Expressiveness of First-Order Description Logics

Natasha Kurtonina[1] and Maarten de Rijke[2]

[1] IRCS, University of Pennsylvania
3401 Walnut Street, Philadelphia, PA 19104-6228, USA
natashak@linc.cis.upenn.edu
[2] Dept. of Computer Science, University of Warwick
Coventry CV4 7AL, England
mdr@dcs.warwick.ac.uk

Abstract

We introduce a method for characterizing the expressive power of first-order description logics. The method is essentially model-theoretic in nature in that it gives preservation results uniquely identifying a wide range of description logics as fragments of first-order logic. The languages studied in the paper all belong to the well-known $\mathcal{FL}^-$ and $\mathcal{AL}$ hierarchies.

1 Introduction

Description logics have been proposed in knowledge representation to specify systems in which structured knowledge can be expressed and reasoned with in a principled way. They provide a logical basis to the well-known traditions of frame-based systems, semantic networks and KL-ONE-like languages, object-oriented representations, semantic data models, and type systems. In the design of description logics two important theoretical considerations are complexity and expressive power. A popular slogan here is 'complexity versus expressiveness': the more expressive a description logic is, the higher the complexity of the reasoning tasks that can be performed in it. The complexity of satisfiability and subsumption problems for description logics has been studied extensively (cf. [DLNS96, DLNN97]), but the problem of expressiveness has hardly been addressed so far; we are aware of only two publications on this topic [Baa96, Bor96]. The purpose of this paper is to help fill this gap. We characterize and compare the expressive power of all logics in two well-known hierarchies of description logics.

The methods we use first identify description logics as fragments of first-order logic, and then characterize these fragments in terms of a unique model-theoretic property. The main technical tool used is preservation under a suitable notion of (bi-)simulation. More precisely, with each description logic $\mathcal{L}$ we associate a characteristic (bi-)simulation such that all and only the $\mathcal{L}$-expressions are preserved under this (bi-)simulation. Then, the expressive power of two description logics can be compared by comparing the model-theoretic behaviour of their expressions with respect to their respective (bi-)simulations. The characteristic (bi-)simulations can then be used to classify the properties that are definable in description logics.

We think that our results are significant for the knowledge representation community because, for the first time, they give exact and explicit model-theoretic characterizations of the expressive power of a wide range of description logics. In addition, they illustrate a *general* method for coping with expressiveness issues; they may be used by the designer of knowledge based systems to help choose the description logic that best fits his or her descriptive requirements.

Baader [Baa96] seems to have been the first to propose a formal definition of the expressive power of description logics; the only other formal paper on the issue is [Bor96]. Our definition of expressive power is somewhat simpler than Baader's. Also, our paper differs from [Baa96, Bor96] in that we give *exact* and *explicit model-theoretic* characterizations of the expressive power of a *wide range* of logics (cf. Section 5 for further discussion). The results in this paper are based on preservation theorems that are similar to ones found in the literature on modal and temporal logic and the modal μ-calculus [Ben85, JW96, KR97b]. However, as description logics often lack some boolean operations, the proofs of our preservation theorems require novel technical tools and methods. Our preservation results are similar in spirit to the characterizations of finite variable fragments in terms of pebble games due to [IK87]. Furthermore, there is a considerable body of work on the expressive power of query languages, but most of this is phrased in terms of complexity classes [AHV95, Imm86]. The results in the present paper, however, are entirely model-theoretic.

We proceeds as follows. In Section 2 we describe the technical prerequisites for the paper, and review our notation. Section 3 then explains our method and the definition of expressive power used. The main results of the paper are contained in Section 4, together with illustrations of their use. Section 6 contains concluding remarks and describes ongoing work. Formal proofs of the main characterization results are included in an appendix.

2 Technical Background

The main ingredients of description logics are *concepts* and *roles*. The former are interpreted as subsets of a given domain, and the latter as binary relations on the domain. Table 1 lists constructors that allow one to build (complex) concepts and roles from (atomic) concept names and role names.

For instance, the concept $\mathsf{Man} \sqcap \exists \mathsf{Child}.\top \sqcap \forall \mathsf{Child}.\mathsf{Human}$ denotes the set of all fathers.

Constructor name	Syntax	Semantics
concept name	A	$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
top	$\top$	$\Delta^{\mathcal{I}}$
bottom	$\perp$	$\emptyset$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
disjunction ($\mathcal{U}$)	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
negation ($\mathcal{C}$)	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
univ. quant.	$\forall R.C$	$\{d_1 \mid \forall d_2\, (d_1, d_2) \in R^{\mathcal{I}} \to d_2 \in C^{\mathcal{I}}\}$
exist. quant. ($\mathcal{E}$)	$\exists R.C$	$\{d_1 \mid \exists d_2\, (d_1, d_2) \in R^{\mathcal{I}} \wedge d_2 \in C^{\mathcal{I}}\}$
number	$(\geq\ n\ R)$	$\{d_1 \mid \lvert\{(d_1, d_2) \in R^{\mathcal{I}}\}\rvert \geq n\}$
restriction ($\mathcal{N}$)	$(\leq\ n\ R)$	$\{d_1 \mid \lvert\{(d_1, d_2) \in R^{\mathcal{I}}\}\rvert \leq n\}$
role name	R	$R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
role conj. ($\mathcal{R}$)	$Q \sqcap R$	$Q^{\mathcal{I}} \cap R^{\mathcal{I}}$

Table 1: Constructors in First-Order Description Logics

Description logics differ in the constructions they admit. By combining constructors taken from Table 1, two well-known hierarchies of description logics may be obtained. The logics we consider here are extensions of $\mathcal{FL}^-$; this is the logic with $\top$, $\perp$, universal quantification, conjunction and unqualified existential quantification $\exists R.\top$.[1] $\mathcal{AL}$ extends $\mathcal{FL}^-$ by negation of concept names (that is, negations of the form $\neg A$, where A is an atomic concept name). Extensions of $\mathcal{FL}^-$ and $\mathcal{AL}$ are denoted by postfixing the name of the constructors being added. For instance, $\mathcal{FLEU}^-$ is $\mathcal{FL}^-$ with (full) existential quantification and disjunction.

Description logics are interpreted on *interpretations* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty domain, and $\cdot^{\mathcal{I}}$ is an interpretation function assigning subsets of $\Delta^{\mathcal{I}}$ to concept names and binary relations over $\Delta^{\mathcal{I}}$ to role names; complex concepts and roles are interpreted using the recipes specified in Table 1. The *semantic value* of an expression E in an interpretation $\mathcal{I}$ is simply the set $E^{\mathcal{I}}$. Two expressions are called *equivalent* if they have the same semantic value in every interpretation.

For further details on both applications and theoretical aspects of description logics, we refer the reader to [DLNS96], or to the description logic home page at `http://dl.kr.org/dl/`.

[1]Some definitions of $\mathcal{FL}^-$ don't include $\top$ and $\perp$ in the logic; cf. [DLNS96]. To simplify the formulation of our results we have decided to include them.

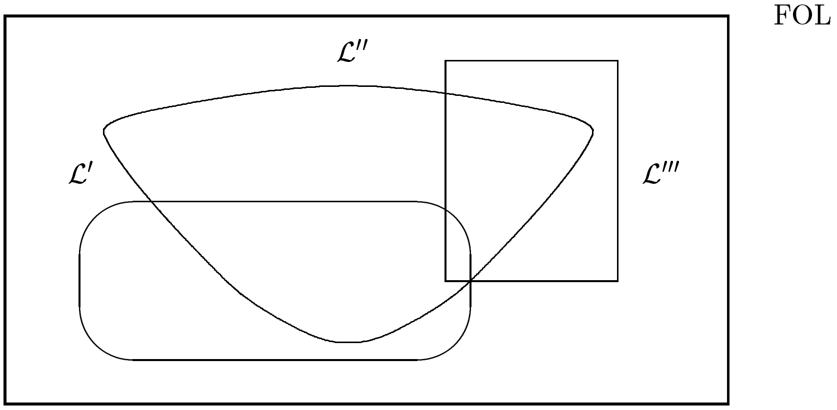


Figure 1: The Method

3 Defining Expressive Power

In this section we define our notion of expressive power, and explain our method for determining the expressive power of a given description logic.

Our aim in this paper is to determine the expressive power of every extension of $\mathcal{FL}^-$ and $\mathcal{AL}$ that can be defined using the constructors in Table 1. We say that a logic $\mathcal{L}_1$ is *at least as expressive as* a logic $\mathcal{L}_2$ if for every expression in $\mathcal{L}_2$ there is an equivalent expression in $\mathcal{L}_1$; notation: $\mathcal{L}_2 \leq \mathcal{L}_1$. If $\mathcal{L}_2 \leq \mathcal{L}_1$ and $\mathcal{L}_1 \not\leq \mathcal{L}_2$, we write $\mathcal{L}_2 < \mathcal{L}_1$; if both $\mathcal{L}_1 \leq \mathcal{L}_2$ and $\mathcal{L}_2 \leq \mathcal{L}_1$ hold, we write $\mathcal{L}_1 \equiv \mathcal{L}_2$.

The method we use for explaining the expressive power of description logics has the following ingredients:

1. a mapping taking description logics to fragments of first-order logic;
2. characterizations of these fragments by model-theoretic means; and
3. comparisons between (the expressive power of) description logics based on comparisons between the corresponding first-order fragments; cf. Figure 1, where the rectangle denotes first-order logic, and the closed curves denote (fragments corresponding to) description logics.

In line with our methodology we will pursue the above items 1, 2, and 3 for each of the description logics considered in this paper. First, item 1 is next to trivial. The semantics given in Table 1 induces translations $(\cdot)^\tau$ and $(\cdot)^\sigma$ taking concepts and roles, respectively, to formulas in a first-order language whose signature consists of unary predicate symbols corresponding to atomic concepts names, and binary predicate symbols corresponding to atomic role names:

$$
\begin{array}{rcl}
A^{\tau_x} & = & Ax \qquad\qquad (C \sqcap D)^{\tau_x} \;=\; C^{\tau_x} \wedge D^{\tau_x} \\
\top^{\tau_x} & = & (x = x) \qquad (C \sqcup D)^{\tau_x} \;=\; C^{\tau_x} \vee D^{\tau_x} \\
\bot^{\tau_x} & = & (x \neq x) \qquad\quad (\neg C)^{\tau_x} \;=\; \neg C^{\tau_x} \\
(\forall R.C)^{\tau_x} & = & \forall y\, (R^{\sigma_{xy}} \to C^{\tau_y}), \text{ where } y \text{ is a fresh variable} \\
(\exists R.C)^{\tau_x} & = & \exists y\, (R^{\sigma_{xy}} \wedge C^{\tau_y}), \text{ where } y \text{ is a fresh variable} \\
(\geq n\ R)^{\tau_x} & = & \exists y_1 \ldots y_n \left(\bigwedge_{i \neq j} y_i \neq y_j \wedge \bigwedge_i R^{\sigma_{xy_i}} \right) \\
 & & \text{where all } y_i \text{ are a fresh variables} \\
(\leq n\ R)^{\tau_x} & = & \forall y_1 \ldots y_{n+1} \left(\bigwedge_{i \neq j} y_i \neq y_j \to \bigvee_i \neg R^{\sigma_{xy_i}} \right) \\
 & & \text{where all } y_i \text{ are a fresh variables} \\
R^{\sigma_{xy}} & = & Rxy \qquad\quad (Q \sqcap R)^{\sigma_{xy}} \;=\; Q^{\sigma_{xy}} \wedge R^{\sigma_{xy}}
\end{array}
$$

Observe that to translate concepts and roles in description logics without number restrictions we only need two individual variables.

To be able to state that concepts and roles are equivalent to their translations under τ and σ, we need to relate the semantics of description logics and first-order logic. But interpretations can naturally be viewed as models for the first-order language we consider here. Thus, we will for example write $\mathcal{I} \models \alpha(x)[d]$ to denote that the first-order formula α is true in $\mathcal{I}$ (viewed as a first-order model), with d assigned to α's free variable x. Below we will exploit this connection, often without making it explicit.

Proposition 3.1 *Let C be a concept and R a role. For any interpretation $\mathcal{I}$ and any d, $e \in \Delta^{\mathcal{I}}$ we have the following equivalences:*

1. $d \in C^{\mathcal{I}}$ *iff* $\mathcal{I} \models C^{\tau_x}[d]$
2. $(d, e) \in R^{\mathcal{I}}$ *iff* $\mathcal{I} \models R^{\sigma_{xy}}[de]$.

Given this proposition we are allowed to simply *identify* description logics with their corresponding first-order fragments, and if no confusion is possible we write C instead of C^{τ}, and R instead of R^{σ}.

Proposition 3.1 settles item 1 of our method. Next comes item 2 — this is much more work. The semantic characterizations that we are after will be formulated in terms of preservation under a suitable relation between interpretations. To make this strategy more concrete we first recast a result from modal logic in description logical terms.

Schild [Sch91] was the first to give a precise formulation of the connection between description logics and modal logics. Readers familiar with multimodal logic will immediately recognize the similarity between existential quantification $\exists R.C$ and the diamond operator $\langle R \rangle C$, and between universal quantification $\forall R.C$ and the box operator $[R]C$. Given this connection between description logics and modal logics, results in the one domain become available to the other. In modal logic, the following notion is now being used as an important model-theoretic tool, even at the textbook level, cf. [Pop94].

