)} &\iff \lim_{n \to \infty} d(x_n, c) = 0 \\ &\iff \lim_{n \to \infty} p(x_n, c) = \lim_{n \to \infty} p(x_n, x_n) = p(c, c). \end{aligned}$$

□

Example 3.3 *In $\Re$ with the usual pmetric the proper limit of the sequence $(1/n)$ is 0.*

From the definition, we immediately see that proper limits are unique if they exist. We return briefly to the set $\mathcal{L}(x_n)$ of limit points of a sequence (x_n). The next lemma shows that $\mathcal{L}(x_n)$ has a nice structure for properly convergent sequences.

Lemma 3.4 *Suppose (x_n) is a sequence in a partial metric space (S, p), and $x_n \to c$ (properly), then $c = \sup \mathcal{L}(x_n)$.*

Proof. Since $c \in \mathcal{L}(x_n)$, we only have to show that if $a \in \mathcal{L}(x_n)$ then $a \sqsubseteq c$. For any $\varepsilon > 0$, we know there exists $N \geq 1$, such that for any $n \geq N$,

$$\begin{aligned} p(a, x_n) &< p(a, a) + \varepsilon/2, \\ d(c, x_n) &< \varepsilon/2. \end{aligned}$$

So, for any $n \geq N$,

$$\begin{aligned} p(a, a) \leq p(a, c) &\leq p(a, x_n) + p(x_n, c) - p(x_n, x_n) \\ &\leq p(a, x_n) + d(x_n, c) \\ &< p(a, a) + \varepsilon. \end{aligned}$$

Since this holds for any $\varepsilon > 0$ then $p(a, a) = p(a, c)$, and $a \sqsubseteq c$, as required.

□

It is important to notice that the converse to the lemma does not hold. For example, if we give $\Re \setminus (-1, 0] \subseteq \Re$, the usual pmetric, then the sequence $(1/n)$ has no proper limit, but $\sup \mathcal{L}(1/n) = -1$.

Separable Spaces

We begin with the bitopological definition of a dense set (which again uses $\mathcal{T}[p] \subseteq \mathcal{T}[d]$), and then discuss an equivalent notion in terms of proper convergence.

Definition 3.5 *We say that a set Y in a partial metric space (S, p) is* **dense** *if Y is dense in (S, d).*

By considering the induced metric space, we see that Y is dense in (S, p) if, and only if, every point in S is the proper limit of a sequence in Y. So we have regained our intuitive idea of approximating every point in S. If we reconsider the motivating example of (2.2), i.e. $S = (-\infty, b] \subseteq \Re$ with the usual pmetric, then we see that $\{b\}$ is not dense in (S, p). As an example of a dense set in (S, p) we have $\mathbf{Q} \cap S$.

Definition 3.6 *A partial metric space (S, p) is* **separable** *if there is a countable dense subset of S.*

Now, a metric space is separable if, and only if, it has a second–countable topology. It is interesting to see that exactly the same relationship holds with respect to *both* topologies for a partial metric space.

Theorem 3.7 *The following three conditions are equivalent:*

1. *(S, p) is separable.*
2. *$\mathcal{T}[d]$ is second countable.*
3. *$\mathcal{T}[p]$ is second countable.*

The equivalence of (1) and (2) follows from considering the induced metric space. We delay the proof of the equivalence of (1) and (3) until the appendices, so as not to interrupt the main flow of the report.

Separability Examples

Example 3.8 *$P\omega$ with the usual pmetric is separable.*

Proof. Consider the countable set $Y = \{y \in P\omega \mid |y| < \infty\} \subseteq P\omega$. For any $x \in P\omega$, we define

$$y_n = x \cap \{1, \ldots n\} \in Y, \quad \text{for all } n \geq 1.$$

Then, for all $n \geq 1$,

$$d(x, y_n) \leq \sum_{i=n+1}^{\infty} 2^{-i} = 2^{-n},$$

so that $y_n \rightarrow x$ (properly), and Y is dense in $P\omega$.

□

Example 3.9 *S^∞ with the usual pmetric is separable if, and only if, S is countable.*

Proof. Suppose first that S is countable, and consider the countable set

$$Y = \{y \in S^\infty \mid l(y) < \infty\}.$$

Then for any $x \in S^\infty$, we can assume that $l(x) = \infty$ (otherwise $x \in Y$ already), and define

$$y_n = (x_0, \ldots, x_{n-1}) \in Y, \quad \text{for all } n \geq 1.$$

Then, for all $n \geq 1$, $y_n \sqsubseteq x$ implies that

$$p(y_n, x) = p(y_n, y_n) = 2^{-n}.$$

So $\lim_{n\to\infty} p(y_n, x) = \lim_{n\to\infty} (y_n, y_n) = 0 = p(x, x)$, and $y_n \to x$ (properly), which implies that Y is dense in S^∞.
If we now assume that S is uncountable, and that Y is dense in S^∞, we show that Y is uncountable, and hence S^∞ cannot be separable. For all $s \in S$, we let $x_s = (s, s, \ldots) \in S^\infty$. Then there exists $y_s \in Y$ such that

$$p(y_s, x_s) < p(x_s, x_s) + 1/2.$$

But this implies that $y_{s,0} = x_{s,0} = s$, and so Y is uncountable.

□

3.2 Continuity

One of the main results of this report is that every partial metric space has an essentially unique completion (in the same sense as metric spaces). To make this statement precise we require a definition of an isometry, and so we look at continuous functions next. We note however, that it is in the area of function spaces that much work remains to be done.