Definition 3.2 Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ be two interpretations. A non-empty relation $Z \subseteq (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}})$ is called a *bisimulation* if it satisfies the following clauses.

1. If $d_1 Z d_2$, then, for every (atomic) concept name A, $d_1 \in A^{\mathcal{I}}$ iff $d_2 \in A^{\mathcal{J}}$.
2. For every (atomic) role name R, if $d_1 Z d_2$ and $R^{\mathcal{I}} d_1 e_1$, then there exists e_2 in $\Delta^{\mathcal{J}}$ such that $R^{\mathcal{J}} d_2 e_2$ and $e_1 Z e_2$.
3. For every (atomic) role name R, if $d_1 Z d_2$ and $R^{\mathcal{J}} d_2 e_2$, then there exists e_1 in $\Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}} d_1 e_1$ and $e_1 Z e_2$.

A first-order formula $\alpha(x)$ is said to be *preserved under bisimulations* if for all interpretations $\mathcal{I}$, $\mathcal{J}$, all objects $d_1 \in \Delta^{\mathcal{I}}$ and $d_2 \in \Delta^{\mathcal{J}}$, and all bisimulations Z between $\mathcal{I}$ and $\mathcal{J}$, we have that $\mathcal{I} \models \alpha[d_1]$ implies $\mathcal{J} \models \alpha[d_2]$ whenever $d_1 Z d_2$.

Bisimulations are aso used extensively in concurrency theory [Mil89], and to a lesser extent in the area of semistructured data [BDHS96].

What is the relevance of bisimulations for purposes of the present paper? Briefly, bisimulations are relations between interpretations that preserve all $\mathcal{ALC}$-concepts. This is clear for atomic concept names (clause 1 in Definition 3.2), and a simple induction shows it to hold for boolean combinations as well. The *back-and-forth* clauses 2 and 3 guarantee preservation of existential and universal quantification, respectively.

The following theorem establishes a kind of converse for this preservation result; it is the starting point for our investigations.

Theorem 3.3 *Let $\alpha(x)$ be a first-order formula. Then $\alpha(x)$ is (equivalent to) an $\mathcal{ALC}$-concept iff it is preserved under bisimulations.*

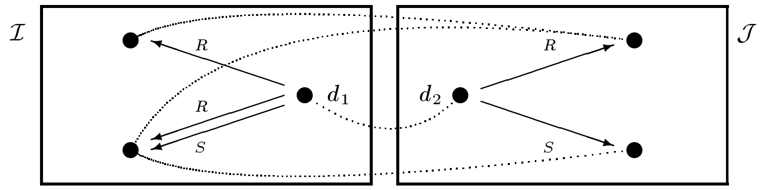
Proof. The proof consists of two parts: as explained above, by a simple induction one can show that $\mathcal{ALC}$-concepts are preserved under bisimulations. The proof of the other direction can be obtained as follows. As was first observed in [Sch91], $\mathcal{ALC}$ is a notational variant of normal multi-modal logic (with full boolean expressivity). The corresponding preservation theorem for mono-modal logic may be found in [Ben85], but it can easily be extended to the multi-modal case. $\dashv$

In words, preservation under bisimulations is the unique model-theoretic property that characterizes $\mathcal{ALC}$ as a fragment of first-order logic. One can put this property to good use in the following way: to show that a description logic $\mathcal{L}$ (extending $\mathcal{ALC}$) is more expressive than $\mathcal{ALC}$, by Theorem 3.3 it suffices to identify an $\mathcal{L}$-concept that is not preserved under bisimulations.

Corollary 3.4 *Let $\mathcal{L}$ be a description logic that can be obtained from $\mathcal{ALC}$ by adding any non-empty combination of $\mathcal{R}$ or $\mathcal{N}$. Then $\mathcal{ALC} < \mathcal{L}$.*

Proof. To prove $\mathcal{ALC} < \mathcal{ALCR}$ (or $\mathcal{ALCN}$ or $\mathcal{ALCRN}$, respectively), it suffices to provide two interpretations $\mathcal{I}$, $\mathcal{J}$ and objects $d_1 \in \Delta^{\mathcal{I}}$, $d_2 \in \Delta^{\mathcal{J}}$ as well as a bisimulation Z linking d_1 and d_2, such that for some $\mathcal{ALCR}$-concept C (or $\mathcal{ALCN}$-concept or $\mathcal{ALCNR}$-concept) we have $d_1 \in C^{\mathcal{I}}$ but $d_2 \notin C^{\mathcal{J}}$.

Consider the interpretations $\mathcal{I}$ and $\mathcal{J}$ depicted below. (The arrows denote the interpretation of R, an object is labeled A if it is in the interpretation of A, and the dotted lines indicate the bisimulation.)



Let $C_1 := (\exists(R \sqcap S).\top)$ and $C_2 := (\geq 2\ R)$. Then, clearly, $d_1 \in C_1^{\mathcal{I}}$ and $d_1 \in C_2^{\mathcal{I}}$, but $d_2 \notin C_1^{\mathcal{J}}$ and $d_2 \notin C_2^{\mathcal{J}}$.

We leave it to the reader to check that the relation indicated by the dotted lines is indeed a bisimulation. It follows from Theorem 3.3 that neither $\exists(R \sqcap S).\top$ nor $(\geq 2\ R)$ is (equivalent to) an $\mathcal{ALC}$-concept. Hence, $\mathcal{ALC} < \mathcal{ALCR}$, $\mathcal{ALCN}$, $\mathcal{ALCRN}$. $\dashv$

Now, what do we need to do to adapt the above result for other extensions of $\mathcal{FL}^-$ defined by Table 1? For logics *less expressive* than $\mathcal{ALC}$ we can not just use bisimulations, as such logics lack negation or disjunction, and these are automatically preserved under bisimulations; moreover, the *proof* of Theorem 3.3 uses the presence of the booleans in an essential way. For logics *more expressive* than $\mathcal{ALC}$ some of their constructors need not be preserved under bisimulations. Therefore we have to develop new notions of (bi-)simulation; this wil be the focus of our attention in Section 4.

4 Separating Description Logics

This section contains the main results of the paper. For $\mathcal{FL}^-$, $\mathcal{AL}$, and all of their extensions that can be defined using the constructors in Table 1, we present semantic characterization analogous to Theorem 3.3. We subsequently use these to separate logics, thus completing items 2 and 3 of the methodology outlined in Section 3, and obtain a complete classification of the full $\mathcal{FL}^-$ and $\mathcal{AL}$-hierarchies.

We proceed as follows. We first consider the 'minimal' logic $\mathcal{FL}^-$, characterize it semantically, and use the characterization to separate $\mathcal{FL}^-$ from richer logics. After that, we treat each of the constructors in Table 1 that are not in $\mathcal{FL}^-$, and examine which changes are needed to characterize the resulting logics. This is followed by a section in which we consider combinations of constructors. Our classification results are summarized in a diagram at the end of the section. Proofs of the characterization results are given in an appendix.

Throughout this section the following abbreviations will prove to be useful; let X, Y be subsets of a given domain.

$$
\begin{array}{lll}
XR^{\uparrow}Y & \text{iff} & \text{for all } d \in X \text{ there exists } e \in Y \text{ such that } Rde \\
XR_{\downarrow}Y & \text{iff} & \text{for all } e \in Y \text{ there exists } d \in X \text{ such that } Rde.
\end{array}
$$

In words, X is $R^{\uparrow}$-related to Y if every object in X 'sees' an object in Y; and X is $R_{\downarrow}$-related to Y if every object in Y is 'seen' by an object in X. As an aside, the two relations $R^{\uparrow}$ and $R_{\downarrow}$ are two particular instances of 'lifting' a binary relation on objects to a binary relation on sets of objects. In the setting of program semantics they are known as the *Hoare power order* and the *Egli-Milner power order*, respectively; cf. [Vic89].

4.1 The Base Case: $\mathcal{FL}^-$

Recall that the logic $\mathcal{FL}^-$ has $\top$, $\bot$, universal quantification $\forall R.C$, conjunction $C \sqcap D$, and unqualified existential quantification $\exists R.\top$.

What do we need to develop a notion of bisimulation that can be used to characterize $\mathcal{FL}^-$? First of all, ordinary bisimulations as defined in Definition 3.2 preserve *negations* of concepts — this is obviously too much for $\mathcal{FL}^-$, as it does not have negations. To destroy preservation of negations we will introduce a direction in the atomic clause of Definition 3.2, and hence make bisimulations non-symmetric. This change will enable us to preserve positive (negation-free) information only.

However, disjunctions would still be preserved under such non-symmetric bisimulations. As $\mathcal{FL}^-$ does not allow disjunctions of concepts, we need to block this as well. To achieve this, we change the format of bisimulations: instead of linking an object to an object, we will link a set of objects to an object. The notion of preservation will then say that if a concept or formula holds for every object in the set, then it must hold in the similar object. If a disjunctive concept or formula holds for all objects in a set (of size at least two) this no longer implies that one of the disjuncts holds for all objects in the set; as a consequence the inductive argument needed to prove Theorem 3.3 may break down.[2]

[2] As an aside, by linking sets (or objects) to sets we would also be able to deal with logics without conjunction.

Definition 4.1 Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ be two interpretations. An $\mathcal{FL}^-$*-simulation* is a non-empty relation $Z \subseteq \mathcal{P}(\Delta^{\mathcal{I}}) \times \Delta^{\mathcal{J}}$ such that the following hold.

1. If $X_1 Z d_2$ then, for every (atomic) concept name A, if $X_1 \subseteq A^{\mathcal{I}}$, then $d_2 \in A^{\mathcal{J}}$.
2. For every (atomic) role name R, if $X_1 (R^{\mathcal{I}})^{\uparrow} Y_1$ and $X_1 Z d_2$, then there exists $e_2 \in \Delta^{\mathcal{J}}$ with $R^{\mathcal{J}} d_2 e_2$.
3. For every (atomic) role name R, if $R^{\mathcal{J}} d_2 e_2$ and $X_1 Z d_2$, then there exists $Y_1 \subseteq \Delta^{\mathcal{I}}$ with $X_1 (R^{\mathcal{I}})_{\downarrow} Y_1$ and $Y_1 Z e_2$.

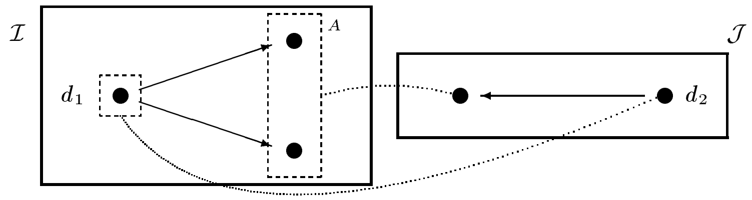
A first-order formula $\alpha(x)$ is *preserved under* $\mathcal{FL}^-$*-simulations* if for all interpretations $\mathcal{I}$ and $\mathcal{J}$, all sets $X \subseteq \Delta^{\mathcal{I}}$ and objects $d_2 \in \Delta^{\mathcal{J}}$, and all $\mathcal{FL}^-$-simulations Z between $\mathcal{I}$ and $\mathcal{J}$, we have that if $X Z d_2$ and for all $d_1 \in X$, $\mathcal{I} \models \alpha[d_1]$, then $\mathcal{J} \models \alpha[d_2]$.

The basic intuition underlying the clauses in Definition 4.1 is that atomic roles need to be preserved (clause 1); we only need to preserve unqualified existential quantifications (clause 2), but we need to preserve full universal quantification $\forall R.C$, where C may itself be a complex concept — this necessitates the 'and $Y_1 Z e_2$' in clause 3. Of course, in addition we need to preserve conjunctive concepts $C \sqcap D$, but this we get for free.

Theorem 4.2 (Characterization of $\mathcal{FL}^-$) *Let $\alpha(x)$ be a first-order formula. Then $\alpha(x)$ is equivalent to an $\mathcal{FL}^-$-concept iff it is preserved under $\mathcal{FL}^-$-simulations.*

Corollary 4.3 *Let $\mathcal{L}$ be either $\mathcal{AL}$ or any description logic that can be obtained from $\mathcal{FL}^-$ or $\mathcal{AL}$ by adding any non-empty combination of $\mathcal{U}$, $\mathcal{C}$, $\mathcal{E}$, $\mathcal{N}$, or $\mathcal{R}$. Then $\mathcal{FL}^- < \mathcal{L}$.*

Proof. We only show this for one logic, and we do so by displaying a concept that can not be equivalent to a concept in $\mathcal{FL}^-$. The concept $\exists R.A$ (which lives in $\mathcal{FLE}^-$ and its extensions) is not equivalent to an $\mathcal{FL}^-$-concept, as it is not preserved under $\mathcal{FL}^-$-simulations. To see why, consider the interpretations $\mathcal{I}$ and $\mathcal{J}$ depicted below. (The dashed boxes indicate sets.)