Definition 3.10 *Suppose (S, p) and (S', p') are partial metric spaces, with induced metrics d and d' respectively, then $f : (S, p) \to (S', p')$ is* **continuous** *if both $f : (S, \mathcal{T}[p]) \to (S', \mathcal{T}[p'])$ and $f : (S, \mathcal{T}[d]) \to (S', \mathcal{T}[d'])$ are continuous.*

This is the first time that we haven't been able to utilise the fact that $\mathcal{T}[p] \subseteq \mathcal{T}[d]$, since a function can be continuous with respect to the metric topologies, but not the pmetric topologies. We immediately see that $f : (S, p) \to (S', p')$ is continuous if, and only if,

1. $x_n \to x$ implies $f(x_n) \to f(x)$.

2. $x_n \to x$ (properly) implies $f(x_n) \to f(x)$ (properly).

The definition of a homeomorphism is just as natural, and we follow it with an example which hopefully gives some justification for our definitions.

Definition 3.11 *Suppose (S,p) and (S',p') are partial metric spaces and $f : (S,p) \to (S',p')$ is a continuous function, with a continuous inverse, then we say that f is a* **homeomorphism**, *and that (S,p) and (S',p') are* **homeomorphic**.

Example 3.12 *Consider $\Re$ and $\Re \setminus (-1,0] \subseteq \Re$ with the usual pmetrics. We define $f : \Re \to \Re \setminus (-1,0]$ by*

$$f(x) = \begin{cases} x, & \text{if } x > 0, \\ x-1, & \text{if } x \leq 0. \end{cases}$$

It is easy to check that f is a homeomorphism with respect to the pmetric topologies, but not the metric topologies. So f is not a homeomorphism in the partial metric sense. This agrees with our intuition, since the sequence $(1/n)$ has a proper limit in the one (i.e. 0), but not in the other.

Now an isometry is a metric concept, but even so, the definition we require is not at all obvious. We will give a definition and then try to justify it with a brief discussion and example. Further evidence comes when we consider completions in section 3.3. A more satisfactory explanation would require a detailed examination of equivalent partial metrics on a given space, which is probably a separate topic in its own right.

Definition 3.13 *Suppose (S,p) and (S',p') are partial metric spaces, then an* **isometry** *is a bijection $f : (S,p) \to (S',p')$ such that*

$$p(x,y) - p(x,x) = p'(f(x),f(y)) - p'(f(x),f(x)), \quad \text{for all } x,y \in S.$$

If f is an isometry from (S,p) to $(f(S),p')$, then we say that f is an **isometry into** *(S',p').*

It is easy to see that an isometry is a homeomorphism, and that if there exists $\delta \in \Re$ such that $f : (S,p) \to (S',p')$ satisfies

$$p'(f(x),f(y)) = p(x,y) + \delta, \quad \text{for all } x,y \in S,$$

then f is an isometry into (S',p'). In the next example, we see why we don't demand that the above equation hold for an isometry with $\delta = 0$.

Example 3.14 *Consider the two sets $[0,1],[1,2] \subseteq \Re$ with the usual pmetric p. Intuitively these spaces really are "identical", and we would expect the map $f : ([0,1],p) \to ([1,2],p)$ defined by $f(x) = x+1$, to be an isometry. Since*

$$p(f(x),f(y)) = p(x,y) - 1, \quad \text{for all } x,y \in [0,1],$$

we see that our definition is the strongest for which f is an isometry.

3.3 Completeness

We turn now to Cauchy sequences and completeness, which were discussed in [5]. We take these ideas one step further, and see that every partial metric space has an essentially unique completion, in the same sense as metric spaces. We will then pursue our investigation of $P\omega$ and S^∞.

Cauchy Sequences

We begin with the bitopological definition of a Cauchy sequence, and see that they can be expressed naturally in terms of the pmetric.

Definition 3.15 *A sequence (x_n) in a partial metric space (S, p) is* **Cauchy** *if it is Cauchy in (S, d).*

Lemma 3.16 *Suppose (x_n) is a sequence in a partial metric space (S, p), then*

$$(x_n) \textit{ is Cauchy if, and only if, } \lim_{n,m\to\infty} p(x_n, x_m) \textit{ exists.}$$

Proof. Suppose that (x_n) is a Cauchy sequence. Since

$$| p(x_n, x_n) - p(x_m, x_m) | \leq d(x_n, x_m), \text{ for all } n, m \geq 1,$$

then $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ implies that $(p(x_n, x_n))$ is a Cauchy sequence in $\Re$, and so $\lim_{n\to\infty} p(x_n, x_n) \in \Re$. Then we have,

$$\begin{aligned} (x_n) \text{ is Cauchy} &\iff \lim_{n,m\to\infty} d(x_n, x_m) = 0 \text{ and } \lim_{n\to\infty} p(x_n, x_n) \in \Re \\ &\iff \lim_{n,m\to\infty} p(x_n, x_m) \in \Re. \end{aligned}$$

□

From considering the induced metric space, we see that every properly convergent sequence is Cauchy. It is not true, however, that every convergent sequence is Cauchy. Consider for example the sequence $(0, 1, 0, 1, 0, \ldots)$ in $\Re$ with the usual pmetric. This converges to 0, but is not Cauchy by the above lemma.

Definition 3.17 *A partial metric space is* **complete** *if the induced metric space is complete.*

So in a complete partial metric space, every Cauchy sequence has a proper limit. Both the usual and flat pmetrics on $\Re$ are complete, since they induce the usual metric on $\Re$. We look now at completions, and lift the definition straight from metric spaces.