Here we have that $\{d_1\} \subseteq (\exists R.A)^{\mathcal{I}}$ but $d_2 \notin (\exists R.A)^{\mathcal{J}}$, even though there is an $\mathcal{FL}^-$-simulation relating $\{d_1\}$ to d_2. Hence, by Theorem 4.2, $\exists R.A$ can not be equivalent to an $\mathcal{FL}^-$-concept. As $\mathcal{FL}^- \leq \mathcal{FLE}^-$ is obvious, it follows that $\mathcal{FL}^- < \mathcal{FLE}^-$. $\dashv$

4.2 Adding Negation

We now consider the changes that need to be made to the basic set-up for $\mathcal{FL}^-$-simulations if some form of negation is present in the logic. In particular, we consider the logic $\mathcal{AL}$; recall that it extends $\mathcal{FL}^-$ by negation of (atomic) concept names. It turns out that only minor changes are required as compared to Definition 4.1.

Definition 4.4 Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ and $\mathcal{J} = (\Delta^\mathcal{J}, \cdot^\mathcal{J})$ be two interpretations. An $\mathcal{AL}$*-simulation* is a non-empty relation $Z \subseteq \mathcal{P}(\Delta^\mathcal{I}) \times \Delta^\mathcal{J}$ such that the following hold.

1. If $X_1 Z d_2$ then, for every (atomic) concept name A, if $X_1 \subseteq A^\mathcal{I}$, then $d_2 \in A^\mathcal{J}$, and if $X_1 \subseteq \neg A^\mathcal{I}$, then $d_2 \in \neg A^\mathcal{J}$.

2. For every (atomic) role name R, if $X_1 (R^\mathcal{I})^\uparrow Y_1$ and $X_1 Z d_2$, then there exists $e_2 \in \Delta^\mathcal{J}$ with $R^\mathcal{J} d_2 e_2$.

3. For every (atomic) role name R, if $R^\mathcal{J} d_2 e_2$ and $X_1 Z d_2$, then there exists $Y_1 \subseteq \Delta^\mathcal{I}$ with $X_1 (R^\mathcal{I})_\downarrow Y_1$ and $Y_1 Z e_2$.

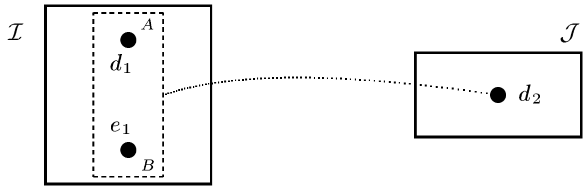
A first-order formula $\alpha(x)$ is *preserved under* $\mathcal{AL}$*-simulations* if for all interpretations $\mathcal{I}$ and $\mathcal{J}$, all sets $X \subseteq \Delta^\mathcal{I}$ and objects $d_2 \in \Delta^\mathcal{J}$, and all $\mathcal{AL}$-simulations Z between $\mathcal{I}$ and $\mathcal{J}$, we have that if $X Z d_2$ and for all $d_1 \in X$, $\mathcal{I} \models \alpha[d_1]$, then $\mathcal{J} \models \alpha[d_2]$.

The intuition underlying the change in clause 1 of Definition 4.4 (as compared to clause 1 of Definition 4.1) is that both positive and negative atomic information information now needs to be preserved in passing from $\mathcal{I}$ to $\mathcal{J}$.

Theorem 4.5 (Characterization of $\mathcal{AL}$) *Let $\alpha(x)$ be a first-order formula. Then $\alpha(x)$ is equivalent to an $\mathcal{AL}$-concept iff it is preserved under $\mathcal{AL}$-simulations.*

Corollary 4.6 *Let $\mathcal{L}$ be a description logic that can be obtained from $\mathcal{AL}$ by adding any non-empty combination of $\mathcal{U}$, $\mathcal{C}$, $\mathcal{E}$, $\mathcal{N}$, or $\mathcal{R}$. Then $\mathcal{AL} < \mathcal{L}$. Also, if $\mathcal{L}$ is obtained from $\mathcal{FL}^-$ by adding one of $\mathcal{U}$, $\mathcal{E}$, $\mathcal{N}$, or $\mathcal{R}$, then $\mathcal{L} \not< \mathcal{AL}$.*

Proof. As in Corollary 4.3, by way of example we only consider one case for the proof of the first claim. We show that $\mathcal{AL}$ is strictly less expressive than $\mathcal{ALU}$ by providing an $\mathcal{ALU}$-concept that is not equivalent to any $\mathcal{AL}$-concept.



The $\mathcal{ALU}$-concept $A \sqcup B$ is not equivalent to an $\mathcal{AL}$-concept. In the two interpretations $\mathcal{I}$, $\mathcal{J}$ depicted above we have that $\{d_1, e_1\} \subseteq (A \sqcup B)^{\mathcal{I}}$, and there exists an $\mathcal{AL}$-simulation linking $\{d_1, d_1'\}$ to d_2, but $d_2 \notin (A \sqcup B)^{\mathcal{J}}$. By Theorem 4.5, then, $(A \sqcup B)$ can not be equivalent to an $\mathcal{AL}$-concept.

Similar arguments may be used to establish the second claim of the corollary. $\dashv$

4.3 Adding Existential Quantification

Next we consider adding full existential quantification as a constructor to $\mathcal{FL}^-$. For the resulting logic $\mathcal{FLE}^-$ we obtain the appropriate notion of simulation by taking Definition 4.1 and adding 'and $Y_1 Z e_2$' as a conjunct to clause 2. Clearly, what we need for $\mathcal{FLE}^-$-concepts to be preserved by an appropriate notion of simulation, is that concepts of the form $\exists R.C$ are preserved, and the additional condition '$Y_1 Z e_2$' achieves this — it simply mirrors clause 3 (which achieves preservation of universal quantifications), and hence to a certain degree it restores symmetry.

Definition 4.7 Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ be two interpretations. An $\mathcal{FLE}^-$*-simulation* is a non-empty relation $Z \subseteq \mathcal{P}(\Delta^{\mathcal{I}}) \times \Delta^{\mathcal{J}}$ such that the following hold.

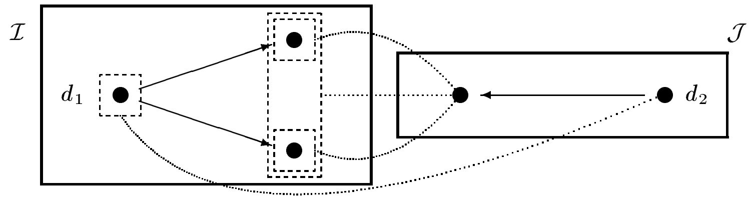
1. If $X_1 Z d_2$ then, for every (atomic) concept name A, if $X_1 \subseteq A^{\mathcal{I}}$, then $d_2 \in A^{\mathcal{J}}$.
2. For every (atomic) role name R, if $X_1 (R^{\mathcal{I}})^{\uparrow} Y_1$ and $X_1 Z d_2$, then there exists $e_2 \in \Delta^{\mathcal{J}}$ with $R^{\mathcal{J}} d_2 e_2$ and $Y_1 Z e_2$.
3. For every (atomic) role name R, if $R^{\mathcal{J}} d_2 e_2$ and $X_1 Z d_2$, then there exists $Y_1 \subseteq \Delta^{\mathcal{I}}$ with $X_1 (R^{\mathcal{I}})_{\downarrow} Y_1$ and $Y_1 Z e_2$.

A first-order formula $\alpha(x)$ is *preserved under* $\mathcal{FLE}^-$*-simulations* if for all interpretations $\mathcal{I}$ and $\mathcal{J}$, all sets $X \subseteq \Delta^{\mathcal{I}}$ and objects $d_2 \in \Delta^{\mathcal{J}}$, and all $\mathcal{FLE}^-$-simulations Z between $\mathcal{I}$ and $\mathcal{J}$, we have that if $X Z d_2$ and for all $d_1 \in X$, $\mathcal{I} \models \alpha[d_1]$, then $\mathcal{J} \models \alpha[d_2]$.

Theorem 4.8 (Characterization of $\mathcal{FLE}^-$) *Let $\alpha(x)$ be a first-order formula. Then $\alpha(x)$ is equivalent to an $\mathcal{FLE}^-$-concept iff it is preserved under $\mathcal{FLE}^-$-simulations.*

Corollary 4.9 *Let $\mathcal{L}$ be a description logic that can be obtained from $\mathcal{FLE}^-$ by adding any non-empty combination of $\mathcal{U}$, $\mathcal{C}$, $\mathcal{N}$, or $\mathcal{R}$. Then $\mathcal{FLE}^- < \mathcal{L}$. Also, if $\mathcal{L}$ is either $\mathcal{AL}$ or obtained from $\mathcal{FL}^-$ by adding one of $\mathcal{U}$, $\mathcal{N}$, or $\mathcal{R}$, then $\mathcal{L} \not\leq \mathcal{FLE}^-$.*

Proof. As before, we will only prove the corollary for one case. We will show that $\mathcal{FLE}^-$ is strictly less expressive than $\mathcal{FLEN}^-$. The interpretations in the following figure show that the $\mathcal{FLEN}^-$-concept $(\geq 2\ R)$ is not equivalent to an $\mathcal{FLE}^-$-concept.



In the above figure we have $\{d_1\} \subseteq (\geq 2\ R)^{\mathcal{I}}$ but $d_2 \notin (\geq 2\ R)^{\mathcal{J}}$ even though there is an $\mathcal{FLE}^-$-simulation (indicated by the dotted lines) that links $\{d_1\}$ to d_2. ⊣

4.4 Adding Disjunction

For $\mathcal{FLU}^-$ we obtain the appropriate notion of simulation by taking Definition 4.1, but instead of linking sets of objects to objects, we now link objects (or: singleton sets) to objects. As explained in the introduction to this section, if a notion of simulations links *sets* of objects to *single* objects, disjunctions need not preserved, the reason being that from the fact that, if X is a set, then $X \subseteq (C \sqcup D)^{\mathcal{I}}$ does not imply $X \subseteq C^{\mathcal{I}}$ or $X \subseteq D^{\mathcal{I}}$. Working with single objects, however, we would of course be able to infer from $d \in (C \sqcup D)^{\mathcal{I}}$ that $d \in C^{\mathcal{I}}$ or $d \in D^{\mathcal{I}}$, and this would allow us to give an inductive proof of a preservation result for disjunctions.

Definition 4.10 Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ be two interpretations. An $\mathcal{FLU}^-$*-simulation* is a non-empty relation $Z \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ such that the following hold.

1. If $d_1 Z d_2$ then, for every (atomic) concept name A, if $d_1 \in A^{\mathcal{I}}$, then $d_2 \in A^{\mathcal{J}}$.

2. For every (atomic) role name R, if $R^{\mathcal{I}} d_1 e_1$ and $d_1 Z d_2$, then there exists $e_2 \in \Delta^{\mathcal{J}}$ with $R^{\mathcal{J}} d_2 e_2$.

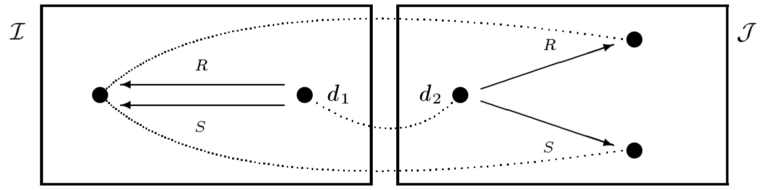
3. For every (atomic) role name R, if $R^{\mathcal{J}} d_2 e_2$ and $d_1 Z d_2$, then there exists $e_1 \in \Delta^{\mathcal{I}}$ with $R^{\mathcal{I}} d_1 e_1$ and $e_1 Z e_2$.

A first-order formula $\alpha(x)$ is *preserved under $\mathcal{FLU}^-$-simulations* if for all interpretations $\mathcal{I}$ and $\mathcal{J}$, all objects $d_1 \in \Delta^{\mathcal{I}}$, $d_2 \in \Delta^{\mathcal{J}}$, and all $\mathcal{FLU}^-$-simulations Z between $\mathcal{I}$ and $\mathcal{J}$, we have that if $d_1 Z d_2$ and $\mathcal{I} \models \alpha[d_1]$, then $\mathcal{J} \models \alpha[d_2]$.

Theorem 4.11 (Characterization of $\mathcal{FLU}^-$) *Let $\alpha(x)$ be a first-order formula. Then $\alpha(x)$ is equivalent to an $\mathcal{FLU}^-$-concept iff it is preserved under $\mathcal{FLU}^-$-simulations.*

Corollary 4.12 *Let $\mathcal{L}$ be a description logic that can be obtained from $\mathcal{FLU}^-$ by adding any non-empty combination of $\mathcal{C}$, $\mathcal{E}$, $\mathcal{N}$, or $\mathcal{R}$. Then $\mathcal{FLU}^- < \mathcal{L}$. Also, if $\mathcal{L}$ is either $\mathcal{AL}$ or obtained from $\mathcal{FL}^-$ by adding one of $\mathcal{E}$, $\mathcal{N}$, or $\mathcal{R}$, then $\mathcal{L} \not< \mathcal{FLU}^-$.*

Proof. As before, we will only prove the corollary for one case. We will separate $\mathcal{FLU}^-$ from $\mathcal{FLUR}^-$. The interpretations in the following figure show that the $\mathcal{FLUR}^-$-concept $\exists(R \sqcap S).\top$ is not equivalent to an $\mathcal{FLU}^-$-concept.



In the above figure we have $d_1 \in (\exists(R \sqcap S).\top)^{\mathcal{I}}$ but $d_2 \notin (\exists(R \sqcap S).\top)^{\mathcal{J}}$ even though there is an $\mathcal{FLU}^-$-simulation (indicated by the dotted lines) that links d_1 to d_2. $\dashv$

4.5 Adding Number Restrictions

To arrive at a notion of simulation for $\mathcal{FLN}^-$ we use the above ideas together with ideas from [Rij96]. The main feature of the notion of $\mathcal{FLN}^-$-simulation is that in order to guarantee preservation of number restrictions it records the size of sets of objects taking part in the simulation. It does this using a whole sequence of relations between sets of sets of objects on the one hand and sets of objects on the other; later on, in the presence of disjunction we will be able to simplify these to relations between sets of objects on both sides.

The following notation will prove to be useful. We write $R^\bullet d_1 Y_1$ if for all $e_1 \in Y_1$, $R d_1 e_1$ holds. As before, since $\mathcal{FLN}^-$ is a logic without disjunction, our notion of simulation for $\mathcal{FLN}^-$ needs to relate sets of objects to objects. But we need a bit more. For, let $\mathcal{I}$ be an interpretation, and let $X_1 \subseteq \Delta^{\mathcal{I}}$ be such that $X_1 \subseteq (\geq n\ R)^{\mathcal{I}}$; then, for each $d_1 \in X_1$ there exists $Y_{d_1} \subseteq \Delta^{\mathcal{I}}$ with $|Y_{d_1}| \geq n$ and $(R^{\mathcal{I}})^\bullet d_1 Y_{d_1}$. Now, to ensure preservation of $(\geq n\ R)$

from X_1 to any object d_2 similar to X_1, we need to consider the collection of all these sets Y_{d_1}, where d_1 ranges over elements of X_1. The following definition captures this idea.

Definition 4.13 Let R be a role name, and $i > 0$. Assume that $X_1 \subseteq \Delta^{\mathcal{I}}$, where $\mathcal{I}$ is some interpretation. An *i-cloud* is a set $\mathcal{X}$ of subsets of $\Delta^{\mathcal{I}}$ such that for all $Y \in \mathcal{X}$, $|Y| = i$.

An i-cloud $\mathcal{X}$ is said to be *R-above* $X_1 \subseteq \Delta^{\mathcal{I}}$ if for all $d_1 \in X_1$ there exists $Y_1 \in \mathcal{X}$ such that $(R^{\mathcal{I}})^\bullet d_1 Y_1$.

A set X_1 is said to be *R-below* an i-cloud $\mathcal{X}$ if for every $Y_1 \in \mathcal{X}$ there exists $d_1 \in X_1$ such that $(R^{\mathcal{I}})^\bullet d_1 Y_1$.

By indexing i-clouds with the set above which they hang, we can ensure that every cloud is above exactly one set only.

We are ready now for the definition of an $\mathcal{FLN}^-$-simulation. We use $\mathcal{P}^{<\omega}(X)$ to denote the collection of finite subsets of X.

Definition 4.14 Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ be two interpretations. An *$\mathcal{FLN}^-$-simulation* between $\mathcal{I}$ and $\mathcal{J}$ is a sequence of relations $Z = (Z_0, Z_1, \ldots, Z_n, \ldots)$ such that the following hold.

1. Z_0 is non-empty;
2. (a) $Z_0 \subseteq \mathcal{P}(\Delta^{\mathcal{I}}) \times \Delta^{\mathcal{J}}$;
   (b) for all $i > 0$, $Z_i \subseteq \mathcal{P}(\mathcal{P}^{<\omega}(\Delta^{\mathcal{I}})) \times \mathcal{P}^{<\omega}(\Delta^{\mathcal{J}})$;
3. for all $i > 0$, if $\mathcal{X} Z_i Y_2$, then, for any $X \in \mathcal{X}$, $|X| = |Y_2| = i$;
4. if $X_1 Z_0 d_2$, then, for any (atomic) concept name A, if $X_1 \subseteq A^{\mathcal{I}}$ then $d_2 \in A^{\mathcal{J}}$;
5. if $X_1 Z_0 d_2$ and $\mathcal{X} \subseteq \mathcal{P}(\mathcal{P}^{<\omega}(\Delta^{\mathcal{I}}))$ is a non-empty i-cloud R-above X_1, where $i > 0$, then there exists $Y_2 \subseteq \Delta^{\mathcal{J}}$ with $(R^{\mathcal{J}})^\bullet d_2 Y_2$ and $\mathcal{X} Z_i Y_2$;
6. if $X_1 Z_0 d_2$ and $(R^{\mathcal{J}})^\bullet d_2 Y_2$, where $|Y_2| = i > 0$, then there exists a non-empty i-cloud $\mathcal{X} \subseteq \mathcal{P}(\mathcal{P}^{<\omega}(\Delta^{\mathcal{I}}))$ such that X_1 is R-below $\mathcal{X}$ and $\mathcal{X} Z_i Y_2$;
7. if $R^{\mathcal{J}} d_2 e_2$ and $X_1 Z_0 d_2$, then there exists a 1-cloud $\mathcal{X}$ such that X_1 is R-below $\mathcal{X}$ and $(\bigcup \mathcal{X})\, Z_0 e_2$.

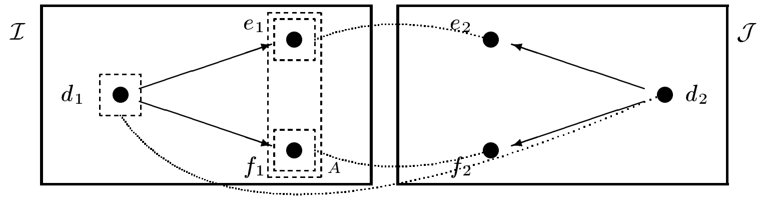
A first-order formula $\alpha(x)$ is *preserved under $\mathcal{FLN}^-$-simulations* if for all interpretations $\mathcal{I}$ and $\mathcal{J}$, all sets of objects $X_1 \subseteq \Delta^{\mathcal{I}}$ and objects $d_2 \in \Delta^{\mathcal{J}}$, and all $\mathcal{FLN}^-$-simulations $Z = (Z_0, Z_1, \ldots)$ between $\mathcal{I}$ and $\mathcal{J}$, we have that if $X_1 Z_0 d_2$ and for all $d_1 \in X_1$, $\mathcal{I} \models \alpha[d_1]$, then $\mathcal{J} \models \alpha[d_2]$.

To grasp the intuition behind Definition 4.14, observe that Z_0 is the 'engine' of the simulation that guarantees preservation, and the other relations Z_1, Z_2, ... are needed for matching finite sets of the same size. Clauses 1–3 of Definition 4.14 are bookkeeping clauses, and clause 4 is the familiar one about preservation of atomic concepts. Clauses 5 and 6 are the back-and-forth clauses that guarantee preservation of number restrictions $(\geq i\ R)$ and $(\leq i\ R)$, respectively. Clause 7 is needed to preserve universal quantifications $\forall R.C$.

Theorem 4.15 (Characterization of $\mathcal{FLN}^-$) *Let $\alpha(x)$ be a first-order formula. Then $\alpha(x)$ is equivalent to an $\mathcal{FLN}^-$-concept iff it is preserved under $\mathcal{FLN}^-$-simulations.*

Corollary 4.16 *Let $\mathcal{L}$ be a description logic that can be obtained from $\mathcal{FLN}^-$ by adding any non-empty combination of $\mathcal{U}$, $\mathcal{C}$, $\mathcal{E}$, or $\mathcal{R}$. Then $\mathcal{FLN}^- < \mathcal{L}$. Also, if $\mathcal{L}$ is either $\mathcal{AL}$ or obtained from $\mathcal{FL}^-$ by adding one of $\mathcal{E}$, $\mathcal{U}$, or $\mathcal{R}$, then $\mathcal{L} \not< \mathcal{FLN}^-$.*

Proof. We only prove the corollary for one case: $\mathcal{FLN}^- < \mathcal{FLNE}^-$. Consider the interpretations $\mathcal{I}$, $\mathcal{J}$ depicted below (the dotted lines indicate Z_0; other relations Z_i, for $i > 0$, are specified in the text below).



Clearly, $\{d_1\} \subseteq (\exists R.A)^{\mathcal{I}}$, but $d_2 \notin (\exists R.A)^{\mathcal{J}}$, so if there exists an $\mathcal{FLN}^-$-simulation linking $\{d_1\}$ and d_2, then $\exists R.A$ cannot be (equivalent to) an $\mathcal{FLN}^-$-concept. We leave it to the reader to show that the following tuple Z is indeed an $\mathcal{FLN}^-$-simulation linking $\{d_1\}$ and d_2: $Z = (Z_0, Z_1, Z_2, \ldots)$, where, for $i > 2$, $Z_i = \emptyset$, while

$$\begin{aligned}
Z_0 &= \{(\{d_1\}, d_2), (\{e_1\}, e_2), (\{f_1\}, f_2), (\{e_1, f_1\}, e_2), (\{e_1, f_1\}, f_2)\} \\
Z_1 &= \{(\{\{e_1, \{f_1\}\}, \{e_2\}), (\{\{e_1, \{f_1\}\}, \{f_2\})\} \\
Z_2 &= \{(\{\{e_1, f_1\}\}, \{e_2, f_2\})\}. \quad \dashv
\end{aligned}$$

4.6 Adding Role Conjunction

Combining ideas from [Hol, KR97a] and the preceding sections, we arrive at a notion of simulation for $\mathcal{FLR}^-$. Its distinguishing feature is that it not only relates sets of objects to objects (as in Definition 4.1), but to cater for role intersection it also links pairs of (sets of) objects to pairs of objects. We will need the following auxiliary notion.

Let X, Y be two sets of objects. A collection of (atomic) role names $\mathcal{R}$ is called *meet closed for X and Y* if $X(\bigsqcap\mathcal{R})^{\uparrow}Y$.

Definition 4.17 Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ be two interpretations. An *$\mathcal{FLR}^{-}$-simulation* is a triple $Z = (Z_0, Z_1, Z_2)$ such that the following hold.

1. (a) $Z_0 \subseteq \mathcal{P}(\Delta^{\mathcal{I}}) \times \Delta^{\mathcal{J}}$;
   (b) $Z_1 \subseteq (\mathcal{P}(\Delta^{\mathcal{I}}) \times \mathcal{P}(\Delta^{\mathcal{I}})) \times (\Delta^{\mathcal{J}} \times \mathcal{P}(\Delta^{\mathcal{J}}))$.
   (c) $Z_2 \subseteq (\mathcal{P}(\Delta^{\mathcal{I}}) \times \mathcal{P}(\Delta^{\mathcal{I}})) \times (\Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}})$.
2. If $X_1 Z_0 d_2$, then, for every (atomic) concept name A, if $X_1 \subseteq A^{\mathcal{I}}$, then $d_2 \in A^{\mathcal{J}}$.
3. (a) If $(X_1, Y_1) Z_1 (d_2, E_2)$ then, for every collection of role names $\mathcal{R}$ that is meet closed for X_1 and Y_1, there exists an $e_2 \in E_2$ such that $(\bigsqcap\mathcal{R})^{\mathcal{I}} d_2 e_2$.
   (b) If $(X_1, Y_1) Z_2 (d_2, e_2)$ then, for every role R, if $R^{\mathcal{J}} d_2 e_2$ holds, then $X_1 (R^{\mathcal{I}})_{\downarrow} Y_1$.
4. (a) If $X_1 Z_0 d_2$, then, for every (atomic) role name R, if $X_1 (R^{\mathcal{I}})^{\uparrow} Y_1$, then there exists $E_2 \subseteq \Delta^{\mathcal{J}}$ with $(X_1, Y_1) Z_1 (d_2, E_2)$.
   (b) If $X_1 Z_0 d_2$, then, for every (atomic) role name R, if $R^{\mathcal{J}} d_2 e_2$, then there exists $Y_1 \subseteq \Delta^{\mathcal{I}}$ with $(X_1, Y_1) Z_2 (d_2, e_2)$.
5. If $(X_1, Y_1) Z_2 (d_2, e_2)$, then $Y_1 Z_0 e_2$.

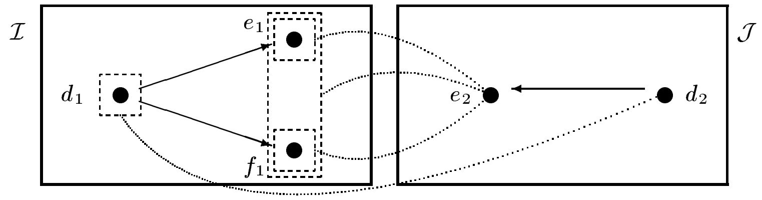
A first-order formula $\alpha(x)$ is *preserved under $\mathcal{FLR}^{-}$-simulations* if for every two interpretations $\mathcal{I}$ and $\mathcal{J}$, all sets $X \subseteq \Delta^{\mathcal{I}}$ and objects $d_2 \in \Delta^{\mathcal{J}}$, and all $\mathcal{FLR}^{-}$-simulations Z between $\mathcal{I}$ and $\mathcal{J}$, we have that if $X_1 Z_0 d_2$ and for all $d_1 \in X_1$, $\mathcal{I} \models \alpha[d_1]$, then $\mathcal{J} \models \alpha[d_2]$.