Definition 3.18 *Suppose (S, p) is a partial metric space, then a* **completion** *of S is a partial metric space $(\hat{S}, \hat{p})$ and a map $i : S \to \hat{S}$ such that*

1. $(\hat{S}, \hat{p})$ *is complete.*

2. i *is an isometry into* $\hat{S}$*.*

3. $i(S)$ *is dense in* $\hat{S}$*.*

In the appendices we prove the main result of this section.

Theorem 3.19 *Every partial metric space* (S, p) *has a unique completion (up to isometry).*

Completeness Examples

Example 3.20 $P\omega$ *with the usual pmetric is complete.*

Proof. Suppose that d is the induced metric as in (2.17), we prove that $(P\omega, d)$ is complete. Let (x_n) be a Cauchy sequence in $(P\omega, d)$. Then, for any $k \geq 1$, there exists $N_k \geq 1$, such that for any $n, m \geq N_k$,

$$\sum_{i \in x_n \triangle x_m} 2^{-i} = d(x_n, x_m) < 2^{-k}.$$

In particular, for any $n, m \geq N_k$, $k \in x_n$ if, and only if, $k \in x_m$. We claim that

$$x = \{k \in \mathbf{N} \mid k \in x_{N_k}\} \in P\omega,$$

is the limit of (x_n) in $(P\omega, d)$.
If $\varepsilon > 0$, there exists $k \geq 1$ such that $0 < 2^{-k} < \varepsilon$. Then, for any $n \geq \max\{N_i \mid 1 \leq i \leq k\}$ and $1 \leq i \leq k$, since $n \geq N_i$ we have

$$i \in x_n \iff i \in x_{N_i} \iff i \in x.$$

So we see that, for any $n \geq \max\{N_i \mid 1 \leq i \leq k\}$,

$$d(x_n, x) = \sum_{i \in x_n \triangle x} 2^{-i} \leq \sum_{i=k+1}^{\infty} 2^{-i} = 2^{-k} < \varepsilon.$$

So $x_n \to x$ in $(P\omega, d)$ which is then complete as required.

□

For S^∞ we will again be considering a Cauchy sequence (x_n) in S^∞. Since this can get notationally confusing (i.e. we have a sequence of sequences), we remark that we will write

$$x_n = (x_{n,0}, x_{n,1}, \ldots) \in S^\infty, \text{ for any } n \geq 1.$$

Example 3.21 S^∞ *with the usual pmetric is complete.*

Proof. Suppose that p is the usual pmetric, we prove that (S^∞, p) is complete directly. Let (x_n) be a Cauchy sequence in (S^∞, p). We must have $\lim_{n,m\to\infty} p(x_n, x_m) = 2^{-t}$ (for some $t \in \mathbf{N} \cup \{\infty\}$, $t \geq 0$), since, for every $n, m \geq 1$, $p(x_n, x_m)$ is of this form.
Now, for any $0 \leq k < t$, there exists $N_k \geq 1$ such that for all $n, m \geq N_k$,

$$\begin{array}{rc} & p(x_n, x_m) < 2^{-k-1} + 2^{-t} \leq 2^{-k} \\ \iff & \sup\{i \in \mathbf{N} \mid i \leq \min\{l(x_n), l(x_m)\}, \forall j < i, x_{n,j} = x_{m,j}\} > k \\ \iff & l(x_n), l(x_m) > k \text{ and } \forall j \leq k, x_{n,j} = x_{m,j}. \end{array}$$

We claim that the following point in S^∞ is the proper limit of (x_n),

$$y = (y_0, y_1, \ldots) \quad \text{where } y_k = x_{N_k,k} \text{ for all } 0 \leq k < t.$$

So, $l(y) = t$ and $p(y, y) = 2^{-t} = \lim_{n,m\to\infty} p(x_n, x_m)$. We consider separately the cases $t < \infty$ and $t = \infty$.
If $t < \infty$, then for all $n \geq \max\{N_i \mid 0 \leq i < t\}$ and $0 \leq i < t$, since $n \geq N_i$, then $l(x_n) > i$ (and so $l(x_n) \geq t$) and

$$x_{n,i} = x_{N_i,i} = y_i.$$

So

$$\sup\{i \in \mathbf{N} \mid i \leq \min\{l(x_n), l(y)\}, \forall j < i, x_{n,j} = y_j\} = t,$$

and $p(x_n, y) = 2^{-t}$.
If $t = \infty$, then given $\varepsilon > 0$, there exists $k \geq 0$ such that $0 < 2^{-k} < \varepsilon$. So, for any $n \geq \max\{N_i \mid 0 \leq i < k\}$ and $0 \leq i < k$, since $n \geq N_i$, then

$$x_{n,i} = x_{N_i,i} = y_i.$$

This implies that for any $n \geq \max\{N_i \mid 0 \leq i < k\}$,

$$p(x_n, y) \leq 2^{-k} < \varepsilon.$$

In each case we have $\lim_{n\to\infty} p(x_n, y) = 2^{-t} = \lim_{n,m\to\infty} p(x_n, x_m) = p(y, y)$, so that $x_n \to y$ (properly).

□

3.4 Compactness

Before looking at compactness, we will justify our generalisation of the partial metric axioms by considering boundedness and monotonic sequences. We then move on to compactness, and finish by looking at $P\omega$ and S^∞ once more.

Boundedness

We give the natural generalisation of a bounded space from metric spaces, which we call p–bounded (for reasons we consider later), and then show how this can be naturally expressed in terms of the pmetric.