Let us briefly explain what the clauses in Definition 4.17 are meant to achieve. Clause 2 is the familiar clause about preservation of atomic concepts. Clause 3(a) is about preservation of intersecting roles from $\mathcal{I}$ to $\mathcal{J}$; there is slight technical complication here: $X_1 R^{\uparrow} Y_1$ and $X_1 S^{\uparrow} Y_1$ does not imply $X_1 (R \sqcap S)^{\uparrow} Y_1$, and this failure forces us to consider only those collections of role names $\mathcal{R}$ (with $X_1 R^{\uparrow} Y_1$, for $R \in \mathcal{R}$) that are closed under intersection in this sense; the notion of meet closure tries to capture this idea. Next, clause 3(b) simply tries to mirror intersections from $\mathcal{J}$ to $\mathcal{I}$. Clauses 4(a) and 4(b) are the real back-and-forth clauses, where simulations between sets and objects extend to pairs of sets and pairs of objects (and sets). Clause 5 relates such simulations between pairs to simulations between sets and objects (but, by analogy with clauses 2 and 3 of Definition 4.1, this is only required in one direction, viz. from $\mathcal{J}$ to $\mathcal{I}$).

Theorem 4.18 (Characterization of $\mathcal{FLR}^-$) *Let $\alpha(x)$ be a first-order formula. Then $\alpha(x)$ is equivalent to an $\mathcal{FLR}^-$-concept iff it is preserved under $\mathcal{FLR}^-$-simulations.*

Corollary 4.19 *Let $\mathcal{L}$ be a description logic that can be obtained from $\mathcal{FLR}^-$ by adding any non-empty combination of $\mathcal{U}$, $\mathcal{C}$, $\mathcal{E}$, or $\mathcal{N}$. Then $\mathcal{FLR}^- < \mathcal{L}$. Also, if $\mathcal{L}$ is either $\mathcal{AL}$ or obtained from $\mathcal{FL}^-$ of $\mathcal{AL}$ by adding one of $\mathcal{E}$, $\mathcal{U}$, or $\mathcal{N}$, then $\mathcal{L} \not< \mathcal{FLR}^-$.*

Proof. We only prove the corollary for the case $\mathcal{FLR}^- < \mathcal{FLRN}^-$. Consider the two interpretations below.



The dotted lines indicate the Z_0-component of an $\mathcal{FLR}^-$-simulation linking $\{d_1\}$ to d_2; it, and the remaining components, are defined as follows:

$$
\begin{aligned}
Z_0 &= \{(\{d_1\}, d_2), (\{e_1\}, e_2), (\{f_1\}, e_2), (\{e_1, f_1\}, e_2)\} \\
Z_1 &= \{((\{d_1\}, \{e_1\}), (d_2, \{e_2\})), ((\{d_1\}, \{f_1\}), (d_2, \{e_2\})) \\
&\quad ((\{d_1\}, \{e_1, f_1\}), (d_2, \{e_2\}))\} \\
Z_2 &= \{((\{d_1\}, \{e_1\}), (d_2, e_2)), ((\{d_1\}, \{f_1\}), (d_2, e_2)), \\
&\quad ((\{d_1\}, \{e_1, f_1\}), (d_2, e_2))\}
\end{aligned}
$$

We leave it to the reader that this (Z_0, Z_1, Z_2) is indeed an $\mathcal{FLR}^-$-simulation such that $\{d_1\} Z_0 d_2$. Clearly, $\{d_1\} \subseteq (\geq 2\ R)^{\mathcal{I}}$, but $d_2 \notin (\geq 2\ R)^{\mathcal{J}}$. It follows that $(\geq 2\ R)$ is not equivalent to an $\mathcal{FLR}^-$-concept. As we obviously have $\mathcal{FLR}^- \leq \mathcal{FLRN}^-$, we conclude $\mathcal{FLR}^- < \mathcal{FLRN}^-$. $\dashv$

4.7 Combinations

The semantic characterization results obtained so far form the basic building blocks for our further results. Briefly, the idea is to obtain semantic characterizations of logics that contain combinations of the constructors $\mathcal{C}$, $\mathcal{U}$, $\mathcal{E}$, $\mathcal{N}$ and $\mathcal{R}$ by combining the characterizations of the logics admitting only one of the constructors. It will turn out that there is surprisingly little interaction between the various characterizations, and where there is interaction this results in a simplification (especially when $\mathcal{U}$ is added) or in restoring symmetry of various clauses (when $\mathcal{E}$ or $\mathcal{C}$ is added). Only in rare cases (such as $\mathcal{FLNR}^-$) does the characteristic notion of simulation become more complex.

Let us briefly consider the various combinations now. So as not to get lost in a plethora of logics, we will focus on extensions of $\mathcal{AL}$, $\mathcal{FLE}^-$, $\mathcal{FLU}^-$, $\mathcal{FLN}^-$, and $\mathcal{FLR}^-$ by the addition of a single construction. By way of example we show how a characteristic notion of simulation for *any* logic in the $\mathcal{FL}^-$ and $\mathcal{AL}$-hierarchy may be obtained from such extensions.

4.7.1 Extensions of $\mathcal{AL}$

As we have seen from the definitions of bisimulation and $\mathcal{AL}$-simulation (Definitions 3.2 and 4.4), in the presence of negation or negated atomic concept names, the clause guaranteeing preservation of atomic concept names either becomes symmetric (in the case of full negation) or we have to add preservation of negated atomic concepts as well.

That is, let $\mathcal{ALX}$ be one of $\mathcal{ALE}$, $\mathcal{ALU}$, $\mathcal{ALN}$, or $\mathcal{ALR}$. To obtain a characteristic notion of simulation for $\mathcal{ALX}$, we simply take the characteristic notion of simulation for $\mathcal{FLX}^-$ and add to the clause for preservation of atomic concept names the clause that negations of atomic concepts should also be preserved (as in Definition 4.4). Then, the relevant preservation theorems may be proved.

4.7.2 Extensions of $\mathcal{FLE}^-$

With full (qualified) existential quantification $\exists R.C$ present in the logic, the back-and-forth conditions that record the presence of roles, have to become symmetric: not only does the relational pattern need to be matched, but it needs to be matched with a *similar* object. For the semantic characterization results for the logics $\mathcal{FLEU}^-$, $\mathcal{FLEN}^-$, and $\mathcal{FLER}^-$, this requires the following.

- $\mathcal{FLEU}^-$-simulations are defined just like $\mathcal{FLU}^-$-simulations (Definition 4.10) except for clause 2, which needs to be

  2′. For every (atomic) role name R, if $R^{\mathcal{I}}d_1e_1$ and d_1Zd_2, then there exists $e_2 \in \Delta^{\mathcal{J}}$ with $R^{\mathcal{J}}d_2e_2$ and e_1Ze_2.

- $\mathcal{FLEN}^-$-simulations are defined like $\mathcal{FLN}^-$-simulations (Definition 4.14) except for clause 7, which needs to be

  7′. (a) If $X_1(R^{\mathcal{I}})^{\uparrow}Y_1$ and $X_1Z_0d_2$, then there exists e_2 such that $R^{\mathcal{J}}d_2e_2$ and $Y_1Z_0e_2$.
  (b) If $R^{\mathcal{J}}d_2e_2$ and $X_1Z_0d_2$, then there exists a 1-cloud $\mathcal{X}$ such that X_1 is R-below $\mathcal{X}$ and $(\bigcup\mathcal{X})\, Z_0e_2$.

- $\mathcal{FLER}^-$-simulations are defined like $\mathcal{FLR}^-$-simulations (Definition 4.17) except for clause 5, which needs to be

  5′. (a) If $(X_1, Y_1)Z_1(d_2, e_2)$, then $Y_1Z_0e_2$.

(b) If $(X_1, Y_1)Z_2(d_2, e_2)$, then $Y_1 Z_0 e_2$.

Using the above definitions, semantic characterizations may be given for each of the languages involved.

4.7.3 Extensions of $\mathcal{FLU}^-$

From a logical point of view having disjunctions of concepts available in a description logic simplifies matters considerably: we no longer have to relate sets of objects to single objects, but can simply relate objects to objects. Extending $\mathcal{FLU}^-$ by number restrictions or role conjunction requires the following changes to arrive at a characteristic notion of simulation.

- $\mathcal{FLUN}^-$-simulations are defined just like $\mathcal{FLN}^-$-simulations (Definition 4.14) except that Z_0 should now be a relation linking objects to objects, and the Z_i $(i > 0)$ should link finite sets to finite sets (of the same size). Clauses 4–7 should then be replaced by

  4′. If $d_1 Z_0 d_2$ then, for any (atomic) concept name A, if $d_1 \in A^{\mathcal{I}}$ then $d_2 \in A^{\mathcal{J}}$;

  5′. if $d_1 Z_0 d_2$ and $(R^{\mathcal{I}})^{\bullet} d_1 Y_1$, where $|Y_1| = i > 0$, then there exists $Y_2 \subseteq \Delta^{\mathcal{J}}$ with $(R^{\mathcal{J}})^{\bullet} d_2 Y_2$ and $Y_1 Z_i Y_2$;

  6′. If $d_1 Z_0 d_2$ and $(R^{\mathcal{J}})^{\bullet} d_2 Y_2$, where $|Y_2| = i > 0$, then there exists $Y_1 \subseteq \Delta^{\mathcal{I}}$ with $(R^{\mathcal{I}})^{\bullet} d_1 Y_1$ and $Y_1 Z_i Y_2$;

  7′. If $R^{\mathcal{J}} d_2 e_2$ and $d_1 Z_0 d_2$, then there exists $e_1 \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}} d_1 e_1$ and $e_1 Z_0 e_2$.

- $\mathcal{FLUR}^-$-simulations are defined just like $\mathcal{FLR}^-$-simulations (Definition 4.17) except that Z_0 should now link objects to objects, and Z_1 and Z_2 should link pairs of objects to pairs of objects. Then, clause 2 should be replace by clause 4′ above, while clauses 3–5 should be replace by

  3′. (a) If $(d_1, e_1)Z_1(d_2, e_2)$ then, for every role name R, if $R^{\mathcal{I}} d_1 e_1$, then $R^{\mathcal{J}} d_2 e_2$;
  (b) If $(d_1, e_1)Z_2(d_2, e_2)$ then, for every role name R, if $R^{\mathcal{J}} d_2 e_2$, then $R^{\mathcal{I}} d_1 e_1$;

  4′. (a) If $d_1 Z_0 d_2$, then for every role name R, if $R^{\mathcal{I}} d_1 e_1$, then there exists $e_2 \in \Delta^{\mathcal{J}}$ with $(d_1, e_1)Z_1(d_2, e_2)$.
  (b) If $d_1 Z_0 d_2$, then for every role name R, if $R^{\mathcal{J}} d_2 e_2$, then there exists $e_1 \in \Delta^{\mathcal{I}}$ with $(d_1, e_1)Z_2(d_2, e_2)$.

  5′. If $(d_1, e_1)Z_2(d_2, e_2)$ then $e_1 Z_0 e_2$.

Using the above ammendements, semantic characterizations may be given for of the languages involved.

4.7.4 Extensions of $\mathcal{FLN}^-$

The only extension of $\mathcal{FLN}^-$ (with a single constructor) that has not been considered so far is $\mathcal{FLNR}^-$. The notion of an $\mathcal{FLNR}^-$-simulation is arrived at by simply adding together the definitions for an $\mathcal{FLN}^-$-simulation and an $\mathcal{FLR}^-$-simulation, respectively. That is, an $\mathcal{FLNR}^-$-simulation is a tuple $(Z_0, Z_1, Z_2, \ldots; Z_1^r, Z_2^r)$ such that $(Z_0, Z_1, Z_2, \ldots)$ is an $\mathcal{FLN}^-$-simulation, and (Z_0, Z_1^r, Z_2^r) is an $\mathcal{FLR}^-$-simulation. Then, the usual semantic characterization results may be given for $\mathcal{FLNR}^-$.

4.7.5 Extensions of $\mathcal{FLR}^-$

Extensions of $\mathcal{FLR}^-$ by one of $\mathcal{C}$, $\mathcal{E}$, $\mathcal{U}$, or $\mathcal{N}$ are all covered in the preceding paragraphs.

4.7.6 Classifying an Arbitrary Description Logic

To obtain a characterization of an arbitrary description logic (defined from Table 1), somply combine the observations listed in Sections 4.7.1–4.7.5. More concretely, one may proceed as follows. Let $\mathcal{L}$ be an arbitrary description logic. First, determine how much negation it admits. If it admits full negation, then we have at least $\mathcal{ALC} \leq \mathcal{L}$ and we can use the ideas in §§4.7.1, 4.7.2; the only further options are that $\mathcal{L}$ admits $\mathcal{N}$ or $\mathcal{R}$, and in that case §4.7.3 applies. If, on the other hand, $\mathcal{L}$ does not admit full negation, we first see whether it does admit $\mathcal{U}$, and we consult §§4.7.1–4.7.3 if it does. If $\mathcal{L}$ does not admit $\mathcal{U}$, then one of §§4.7.1, 4.7.2 and 4.7.4 applies.