Definition 3.22 *A partial metric space* (S, p) *is* **p–bounded** *if* (S, d) *is bounded.*

Lemma 3.23 *Suppose* (S, p) *is a partial metric space, then* (S, p) *is p–bounded if, and only if, there exists* $K_1, K_2 \in \Re$ *such that*

$$K_1 \leq p(x, y) \leq K_2, \quad \textit{for all } x, y \in S.$$

Proof. Suppose first, that (S, p) is p–bounded, then there exists $K \in [0, \infty)$ such that

$$d(x, y) \leq K, \text{ for all } x, y \in S.$$

If we fix $a \in S$, we have for any $x, y \in S$,

$$p(a, a) - K \leq p(a, a) - d(x, a) = 2\,[p(a, a) - p(a, x)] + p(x, x) \leq p(x, y)$$

$$\begin{array}{rrcl} \text{and} & p(x, y) & \leq & p(x, a) + p(a, y) - p(a, a) \\ & & \leq & d(x, a) + d(a, y) + p(a, a) \\ & & \leq & 2K + p(a, a). \end{array}$$

Conversely, if there exists $K_1, K_2 \in \Re$ then

$$d(x, y) = 2p(x, y) - p(x, x) - p(y, y) \leq 2(K_2 - K_1),$$

so (S, d) is bounded, and (S, p) is p–bounded.

□

Using the notation of the lemma, we say that (S, p) is p–bounded above if K_2 exists, and p–bounded below if K_1 exists. We will also say that the pmetric is **unbounded** if neither K_1 nor K_2 exist.

But notice that we already have another notion of boundedness in our spaces, from the induced order. It is clear, if a little confusing, that if a set is bounded below (in the induced order) then it is p–bounded above. Similarly if it is bounded above then it is p–bounded below. The converse to these statements are not true.

We can now see that the generalisation of the partial metric axioms, that we introduced in section 2.1, was to allow unbounded pmetrics. In [5], there was an asymmetry in the axioms, since all the partial metric spaces were p–bounded below (by 0), but not necessarily p–bounded above. We consider what effect this had by looking at monotonic sequences.

Definition 3.24 *A sequence* (x_n) *in a partial metric space is* **monotonic increasing (decreasing)** *if* $x_n \sqsubseteq x_{n+1}$ $(x_{n+1} \sqsubseteq x_n)$*, for all* $n \geq 1$*. It is* **monotonic** *if it has either of these properties.*

The following lemma shows immediately that in [5], all monotonic increasing sequences are Cauchy, whereas not all monotonic decreasing sequences need be. We then deduce that any satisfactory pmetric on $\Re$ with the usual order and metric, must be unbounded.

Lemma 3.25 *Suppose* (S, p) *is a partial metric space, then every monotonic increasing (decreasing) sequence* (x_n) *that is p–bounded below (above) is a Cauchy sequence. Furthermore, if* $x_n \to c$ *(properly) then* $c = \sup\{x_n \mid n \geq 1\}$ $(c = \inf\{x_n \mid n \geq 1\})$*.*

Proof. Suppose (x_n) is monotonic increasing, then for all $n \leq m$,

$$p(x_n, x_n) = p(x_n, x_m) \geq p(x_m, x_m).$$

So we have,

$$\lim_{n,m \to \infty} p(x_n, x_m) = \lim_{n \to \infty} p(x_n, x_n) = \inf\{p(x_n, x_n) \mid n \geq 1\},$$

which exists since (x_n) is p–bounded below. So (x_n) is a Cauchy sequence. Similarly if (x_n) is monotonic decreasing.
For the second part of the lemma, we first assume that (x_n) is a monotonic increasing sequence with proper limit c. It is clear that for all $k \geq 1$, $\lim_{n \to \infty} p(x_k, x_n) = p(x_k, x_k)$, so $x_n \to x_k$, then by (3.4), $x_k \sqsubseteq c$ and c is an upper bound for $\{x_k \mid k \geq 1\}$.
Suppose $c' \in S$ is also an upper bound for this set, then

$$\begin{array}{rcll} p(c,c) \leq p(c,c') & \leq & p(c,x_n) + p(x_n,c') - p(x_n,x_n) & \forall n \geq 1 \\ & = & p(c,x_n) & \\ & \to & p(c,c) \quad \text{as } n \to \infty. & \end{array}$$

So $p(c, c') = p(c, c)$, and $c \sqsubseteq c'$, which implies that $c = \sup\{x_n \mid n \geq 1\}$.
Now we assume that (x_n) is a monotonic decreasing sequence, with proper limit c. We fix $k \geq 1$, and have

$$\begin{array}{rcll} p(c,c) \leq p(c,x_k) & \leq & p(c,x_n) + p(x_n,x_k) - p(x_n,x_n) & \forall n \geq k \\ & = & p(c,x_n) & \\ & \to & p(c,c) \qquad \text{as } n \to \infty & \end{array}$$

So $p(c, c) = p(c, x_k)$, and c is a lower bound for $\{x_k \mid k \geq 1\}$.
Suppose c' is also a lower bound for this set, then

$$\begin{array}{rcll} p(c',c') \leq p(c,c') & \leq & p(c,x_n) + p(x_n,c') - p(x_n,x_n) & \forall n \geq 1 \\ & = & p(c,c) + p(c',c') - p(x_n,x_n) & \\ & \to & p(c',c') \qquad \text{as} \quad n \to \infty & \end{array}$$

So $p(c, c') = p(c', c')$ and $c' \sqsubseteq c$ which implies that $c = \inf\{x_n \mid n \geq 1\}$.