As a concrete example, consider $\mathcal{L} = \mathcal{ALENR}$. As $\mathcal{AL} \leq \mathcal{L}$, the atomic clause in the notion of an $\mathcal{L}$-simulation needs to preserve both atomic concepts and their negations. On top of that we need to ensure preservation of $\mathcal{E}$ (as explained in §4.7.2), and of $\mathcal{N}$ and $\mathcal{R}$ (as explained in §4.7.4). Putting things together, we get that the notion of simulation needed to characterize $\mathcal{ALENR}$, is a tuple $(Z_0, Z_1, \ldots; Z_1^r, Z_2^r)$, where $(Z_0, Z_1, \ldots)$ is an $\mathcal{ALEN}$-simulation (which is just like $\mathcal{FLEN}^-$-simulations, except for the atomic clause), and where (Z_0, Z_1^r, Z_2^r) is an $\mathcal{ALER}^-$-simulation (which is just like $\mathcal{FLEN}^-$-simulations, except for the atomic clause).

4.8 Harvest

We summarize our results in Figure 2. The way one should read the diagram is as follows. Every logic coincides with one of the logics in the diagram, and if a description logic $\mathcal{L}_1$ is above a logic $\mathcal{L}_2$ (via a sequence of one or more arcs), then $\mathcal{L}_2 < \mathcal{L}_1$. If two logics are incomparable in the diagram, then they are incomparable with respect to their expressive power.

Several comments are in order. First, the diagram does not mention all possible combinations of the constructors listed in Table 1. The reason for this is that some logics coincide with others (for example, $\mathcal{FLC}^-$ coincides with $\mathcal{ALEU}^-$).

Second, it should be noted that the classification obtained in Figure 2 is exactly the classification that one would expect from an intuitive point of view (where one logic is more expressive than another if it has more constructors). We view this absence of suprises both as an intuitive justification of our results, and as an indication that we have provided a mathematical underpinning for the basic intuitions one has concerning the expressive power of description logics.

And finally, we should point out that expressive power and complexity do not induce the same classifications of description logics. There are description logics that have the same expressive power but different complexity results for their satisfiability problems. For example, $\mathcal{ALC}$ and $\mathcal{ALC}^i$ (that is, $\mathcal{ALC}$ with inverse of roles) both have a PSPACE-complete satisfiability problem, but the latter is more expressive than the former. The precise relation between these alternative ways of classifying description logics remains to be investigated.

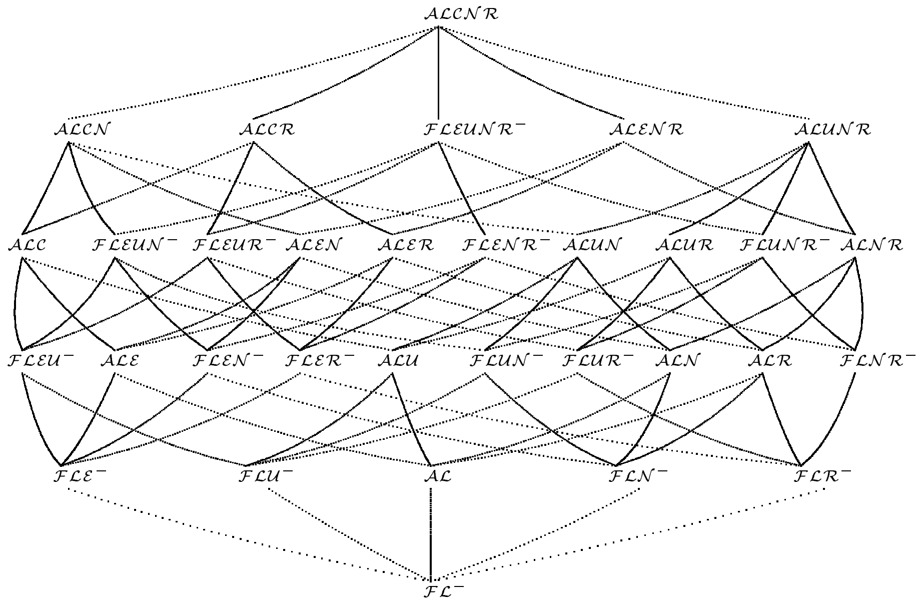


Figure 2: Classifying Description Logics

5 Discussion

We see two major lines of work related to this paper, the first one centered around the use of model-theoretic methods similar to the ones we have used, the second one focusing on the expressive power of description logics.

As to the first theme, the technique of Ehrenfeucht-Fraïssé games in first-order logic is closely related to our simulations, and it has been used to obtain numerous separation and preservation results; see [Doe96]. [IK87] use pebble games to obtain model-theoretic expressivity results about finite variable logics, and related techniques have been used in modal logic as well; for instance, [KR97b] use various kinds of bisimulations to characterize temporal logics with *Since* and *Until*. Also, [TN97] use similar methods to separate query languages. One of the principle advantages shared by these methods is their explicit and intuitive descriptions of the languages being studied. The results in this paper are different from the above ones, as we are interested in relatively poor languages with limited expressive power and without closure under some of the boolean operators; this focus necessitates both new notions of simulations and novel techniques for proving the characterization results.

As to the second theme — expressiveness of description logics —, we know of only two earlier references: [Baa96] and [Bor96]. We will briefly discuss each of these. Baader's work is different from ours in two important ways. First, Baader's definition [Baa96, Definition 3.2] of expressive power differs from ours. Recall that we we define a logic $\mathcal{L}_1$ to be at least as expressive as a logic $\mathcal{L}_2$ if for every $\mathcal{L}_2$-expression there is an equivalent $\mathcal{L}_1$-expression over the same vocabulary. Intuitively, Baader's definition allows $\mathcal{L}_1$ to use additional concepts and roles in finding $\mathcal{L}_1$-equivalents for every $\mathcal{L}_2$-expression. More formally, let Γ be a collection of concepts, and let $\mathrm{Voc}(\Gamma)$ denote the collection of all atomic concepts and roles occurring in Γ. Further, assume that we have a mapping $f : \mathrm{Voc}(\Gamma_1) \to \mathrm{Voc}(\Gamma_2)$, and interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ that satisfy all of $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. Then f embeds $\mathcal{I}_1$ in $\mathcal{I}_2$ if for all $S \in \mathrm{Voc}(\Gamma_1)$ we have $S^{\mathcal{I}_1} = f(S)^{\mathcal{I}_2}$. Then, Γ_2 *can be expressed* by Γ_1 if there exists $f : \mathrm{Voc}(\Gamma_2) \to \mathrm{Voc}(\Gamma_1)$ such that

1. every interpretation that validates all of Γ_2 can be embedded by f in some interpretation that validates all of Γ_1, and
2. for every interpretation $\mathcal{I}_1$ that validates all of Γ_1 there exists an interpretation $\mathcal{I}_2$ that validates all of Γ_2 and that can be embedded in $\mathcal{I}_1$ by f.

Then, $\mathcal{L}_1$ is at least as expressive as $\mathcal{L}_2$ (according to Baader) if every collection of $\mathcal{L}_2$-concepts can be expressed by some collection of $\mathcal{L}_1$-concepts.

Clearly, this more involved definition allows one to equate more description logics with respect to their expressive power than ours does; for instance, under Baader's definition negation of atomic concepts can be simulated by number restrictions over additional roles, whereas according to our results negations of atomic concepts can't be expressed using number restrictions (over the same vocabulary).[3] While we agree that it may be useful to be

[3]As an aside, the difference between our definition and Baader's is analogous to the

able to use additional concepts and roles in finding equivalent expressions, as Baader himself points out, what is lacking from his definition is a measure on how much additional material one may use and on the complexity of the function that maps $\mathcal{L}_2$-expressions to equivalent $\mathcal{L}_1$-expressions over a richer vocabulary.

A second important difference between Baader's work and ours lies in the results that have been obtained. Baader only establishes a small number of separation results, whereas we provide a *complete* classification of all languages definable using the constructors in Table 1. More importantly, our separation results our based on semantic characterizations; this gives a deeper insight into the properties of logics than mere separation results.

In [Bor96] the author shows that certain description logics have the same expressive power as the two or three variable fragment of first-order logic (over the same vocabulary). Two remarks are in order. First, it is well-known that there is a correspondence between some description logics and modal logics (see [Sch91]), and modal logicians have considered the links with finite variable fragments for quite some time (see [Gab81]). Thus, Borgida's results could also have been obtained this way. Secondly, the description logics considered in this paper are all expressible in the two variable fragment of first-order logic (possibly with counting), however, none coincides with the full two-variable fragment.

6 Conclusion

In this paper we have introduced a model-theoretic method for determining the expressive power of description logics. The method consists of three components: a translation into a common background logic (here first-order logic over a suitable vocabulary), semantic characterizations of the translated logics, and using these characterizations to separate logics. The method was successfully applied to obtain expressiveness results for all logics in the $\mathcal{FL}^-$ and $\mathcal{AL}$ hierarchies.

The main benefits of our methods are that they give exact and explicit characterizations of description logics that explain in semantic terms *why* one logic is or is not different from another. While the proofs of the semantic characterizations in terms of various notions of (bi)-simulation are admittedly somewhat technical, the *use* of the characterizations in separating logics is fairly intuitive, as we hope to have demonstrated with our examples. As summarized in the diagram in Figure 2, our mathematical findings corroborate the intuitions one has concerning the expressive power of description logics; we view this as additional evidence in support of our methods.

difference between definability and projective definability in the area of model-theoretic logics; see [BF85].

It should be noted that the role of our semantic characterization results is in *separating* the expressive power of description logics, not in showing that they coincide with respect to their expressive power. For the latter, we use explicit syntactic definitions of the constructions of one logic in terms of the constructions of the other.

Future research in this area will concentrate on the following themes. First, as was pointed out above, the proofs for our characterization results use first-order techniques in an essential way. We aim to avoid these techniques, and thus to extend our methods method to description logics with non-first-order features (like transitive closure). Second, we want to gain a better understanding of the difference between our approach and that of [Baa96]. In particular, we want to extend our model-theoretic tools in ways that will characterize the expressive power of description logics in Baader's sense. Third, there is an influential line of work in the database literature that characterizes the expressive power of query languages in terms of the complexity of the recognition problem associated with queries expressible in the language at hand; see, for instance, [AHV95]. Can this approach be adapted to description logics? And if it can, would it induce the same classification of description logics as ours? Finally, what is the complexity of separating description logics. it is known from the literature on bisimulation that, in general, even the question whether two given interpretations are bisimilar, is undecidable, but for finite interpretations the question becomes decidable. In our case, the question is not just to check bisimilarity, but to determine whether there exists an $\mathcal{L}_1$-concept that is not preserved under $\mathcal{L}_2$-relations. Are there special cases of this question that become decidable?

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A Appendix: Proofs of the Main Results

For each of the logics obtained by adding constructors from Table 1 we will now prove the main semantic characterization theorems announced in Section 4. The proofs all follow the same basic strategy. One half of the result is proved by a simple induction; the other half is more involved and uses compactness arguments, and, in some cases, additional techniques from first-order logic.

Theorem 4.2 *Let $\alpha(x)$ be a first-order formula. Then $\alpha(x)$ is equivalent to an $\mathcal{FL}^-$-concept iff it is preserved under $\mathcal{FL}^-$-simulations.*

Proof. The implication from left to right is proved by induction on concepts. The atomic case is immediate from the definition of $\mathcal{FL}^-$-simulations, and conjunction is easy. Let us consider the existential case. Assume $X_1 Z d_2$. Suppose that $X_1 \subseteq (\exists R.\top)^{\mathcal{I}}$. Let $Y_1 := \{v \in \Delta^{\mathcal{I}} \mid \exists w \in X_1 \, (R^{\mathcal{I}} wv)\}$. Then $X_1 R^{\uparrow} Y_1$. So by clause 2 of Definition 4.1 there exists e_2 with $R^{\mathcal{J}} d_2 e_2$. Clearly, $d_2 \in (\exists R.\top)^{\mathcal{J}}$, as required. The universal case is next. Assume that $X_1 \subseteq (\forall R.C)^{\mathcal{I}}$. Suppose that there is a e_2 with $R^{\mathcal{J}} d_2 e_2$ but $e_2 \notin C^{\mathcal{J}}$. Then, by clause 3 of Definition 4.1 there exists $Y_1 \subseteq \Delta^{\mathcal{I}}$ with $X_1 R^{\downarrow} Y_1$ and $Y_1 Z e_2$. By induction hypothesis, $e_2 \notin C^{\mathcal{J}}$ implies $Y_1 \not\subseteq C^{\mathcal{I}}$. This contradicts $X_1 \subseteq (\forall R.C)^{\mathcal{I}}$.

Now, for the right to left implication, assume that $\alpha(x)$ is preserved under $\mathcal{FL}^-$-simulations, and let $Con(\alpha)$ be the set of its $\mathcal{FL}^-$-consequences.

Claim A.1 *$Con(\alpha) \models \alpha$.*

If we can prove Claim A.1, then, by compactness, there exists a finite conjunction of elements of $Con(\alpha)$ that is equivalent to $\alpha(x)$. So let us prove Claim A.1. Assume that $\mathcal{I} \models Con(\alpha)[w]$. We need to show that $\mathcal{I} \models \alpha[w]$. Let $\Gamma = \{\neg C \mid C \text{ is in } \mathcal{FL}^- \text{ and } w \notin C^{\mathcal{I}}\}$.