$\square$

Corollary 3.26 *Any satisfactory pmetric on $\Re$, with the usual metric and order is unbounded.*

Proof. Suppose for a contradiction that p is a satisfactory pmetric on $\Re$ such that $(\Re, p)$ is bounded below (wlog). Then the sequence (n) is a monotonic increasing sequence in $(\Re, p)$, which is bounded below, and so is Cauchy. But p induces the usual metric on $\Re$, which is complete, so that $(\Re, p)$ is complete. So (n) has a proper limit, which is the supremum of $\mathbf{N}$ by the lemma. This is clearly a contradiction.

$\square$

Now, it could certainly be argued from a Computer Science perspective, that we don't need unbounded pmetrics. However this does not mean that we should force *all* pmetrics to be bounded in the axioms. What we are really claiming, is that partial metric spaces may be of interest in other areas of mathematics, without the bounded restriction. This is certainly the case for $\Re$ with the usual ordering and metric.

Compact Spaces

Definition 3.27 *A partial metric space (S, p) is* **compact** *if (S, d) is compact.*

When we look at $P\omega$ and S^∞, we will find it useful to consider equivalent notions of compactness. For this reason, we generalise finite ε–nets to partial metric spaces, and then consider an essential difference from metric spaces.

Definition 3.28 *Suppose (S, p) is a partial metric space and $\varepsilon > 0$, then a* **finite ε–net** *of (S, p) is a finite set $A \subseteq S$ such that*

$$S \subseteq \bigcup_{x \in A} B_\varepsilon(x; p).$$

The existence of a finite ε–net in a metric space, implies the existence of one for any subspace. This property does *not* hold for partial metric spaces. Consider for example the pmetric on the flat domain $(S^\perp, \leq)$, with S an infinite set. Now $\{\perp\}$ is a finite $\frac{1}{2}$–net of $(S^\perp, p)$ since $B_{1/2}(\perp; p) = S^\perp$. However, (S, p) has no finite $\frac{1}{2}$–net since $B_{1/2}(x; p) = \{x\}$, for all $x \in S$.

Before proving our main theorem on compactness, we need a simple lemma involving sequences in a p–bounded partial metric space.

Lemma 3.29 *Suppose (S, p) is a p–bounded partial metric space, then every sequence (x_n) has a subsequence (x_{n_k}) such that $\lim_{k \to \infty} p(x_{n_k}, x_{n_k})$ exists.*

Proof. By (3.23), there exists $K_1, K_2 \in \Re$ such that the sequence $(p(x_n, x_n))$ lies in the compact interval $[K_1, K_2] \subseteq \Re$, which must then have a convergent subsequence.

□

Theorem 3.30 *Suppose (S,p) is a partial metric space, then the following three conditions are equivalent:*

1. *(S,p) is compact.*
2. *Every sequence in (S,p) has a properly convergent subsequence.*
3. *(a) (S,p) is complete.*
   (b) (S,p) is p–bounded.
   (c) For any $\varepsilon > 0$, and $R \subseteq S$, there exists a finite ε–net for (R,p).

Proof. The equivalence of (1) and (2) follows immediately from considering sequential compactness on the induced metric space. We prove that if (S,d) is totally bounded, then (3b) and (3c) hold, and then consider the induced metric space to see that (1) implies (3). We finish by proving that (3) implies (2).
Suppose first that (S,d) is totally bounded, then for any $R \subseteq S$, (R,d) is totally bounded. But we also have, for any $\varepsilon > 0$ and $x \in S$, $B_\varepsilon(x;d) \subseteq B_\varepsilon(x;p)$, so that (R,p) has a finite ε–net. Since a totally bounded metric space is bounded, then (S,p) is p–bounded.
Now, suppose that (3) holds, and that (x_n) is a sequence in (S,p). By lemma (3.29) and (3b) we can assume that $\lim_{n\to\infty} p(x_n,x_n)$ exists. We let $R_1 = \{x_n \mid n \geq 1\} \subseteq S$, and $\{x_{\alpha_1},\ldots,x_{\alpha_m}\}$ be a finite $\frac{1}{2}$–net for (R_1,p). So there exists an α_j such that $B_{1/2}(x_{\alpha_j};p)$ contains infinitely many x_n. We let $(x_{n,1})$ be a subsequence of (x_n) belonging to $B_{1/2}(x_{\alpha_j};p)$, with $x_{1,1} = x_{\alpha_j}$.
Suppose inductively that for $i = 1,2,\ldots,k-1$, there exists a subsequence $(x_{n,i})$ of (x_n) such that

$$p(x_{n,i},x_{m,i}) \leq p(x_{1,i},x_{1,i}) + 1/i, \quad \text{for all } n,m \geq 1,$$

and that $(x_{n,i})$ is a subsequence of $(x_{n,i-1})$ for $i = 2,3,\ldots,k-1$.
We let $R_k = \{x_{n,k-1} \mid n \geq 2\} \subseteq S$, and $\{x_{\alpha_1,k-1},\ldots x_{\alpha_m,k-1}\}$ be a finite $\frac{1}{2k}$–net for (R_k,p). Then there exists an α_j such that $B_{1/2k}(x_{\alpha_j,k-1};p)$ contains infinitely many $x_{n,k-1}$. We let $(x_{n,k})$ be a subsequence of $(x_{n,k-1})$ belonging to $B_{1/2k}(x_{\alpha_j,k-1};p)$, with $x_{1,k} = x_{\alpha_j,k-1}$.
If we consider the subsequence $(x_{1,n})$ of (x_n), we see that for all $m \geq n$,

$$p(x_{1,n},x_{1,m}) \leq p(x_{1,n},x_{1,n}) + 1/n.$$

Since $\lim_{n\to\infty} p(x_{1,n},x_{1,n})$ exists, we see that $(x_{1,n})$ is a Cauchy subsequence of (x_n). We then use (3a) to finish the proof.