Claim A.2 *For every $\neg C \in \Gamma$, the set $\{\alpha(x), \neg C\}$ is consistent.*

If the claim were false, then C would be a consequence of α, contradicting the definition of Γ. As a corollary we find, for every $\neg C \in \Gamma$, an interpretation $\mathcal{I}_C$ and element $v_C \in \Delta^{\mathcal{I}_C}$ such that $v_C \in \alpha(x)^{\mathcal{I}_C} \cap (\neg C)^{\mathcal{I}_C}$.

Let $\mathcal{J}$ be the disjoint union of the pairs $(\mathcal{I}_C, v_C)$, where $\neg C \in \Gamma$.[4] By results from standard modal logic (cf. [Ben85]), it follows that for every $\neg C \in \Gamma$ there is a bisimulation linking v_C in $\Delta^{\mathcal{I}_C}$ to $v_C \in \Delta^{\mathcal{J}}$. Then, for every $\neg C \in \Gamma$ there is an $\mathcal{FL}^-$-simulation linking $\{v_C\}$ in $\Delta^{\mathcal{I}_C}$, and an $\mathcal{FL}^-$-simulation linking $\{v_C\}$ in $\Delta^{\mathcal{J}}$ to v_C in $\Delta^{\mathcal{I}_C}$ — simply link every singleton $\{d\}$ in the one interpretation to the copy of d in the other interpretation.

By assumption, $\alpha(x)$ is preserved under $\mathcal{FL}^-$-simulations, so $\{v_C\} \subseteq \alpha(x)^{\mathcal{I}_C}$ implies $v_C \in \alpha(x)^{\mathcal{J}}$, for every v_C. Also, as there is an $\mathcal{FL}^-$-simulation linking $\{v_C\}$ in $\Delta^{\mathcal{J}}$ to $v_C \in \Delta^{\mathcal{I}_C}$, the fact that $v_C \notin C^{\mathcal{I}_C}$ implies $v_C \notin C^{\mathcal{J}}$.

Claim A.3 *For every $\mathcal{FL}^-$-concept D, if for all v_C (with $\neg C \in \Gamma$), $v_C \in D^{\mathcal{J}}$, then $w \in D^{\mathcal{I}}$.*

To see why, assume $w \notin D^{\mathcal{I}}$. Then $\neg D \in \Gamma$, so there exists $v_D \in \mathcal{J}$ with $v_D \notin D^{\mathcal{J}}$. Next, define a relation $Z \subseteq \mathcal{P}(\Delta^{\mathcal{J}}) \times \Delta^{\mathcal{I}}$ by putting $X_1 Z d_2$ iff for all $\mathcal{FL}^-$-concepts D, $X_1 \subseteq D^{\mathcal{J}}$ implies $d_2 \in D^{\mathcal{I}}$.

Claim A.4 *The relation Z is an $\mathcal{FL}^-$-simulation.*

Clause 1 of Definition 4.1 is trivially satisfied. For the second clause, suppose that $X_1 R^{\uparrow} Y_1$ and $X_1 Z d_2$; we have to show that there exists $e_2 \in \Delta^{\mathcal{I}}$ with $R^{\mathcal{I}} d_2 e_2$. This is easy: if $X_1 R^{\uparrow} Y_1$, then $X_1 \subseteq (\exists R.\top)^{\mathcal{J}}$; so from $X_1 Z d_2$ we get $d_2 \in (\exists R.\top)^{\mathcal{I}}$, so the required e_2 exists. For the third clause, assume that $R^{\mathcal{I}} d_2 e_2$ and $X_1 Z d_2$; we need to find a $Y_1 \subseteq \Delta^{\mathcal{I}}$ with $X_1 (R^{\mathcal{I}})_{\downarrow} Y_1$ and $Y_1 Z e_2$. Let C be any concept with $e_2 \notin C^{\mathcal{I}}$; then $d_2 \notin (\forall R.C)^{\mathcal{I}}$. So from $X_1 Z d_2$ we get $X_1 \not\subseteq (\forall R.C)^{\mathcal{J}}$. Therefore, there exists $d_1 \in X_1$ and $e_1 \in \Delta^{\mathcal{J}}$ with $R^{\mathcal{J}} d_1 e_1$ but $e_1 \notin C^{\mathcal{J}}$. If we repeat this argument for every concept C with $d_2 \notin C^{\mathcal{I}}$, we obtain a set $Y_1 \subseteq \Delta^{\mathcal{I}}$ with $X_1 R_{\downarrow} Y_1$ and $Y_1 Z e_2$, as desired.

Finally, then, as a corollary to Claims A.3 and A.4 there is an $\mathcal{FL}^-$-simulation relating $\{v_C \in \Delta^{\mathcal{J}} \mid \neg C \in \Gamma\}$ and w. As for every $v_C \in \Delta^{\mathcal{J}}$

[4]That is, $\Delta^{\mathcal{J}}$ is the disjoint union of the sets $\Delta^{\mathcal{I}_C}$; for every concept D, $D^{\mathcal{J}}$ is the disjoint union of the sets $\mathcal{D}^{\mathcal{I}_C}$; and for every role R, $R^{\mathcal{J}}$ is the disjoint union of the sets $R^{\mathcal{I}_C}$.

with $\neg C \in \Gamma$ we have $\mathcal{J} \models \alpha(x)[v_C]$ and as $\alpha(x)$ is preserved under $\mathcal{FL}^-$-simulations, it follows that $\mathcal{I} \models \alpha(x)[w]$. This proves Claim A.1, and hence the theorem. $\dashv$

Theorem 4.5 *Let $\alpha(x)$ be a first-order formula. Then $\alpha(x)$ is equivalent to an $\mathcal{AL}$-concept iff it is preserved under $\mathcal{AL}$-simulations.*

Proof. Repeat all of the Claims A.1, A.2, A.3 and A.4 verbatim, but with $\mathcal{FLE}^-$ instead of $\mathcal{FL}^-$. $\dashv$

The key result used in the proofs of Theorems 4.2 and 4.5 is the compactness theorem. To prove characterization results for languages that are richer than $\mathcal{FL}^-$ we need additional semantic tools, over and above the compactness theorem. The proof of our characterization result for $\mathcal{FLE}^-$, Theorem 4.8, uses so-called ω-saturated models. Briefly, an interpretation $\mathcal{I}$ for a first-order language $\mathcal{L}$ is *ω-saturated* if whenever Δ is a set of first-order formulas in a language $\mathcal{L}'$, where $\mathcal{L}'$ extends $\mathcal{L}_1$ by the addition of finitely many new individual constants, and each finite subset of Δ is satisfiable in an $\mathcal{L}'$-expansion of $\mathcal{I}$, then Δ is satisfiable in this expansion.

A key result about ω-saturated models that will be used in our proofs below says that, in a countable language, every interpretation $\mathcal{I}$ has an ω-saturated elementary extension $\mathcal{I}^*$; that is, for every interpretation $\mathcal{I}$ there is an ω-saturated interpretation $\mathcal{I}^*$ such that $\Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}^*}$ and for every first-order formula $\alpha(x_1, \ldots, x_n)$ and any objects $d_1, \ldots, d_n \in \Delta^{\mathcal{I}}$, $\mathcal{I} \models \alpha[d_1, \ldots, d_n]$ iff $\mathcal{I}^* \models \alpha[d_1, \ldots, d_n]$. We refer the reader to any textbook on model theory for further details; see e.g., [Hod93].

Theorem 4.8 *Let $\alpha(x)$ be a first-order formula. Then $\alpha(x)$ is equivalent to an $\mathcal{FLE}^-$-concept iff it is preserved under $\mathcal{FLE}^-$-simulations.*

Proof. We leave the left to right direction to the reader, and only give a sketch of the right to left direction to the extent that it differs from the proof of Theorem 4.2.

As in the proof of Theorem 4.2 we assume that $\alpha(x)$ is preserved under $\mathcal{FLE}^-$-simulations, and we concentrate on proving that $Con(\alpha) \models \alpha$, where $Con(\alpha)$ is the set of $\mathcal{FLE}^-$-consequences of $\alpha(x)$. So, we assume that $\mathcal{I} \models Con(\alpha)[w]$, and we need to show that $\mathcal{I} \models \alpha(x)[w]$. Let $\Gamma = \{\neg C \mid C$ is in $\mathcal{FLE}^-$ and $w \notin C^{\mathcal{I}}\}$. As in Claim A.2 one can show that for every $\neg C \in \Gamma$, the set $\{\alpha(x), \neg C\}$ is consistent. Consequently, for every $\neg C \in \Gamma$ there are interpretations $\mathcal{I}_C$ and objects v_C such that $v_C \in \alpha(x)^{\mathcal{I}_C} \cap (\neg C)^{\mathcal{I}_C}$.

Let $\mathcal{J}$ be the disjoint union of the interpretations $\mathcal{I}_C$. The relation $\{(X_d, d) \mid d \in X_d\}$ is an $\mathcal{FLE}^-$-simulation linking $\{v_C\}$ in $\mathcal{J}$ (or $\mathcal{I}_C$) to v_C in $\mathcal{I}_C$ (or $\mathcal{J}$). It follows that $v_C \in \alpha(x)^{\mathcal{J}} \setminus C^{\mathcal{J}}$.

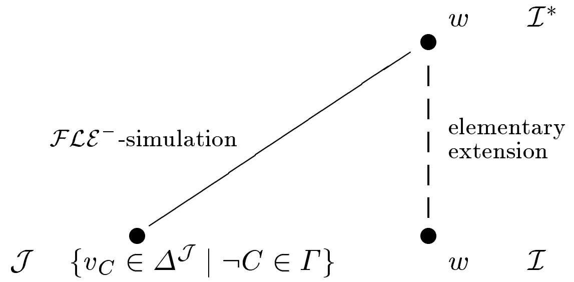
We leave it to the reader to establish an analog of Claim A.3. As explained above, there exists an ω-saturated elementary extension $\mathcal{I}^*$ of $\mathcal{I}$. It follows that for every $\mathcal{FLE}^-$-concept D, $v \in D^{\mathcal{I}}$ iff $v \in D^{\mathcal{I}^*}$.

Next, define a relation $Z \subseteq (\mathcal{P}(\Delta^{\mathcal{J}}) \times \Delta^{\mathcal{I}^*})$ by putting $X_1 Z d_2$ iff for all $\mathcal{FLE}^-$-concepts D, $X_1 \subseteq D^{\mathcal{J}}$ implies $d_2 \in D^{\mathcal{I}^*}$.

Claim A.5 *The relation Z is an $\mathcal{FLE}^-$-simulation.*

We only check clause 2 of Definition 4.7. (Clause 1 is easy, and clause 3 is similar to clause 2 in the proof of Claim A.4.) Assume that $X_1(R^{\mathcal{J}})^{\uparrow}Y_1$ and $X_1 Z d_2$. We need to find an $e_2 \in \Delta^{\mathcal{I}^*}$ with $R^{\mathcal{I}^*} d_2 e_2$ and $Y_1 Z e_2$. Let $C_1 \sqcap \cdots \sqcap C_n$ be an arbitrary finite conjunction of concepts such that $Y_1 \subseteq (C_1 \sqcap \cdots \sqcap C_n)^{\mathcal{J}}$. Clearly, then, $X_1 \subseteq (\exists R.(C_1 \sqcap \cdots \sqcap C_n))^{\mathcal{J}}$. By the definition of Z we find that $d_2 \in (\exists R.(C_1 \sqcap \cdots \sqcap C_n))^{\mathcal{I}^*}$. This implies that there exists $e_2 \in \Delta^{\mathcal{I}^*}$ such that $e_2 \in (C_1 \sqcap \cdots \sqcap C_n)^{\mathcal{I}^*}$. At this point we use the fact that $\mathcal{I}^*$ is ω-saturated. As we have been able to find an object e_2 in $\mathcal{I}^*$ that satisfies $R^{\mathcal{I}^*} d_2 e_2$ together with an arbitrary finite collection of concepts satisfied by all the objects in Y_1, by ω-saturation we can in fact find an object e_2 in $\mathcal{I}^*$ with $R^{\mathcal{I}^*} d_2 e_2$ that satisfies *all* concepts satisfied by the objects in Y_1. This means that $Y_1 Z e_2$, as required.

With the proof of Claim A.5 completed, we have found an $\mathcal{FLE}^-$-simulation between $\mathcal{J}$ and $\mathcal{I}^*$ that relates $\{v_C \in \Delta^{\mathcal{J}} \mid \neg C \in \Gamma\}$ and w. Hence we have the situation depicted in the following diagram.



A walk around the diagram completes the proof. From $\{v_C \in \Delta^{\mathcal{J}} \mid \neg C \in \Gamma\} \subseteq \alpha(x)^{\mathcal{J}}$ and the fact that there is an $\mathcal{FLE}^-$-simulation linking $\{v_C \in \Delta^{\mathcal{J}} \mid \neg C \in \Gamma\}$ to w in $\mathcal{I}^*$, it follows that $w \in \alpha(x)^{\mathcal{I}^*}$. As $\mathcal{I}^*$ is an elementary extension of $\mathcal{I}$, we get $w \in \alpha(x)^{\mathcal{I}}$, and we are done. $\dashv$

Recall that the change required to prove a characterization result for $\mathcal{FLU}^-$ is that we no longer work with simulations involving sets, but with ones involving single objects only. for this characterization result we will also need to use ω-saturated models.