□

Compactness Examples

Example 3.31 *$P\omega$ with the usual pmetric is compact.*

Proof. We know that $(P\omega, d)$ is complete, if we can show that $(P\omega, d)$ is totally bounded then we are done. Given $\varepsilon > 0$, we let $k \geq 1$ be such that $0 < 2^{-k} < \varepsilon$, and define

$$A = \{x \in P\omega \mid x \subseteq \{1, \ldots, k\}\}.$$

Then $|A| = 2^k < \infty$.
Now for any $y \in P\omega$, there exists $x \in A$ such that for all $1 \leq i \leq k$, $i \in x$ if, and only if, $i \in y$. So

$$d(x, y) \leq \sum_{i=k+1}^{\infty} 2^{-i} = 2^{-k} < \varepsilon,$$

and $y \in B_\varepsilon(x; d)$. So $(P\omega, d)$ is totally bounded as required.

$\square$

Example 3.32 *S^∞ with the usual pmetric is compact if, and only if, S is a finite set.*

Proof. Suppose, first, that S is an infinite set, and let $s_1, s_2, \ldots$ be a countable collection of distinct elements of S. We define $x_n = (s_n, s_n, \ldots) \in S^\infty$. Then (x_n) is a sequence in S^∞ such that

$$\begin{aligned} p(x_n, x_n) &= 0, \qquad \text{for all } n \geq 1, \\ p(x_n, x_m) &= 1, \qquad \text{for all } n, m \geq 1 \ (n \neq m). \end{aligned}$$

Clearly (x_n) can have no Cauchy subsequence, and hence no properly convergent subsequence, so S^∞ cannot be compact.
Now suppose that S is finite. We know that S^∞ is complete and p–bounded below by 0, and above by 1, so we are left to prove condition (3c) of the theorem. Suppose we are given $\varepsilon > 0$, we let $k \geq 1$ be such that $0 < 2^{-k} < \varepsilon$, and define

$$A = \{x \in S^\infty \mid l(x) \leq k\},$$

then $|A| = |S|^k < \infty$.
Suppose $R \subseteq S^\infty$, we define $A' \subseteq R$ as follows. For each $x \in A$, if there exists a $y \in R$ with $x \sqsubseteq y$, then we choose one such y for A'. Clearly $|A'| \leq |A| < \infty$. Then for any $z \in R$, there exists $x \in A$ such that $x_i = z_i$, for all $0 \leq i < k$. For this $x \in A$, there exists $y \in A'$ such that $x_i = y_i$, for all $0 \leq i < k$. Then we have

$$p(y, z) \leq 2^{-k} < \varepsilon,$$

and $z \in B_\varepsilon(y; p)$. So A' is a finite ε–net for (R, p), and we are done.

$\square$

4 Conclusions

Although the central argument of this report is that bitopology is the correct context in which to view partial metric spaces, we will use this section to briefly summarise some of the other points raised. Foremost among these, is the case we present for extending the axioms to allow unbounded pmetrics. While acknowledging that this may be of little relevance *within* Computer Science, we must agree with a general point of Sutherland's (in [8]) in wishing "to prove any given result in the appropriate context", which is why we take the broader view.

In the text, we stress that section 3 is a case of putting our ideas on generalisations into practice. However, we are in danger of losing sight of the fact that we are now in possession of many useful results from metric spaces. This process of generalisation is certainly far from over, but at some point we will have sufficient theory to tackle applications, and ultimately test the utility of the theory developed.

In the same vein, we should also be aware that rather than just having a number of motivating examples, we have actually made a study of some interesting partial metric spaces. If these prove to be useful, then we could be closer than anticipated to putting our new theory into practice.

A Proofs

We return now to the proofs we left out of the main text.

Separable Spaces

Lemma A.1 *If (S,p) is a separable partial metric space then $\mathcal{T}[p]$ is second countable.*

Proof. Suppose Y is a countable dense subset of S. We define

$$\mathcal{B} = \{B_r(x;p) \mid x \in Y, r \in \mathbf{Q}(r > 0)\},$$

which is clearly a countable collection of sets in $\mathcal{T}[p]$. We show that $\mathcal{B}$ is a basis for $\mathcal{T}[p]$. Let $y \in S$, and $U \in \mathcal{T}[p]$ containing y. We want to find an element of $\mathcal{B}$ containing y, lying inside U. We know that there exists $r \in \Re(r > 0)$, so that $y \in B_r(y;p) \subseteq U$. We fix

$$r/4 < s < 3r/4 \quad \text{with } s \in \mathbf{Q}. \tag{1}$$

Now there exists a sequence (x_n) in Y with proper limit y. So there exists $N \geq 1$ such that $B_s(x_N;p) \in \mathcal{B}$, and

$$p(x_N, y) - p(y, y) < r/4, \tag{2}$$

$$p(y, y) - p(x_N, x_N) < s - r/4. \tag{3}$$

We first show that y is in this open ball,

$$\begin{aligned} p(x_N, y) &< p(y, y) + r/4 && \text{by (2)} \\ &< p(x_N, x_N) + s && \text{by (3)} \end{aligned}$$

and so $y \in B_s(x_N; p)$. Now suppose that $z \in B_s(x_N; p)$. We show that z is in U by showing that it is in $B_r(y; p)$,

$$\begin{aligned} p(y,z) &\leq p(y,x_N) + p(x_N,z) - p(x_N,x_N) \\ &< p(y,x_N) + s \\ &< p(y,y) + r/4 + s \qquad \text{by (2)} \\ &< p(y,y) + r \qquad \text{by (1)} \end{aligned}$$

and so $z \in B_r(y; p)$, and hence

$$y \in B_s(x_N; p) \subseteq B_r(y; p) \subseteq U.$$

Which proves that $\mathcal{B}$ is a countable base for $\mathcal{T}[p]$ as required.

□

Lemma A.2 *If $\mathcal{T}[p]$ is second countable then (S, p) is separable.*

Proof. Since $\mathcal{T}[p]$ is second countable, then the collection of open balls, which is a basis, has a countable sub–basis $\mathcal{B} = \{B_1, B_2, \ldots\}$. For each $n \geq 1$ we suppose that

$$B_n = B_{\varepsilon_n}(x_n; p)$$

where $x_n \in S$ and $\varepsilon_n > 0$. Since every open ball is p–bounded above, we can define

$$\alpha_n = \sup\{p(x,x) \mid x \in B_n\}.$$

We then define a countable set $Y = \{y_n \mid n \geq 1\}$ as follows: For all $n \geq 1$, if $\inf B_n \in B_n$ then we define $y_n = \inf B_n$, otherwise we let $y_n \in B_n$ be such that

$$\alpha_n - 1/n < p(y_n, y_n) \leq \alpha_n.$$

We show that Y is dense is (S, p). Suppose $x \in S$ and $x \notin Y$. We define

$$\beta = \inf\{n \geq 1 \mid x \in B_n\}.$$

Now, for each $m \geq 1$ there exists $n_m \geq 1$ such that

$$x \in B_{n_m} \subseteq B_{1/m}(x; p).$$

It is clear that (y_{n_m}) is a sequence in Y with limit x. But we still have to find a sequence in Y which has proper limit x. We claim that there must be a subsequence of (y_{n_m}) with this property.

Now, for any $k \geq 1$, we let $k' \geq \max\{\beta, k\}$ and define

$$V_k = \bigcap_{\substack{i=1 \\ x \in B_i}}^{k'} B_i,$$

which is open in $\mathcal{T}[p]$, and has $x \in V_k$.
There clearly exists $J_1 \geq 1$ such that

$$x \in B_{n_{J_1}} \subseteq B_{1/J_1}(x;p) \subseteq V_k.$$

If $B_{1/j}(x;p) = V_k$ for all $j \geq J_1$ then $x = \inf V_k = \inf B_{n_{J_1}}$, and so $x = y_{n_{J_1}} \in Y$, a contradiction. So there must exist $J_2 \geq J_1$ such that for all $j \geq J_2$,

$$x \in B_{n_j} \subseteq B_{1/j}(x;p) \subseteq B_{1/J_2}(x;p) \neq V_k,$$

and so $n_j > k' \geq k$.
For notational convenience, we will let (z_m) denote the sequence (y_{n_m}). So we can find a subsequence (z_{m_k}) of (z_m) such that, for all $k \geq 1$, we have $n_{m_k} > k$. Then

$$\begin{aligned} p(z_{m_k}, z_{m_k}) - p(x,x) &\leq p(z_{m_k}, x) - p(x,x) \\ &< 1/m_k \leq 1/k, \\ p(x,x) - p(z_{m_k}, z_{m_k}) &\leq \alpha_{n_{m_k}} - p(z_{m_k}, z_{m_k}) \\ &< 1/n_{m_k} < 1/k. \end{aligned}$$

So $\mid p(z_{m_k}, z_{m_k}) - p(x,x) \mid < 1/k$ for all $k \geq 1$, and (z_{m_k}) has proper limit x as required.

□

Completions

Consider first a simple, but useful, lemma.

Lemma A.3 *Suppose (S,p) is a partial metric space with induced metric d, then*

$$|p(x,y) - p(z,w)| \leq d(x,z) + d(y,w).$$

Proof. Clearly,

$$\begin{aligned} p(x,y) - p(z,w) &\leq p(x,z) + p(z,y) - p(z,z) - p(z,w) \\ &\leq d(x,z) + p(z,w) + p(w,y) - p(w,w) - p(z,w) \\ &\leq d(x,z) + d(y,w). \end{aligned}$$

Similarly for $p(z,w) - p(x,y)$, and so the result holds.

□

We will now prove that every partial metric space has a unique completion (up to isometry). We do this be generalising the respective proof for metric spaces from [8]. We prove first that every partial metric has a completion.

Theorem A.4 *Every partial metric space (S, p) has a completion.*

Proof. Suppose d is the induced metric on S. We let $\hat{S}$ be the set of equivalence classes of Cauchy sequences in S, where $(x_n) \sim (y_n)$ if, and only if $d(x_n, y_n) \to 0$ as $n \to \infty$. Then for any $\hat{x}, \hat{y} \in \hat{S}$, represented by (x_n) and (y_n) respectively,

$$\hat{d}(\hat{x}, \hat{y}) = \lim_{n \to \infty} d(x_n, y_n),$$

is a well–defined metric, and $(\hat{S}, \hat{d})$ is a completion of (S, d), in the metric sense.
Now, for any $\hat{x}, \hat{y} \in \hat{S}$, represented by (x_n) and (y_n) respectively, we define

$$\hat{p}(\hat{x}, \hat{y}) = \lim_{n \to \infty} p(x_n, y_n).$$

We show that $(\hat{S}, \hat{p})$ is a completion of (S, p).
Our first task is to show that $\lim_{n \to \infty} p(x_n, y_n)$ exists. For any $n, m \geq 1$, we have

$$\begin{aligned} 0 \leq |p(x_n, y_n) - p(x_m, y_m)| &\leq d(x_n, x_m) + d(y_m, y_n) \quad \text{by (A.3)} \\ &\to 0 \quad \text{as } n, m \to \infty. \end{aligned}$$

So $(p(x_n, y_n))$ is Cauchy in $\Re$, and $\lim_{n \to \infty} p(x_n, y_n)$ exists.
To see that $\hat{p}$ is well–defined, we suppose that (x'_n) also represents $\hat{x}$. Then

$$\begin{aligned} 0 \leq |p(x'_n, y_n) - p(x_n, y_n)| &\leq d(x'_n, x_n) \quad \text{by (A.3)} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

So we must have $\lim_{n \to \infty} p(x'_n, y_n) = \lim_{n \to \infty} p(x_n, y_n)$, and $\hat{p}$ is well–defined.
To see that $\hat{p}$ is a pmetric, we note that

$$\begin{aligned} \hat{p}(\hat{x}, \hat{x}) = \hat{p}(\hat{x}, \hat{y}) = \hat{p}(\hat{y}, \hat{y}) &\iff \lim_{n \to \infty} p(x_n, x_n) = \lim_{n \to \infty} p(x_n, y_n) = \lim_{n \to \infty} p(y_n, y_n) \\ &\iff \lim_{n \to \infty} d(x_n, y_n) = 0 \\ &\iff (x_n) \sim (y_n) \\ &\iff \hat{x} = \hat{y}. \end{aligned}$$

The other axioms are all immediate from the axioms for p and taking limits. It is also clear that $\hat{p}$ has induced metric $\hat{d}$, so that $(\hat{S}, \hat{p})$ is complete.
Finally, we define a function $i : S \to \hat{S}$ by $i(x) = (x, x, x, \ldots)$. Since

$$p(x, y) = \lim_{n \to \infty} p(x, y) = \hat{p}(i(x), i(y)), \quad \text{for any } x, y \in S,$$

then i is an isometry into $\hat{S}$, and $i(S)$ is dense in $(\hat{S}, \hat{p})$ since it is dense in $(\hat{S}, \hat{d})$.

□

To prove the uniqueness of the completion, we need a preliminary result. Again we rely heavily on the corresponding result from metric spaces (see [8]). We remark first that an isometry between two partial metric spaces is a (metric) isometry between their induced metric spaces.

Lemma A.5 *If X is a dense subset of a partial metric space (S_1, p_1), and $f : X \to S_2$ is an isometry into a complete partial metric space (S_2, p_2), then f extends uniquely to an isometry of S_1 into S_2.*

Proof. If $x \in S_1$, then there exists (x_n) in X properly converging to x, since X is dense in (S_1, p_1). By considering the induced metric spaces, we see that the function $g : S_1 \to S_2$, where $g(x)$ is defined to be the proper limit of the sequence $(f(x_n))$ in (S_2, p_2), is a well-defined extension of f.
Now, if $x, y \in S_1$ and (x_n), (y_n) are sequences in X properly converging to x, y, then we use (A.3) to see that $\lim_{n \to \infty} p_2(f(x_n), f(y_n)) = p_2(g(x), g(y))$ (and the corresponding result in (S_1, p_1)). We then have

$$\begin{aligned} p_2(g(x), g(y)) - p_2(g(x), g(x)) &= \lim_{n \to \infty} [p_2(f(x_n), f(y_n)) - p_2(f(x_n), f(x_n))] \\ &= \lim_{n \to \infty} [p_1(x_n, y_n) - p_1(x_n, x_n)] \\ &= p_1(x, y) - p_1(x, x). \end{aligned}$$

So g is an isometry into (S_2, p_2). To see that g is unique, we note that if g' is any continuous extension of f, and (x_n) is a sequence in X properly converging to $x \in S_1$, then $g'(x)$ is the proper limit of $(g'(x_n))$, which is the same sequence as $(f(x_n))$, and so $g'(x) = g(x)$.

□

We finish with the following theorem, which gives (essential) uniqueness of completions. We omit the proof, since this is now a standard argument which follows from the lemma using commutative diagrams and the relevant isometries.

Theorem A.6 *Suppose (S, p) is a partial metric space, with completions $(\hat{S}, \hat{p})$ and (S', p') via i and i' respectively. Then there is a unique isometry $j : \hat{S} \to S'$ such that $j \circ i = i'$.*

B Partial Orders

Definition B.1 *A* **partial order** *is a binary relation $\leq$ on a set S such that for any $x, y, z \in S$,*

1. *(Reflexive) $x \leq x$.*
2. *(Antisymmetric) $x \leq y$ and $y \leq x$ implies that $x = y$.*
3. *(Transitive) $x \leq y$ and $y \leq z$ implies that $x \leq z$.*

In a partially ordered set $(S, \leq)$, an **upper bound** for $A \subseteq S$ is an element $x \in S$ such that $a \leq x$, for all $a \in A$. Similarly for a lower bound. The **supremum** of A ($\sup A$) is the least upper bound of A, and the **infimum** of A ($\inf A$) is the greatest lower bound.

Definition B.2 *A partially ordered set $(S, \leq)$ is* **flat** *if, for every $x, y \in S$,*

$$x \leq y \text{ if, and only if, } x = y.$$

Definition B.3 *If X is a topological space, then the* **specialisation order** *on X is defined as*

$$x \sqsubseteq y \iff \quad x \in U,\ U \text{ open, implies } y \in U.$$

Lemma B.4 *Suppose X is a topological space with specialisation order $\sqsubseteq$, it follows that*

1. *If X is a T_0–space, then $\sqsubseteq$ is a partial order.*
2. *If X is a T_2–space, then $\sqsubseteq$ is a flat partial order.*

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