Theorem 4.11 *Let $\alpha(x)$ be a first-order formula. Then $\alpha(x)$ is equivalent to an $\mathcal{FLU}^-$-concept iff it is preserved under $\mathcal{FLU}^-$-simulations.*

Proof. As before, we leave the left to right direction to the reader, and only give a sketch of the right to left direction to the extent that it differs from previous proofs (Theorems 4.2, 4.8). Assume that $\alpha(x)$ is preserved under $\mathcal{FLU}^-$-simulations, and consider the set of its consequences in $\mathcal{FLU}^-$, $Con(\alpha)$. As before it suffices to prove that $Con(\alpha) \models \alpha$. So, we assume that $\mathcal{I} \models Con(\alpha)[w]$, and we need to show that $\mathcal{I} \models \alpha(x)[w]$. Let $\Gamma = \{\neg C \mid C$ is in $\mathcal{FLU}^-$ and $w \notin C^{\mathcal{I}}\}$.

Claim A.6 *The set $\{\alpha(x)\} \cup \Gamma$ is consistent.*

If the claim were false, there would be concepts $\neg C_1, \ldots, \neg C_n \in \Gamma$ such that $\alpha \models \neg(\neg C_1 \sqcap \cdots \sqcap \neg C_n)$, or, in other words, $\alpha \models C_1 \sqcup \cdots \sqcup C_n$. So $w \in (C_1 \sqcup \cdots \sqcup C_n)^{\mathcal{I}}$ as $w \in \alpha(x)^{\mathcal{I}}$, and hence $w \in C_i^{\mathcal{I}}$ for some i with $1 \leq i \leq n$. But, as $C_1, \ldots, C_n$ are $\mathcal{FLU}^-$-concepts, then $\neg C_i \notin \Gamma$, which is a contradiction. This proves Claim A.6.

As a corollary we find an interpretation $\mathcal{J}$ and an object $v \in \Delta^{\mathcal{J}}$ with $v \in (\alpha(x)^{\mathcal{J}} \cap \bigcap\{\neg C^{\mathcal{J}} \mid \neg C \in \Gamma\})$.

Claim A.7 *For every $\mathcal{FLU}^-$-concept D, if $v \in D^{\mathcal{J}}$, then $w \in D^{\mathcal{I}}$.*

Now, let $\mathcal{J}^*$ be an ω-saturated elementary extension of $\mathcal{J}$. It follows that for every $\mathcal{FLU}^-$-concept D, $v \in D^{\mathcal{J}}$ iff $v \in D^{\mathcal{J}^*}$. Next, define a relation $Z \subseteq (\Delta^{\mathcal{J}^*} \times \Delta^{\mathcal{I}})$ by putting $d_1 Z d_2$ iff for all $\mathcal{FLU}^-$-concepts D, $d_1 \in D^{\mathcal{J}^*}$ implies $d_2 \in D^{\mathcal{I}}$.

Claim A.8 *The relation Z is an $\mathcal{FLU}^-$-simulation.*

We only check clauses 2 and 3 of Definition 4.10. Assume that $R^{\mathcal{J}} d_1 e_1$ and $d_1 Z d_2$. We need to find an $e_2 \in \Delta^{\mathcal{I}}$ with $R^{\mathcal{I}^*} d_2 e_2$. But this is almost trivial: given the existence of e_1 we have $d_1 \in (\exists R.\top)^{\mathcal{J}^*}$, and hence $d_2 \in (\exists R.\top)^{\mathcal{I}}$, as $d_1 Z d_2$; from this the existence of the required e_2 follows.

As to clause 3, assume $R^{\mathcal{I}} d_2 e_2$ and $d_1 Z d_2$. We need an $e_1 \in \Delta^{\mathcal{J}^*}$ with $R^{\mathcal{J}^*} d_1 e_1$ and $e_1 Z e_2$. Let $C_1, \ldots, C_n$ be an arbitrary finite number of $\mathcal{FLU}^-$-concepts such that $e_2 \notin (C_1 \sqcap \cdots \sqcap C_n)^{\mathcal{I}}$. Then, $d_2 \notin (\forall R.(C_1 \sqcap \cdots \sqcap C_n))^{\mathcal{I}}$. By the definition of Z we find that $d_1 \notin (\forall R.(C_1 \sqcap \cdots \sqcap C_n))^{\mathcal{J}^*}$. So there exists $e_1 \in \Delta^{\mathcal{J}^*}$ with $e_2 \notin (C_1 \sqcap \cdots \sqcap C_n)^{\mathcal{J}^*}$. By ω-saturation of $\mathcal{J}^*$ this argument can be generalized to the collection of all $\mathcal{FLU}^-$-concepts not satisfied by e_2. So, there exists an $e_1 \in \Delta^{\mathcal{J}^*}$ such that $R^{\mathcal{J}^*} d_1 e_1$ and, for any D, $e_2 \notin D^{\mathcal{I}}$ implies $e_1 \notin D^{\mathcal{J}^*}$. Hence, $e_1 Z e_2$. This proves Claim A.8.

The proof may now be completed in the same way as the proof of Theorem 4.8. $\dashv$

Theorem 4.15 *Let $\alpha(x)$ be a first-order formula. Then $\alpha(x)$ is equivalent to an $\mathcal{FLN}^-$-concept iff it is preserved under $\mathcal{FLN}^-$-simulations.*

Proof. We first prove the left to right direction. We prove by induction on $\mathcal{FLN}^-$-concepts that if $X_1 Z_0 d_2$ and $X_1 \subseteq D^{\mathcal{I}}$, then $d_2 \in D^{\mathcal{J}}$. We only treat the quantificational cases. First, assume that $X_1 Z_0 d_2$ and $X_1 \subseteq (\geq i\ R)^{\mathcal{I}}$. Then, for every $d_1 \in X_1$ there exists $Y_{d_1} \subseteq \Delta^{\mathcal{I}}$ with $(R^{\mathcal{I}})^\bullet d_1 Y_1$ and $|Y_{d_1}| = i$. Collect these sets Y_{d_j} together into a collection $\mathcal{X} \subseteq \mathcal{P}(\mathcal{P}^{<\omega}(\Delta^I))$; then $\mathcal{X}$ is a an i-cloud that is R-above X_1. So, by clause 5 of Definition 4.14 there exists $Y_2 \subseteq \Delta^{\mathcal{J}}$ with $(R^{\mathcal{J}})^\bullet d_2 Y_2$ and $\mathcal{X} Z_i Y_2$. By clause 3 it follows that $|Y_2| = i$, as required.

Next, to prove preservation of $(\leq i\ R)$, assume $X_1 Z_0 d_2$ as before, while $d_2 \notin (\leq i\ R)^{\mathcal{J}}$. Choose $Y_2 \subseteq \Delta^{\mathcal{J}}$ such that $|Y_2| = i+1$ and $(R^{\mathcal{J}})^\bullet d_2 Y_2$. By clause 6 of Definition 4.14 there exists an $i+1$-cloud $\mathcal{X} \subseteq \mathcal{P}(\mathcal{P}^{<\omega}(\Delta^I))$ such that X_1 is R-below $\mathcal{X}$. Then, for all $Y_1 \in \mathcal{X}$, $|Y_1| = i+1$, by clause 3, so $X_1 \not\subseteq (\leq i\ R)^{\mathcal{I}}$, as required.

Finally, we have to prove preservation of concepts of the form $\forall R.C$ (the case $\exists R.\top$ is covered by $(\geq 1\ R)$). Assume $X_1 Z_0 d_2$, $R^{\mathcal{J}} d_2 e_2$ and $e_2 \notin C^{\mathcal{J}}$. By clause 7 of Definition 4.14 there exists a 1-cloud $\mathcal{X} \subseteq \mathcal{P}(\mathcal{P}^{<\omega}(\Delta^{\mathcal{I}}))$ such that X_1 is R-below $\mathcal{X}$ and $(\bigcup \mathcal{X})\, Z_0 e_2$. By induction hypothesis, $(\bigcup \mathcal{X}) \not\subseteq C^{\mathcal{I}}$. That is, there exists $e_1 \in \bigcup \mathcal{X}$ such that $e_1 \notin C^{\mathcal{I}}$. As X_1 is R-below $\mathcal{X}$, there exists $d_1 \in X_1$ with $(R^{\mathcal{I}})^\bullet d_1 \{e_1\}$, or in other words, $R^{\mathcal{I}} d_1 e_1$. It follows that $d_1 \notin (\forall R.C)^{\mathcal{I}}$, and therefore $X_1 \not\subseteq (\forall R.C)^{\mathcal{I}}$, and we are done.

Now, to prove the harder right to left direction, assume that $\alpha(x)$ is preserved under $\mathcal{FLN}^-$-simulations. As in the proofs of our previous preservation results, we proceed to prove that $Con(\alpha) \models \alpha$, where $Con(\alpha)$ is the set of $\mathcal{FLN}^-$-consequences of $\alpha(x)$. So, we assume that $\mathcal{I} \models Con(\alpha)[w]$, and we need to show that $\mathcal{I} \models \alpha(x)[w]$. Let $\Gamma = \{\neg C \mid C \text{ is in } \mathcal{FLN}^- \text{ and } w \notin C^{\mathcal{I}}\}$. As in Claim A.2 we find interpretations $\mathcal{I}_C$ and objects v_C such that $v_C \in \alpha(x)^{\mathcal{I}_C} \cap (\neg C)^{\mathcal{I}_C}$, and we form the disjoint union $\mathcal{J}$ of the interpretations $\mathcal{I}_C$. Clearly, the relation $\{(X_d, d) \mid d \in X_d\}$ is the 'Z_0'-component of an $\mathcal{FLN}^-$-simulation linking $\{v_C\}$ in $\mathcal{J}$ (or $\mathcal{I}_C$) to v_C in $\mathcal{I}_C$ (or $\mathcal{J}$). As a consequence, we obtain that $v_C \in \alpha(x)^{\mathcal{J}} \setminus C^{\mathcal{J}}$.

We leave it to the reader to establish an analog of Claim A.3. Define the following sequence of relations Z_0, Z_1, ...

$$\begin{aligned} Z_0 &:= \{(X_1, d_2) \mid X_1 \subseteq \Delta^{\mathcal{J}},\ d_2 \in \Delta^{\mathcal{I}}, \text{ and for all } D,\ X_1 \subseteq D^{\mathcal{J}} \\ &\qquad \text{implies } d_2 \in D^{\mathcal{I}}\} \\ Z_i &:= \{(\mathcal{X}, Y_2) \mid i > 0,\ \mathcal{X} \subseteq \mathcal{P}(\mathcal{P}^{<\omega}(\Delta^{\mathcal{J}})),\ Y_2 \subseteq \Delta^{\mathcal{I}}, \text{ and for all} \\ &\qquad Y_1 \in \mathcal{X},\ |Y_1| = |Y_2| = i\}. \end{aligned}$$

We tacitly assume that all the collections of sets $\mathcal{X}$ occurring in the above definition are i-clouds above some set $X_1 \subseteq \Delta^{\mathcal{J}}$, for some i.

Claim A.9 *The tuple $Z = (Z_0, Z_1, \ldots, Z_n, \ldots)$ is an $\mathcal{FLN}^-$-simulation.*

To prove the claim, observe first that clauses 1–4 of Definition 4.14 are trivially fulfilled, so we only have to check clauses 5, 6 and 7. As to clause 5, assume that $X_1 Z_0 d_2$, and that $\mathcal{X} \subseteq \mathcal{P}(\mathcal{P}^{<\omega}(\Delta^{\mathcal{J}}))$ is an i-cloud which is R-above X_1. We need a set $Y_2 \subseteq \Delta^{\mathcal{I}}$ with $(R^{\mathcal{I}})^\bullet d_2 Y_2$ and $|Y_2| = i$. Clearly, we have that for every $d_1 \in X_1$, $d_1 \in (\geq i\ R)^{\mathcal{J}}$, hence $X_1 \subseteq (\geq i\ R)^{\mathcal{J}}$. Then, $X_1 Z_0 d_2$ gives $d_2 \in (\geq i\ R)^{\mathcal{I}}$. This implies the existence of the required Y_2.

Next comes clause 6 of Definition 4.14. Assume $X_1 Z_0 d_2$, and $(R^{\mathcal{I}})^{\bullet} d_2 Y_2$, where $|Y_2| = i > 0$. We need to find an i-cloud $\mathcal{X} \subseteq \mathcal{P}(\mathcal{P}^{<\omega}(\Delta^{\mathcal{J}}))$ such that X_1 is R-below $\mathcal{X}$ and $\mathcal{X} Z_i Y_2$. Reason as follows: as $d_2 \notin (\leq i-1\ R)^{\mathcal{I}}$, we get $X_1 \notin (\leq i-1\ R)^{\mathcal{J}}$, and it follows that for some $d_1 \in \Delta^{\mathcal{J}}$ we have $d_1 \notin (\leq i-1\ R)^{\mathcal{J}}$, so $d_1 \in (\geq i\ R)^{\mathcal{J}}$. Let

$$\mathcal{X}_{d_1} = \{Y \subseteq \Delta^{\mathcal{J}} \mid |Y| = i \text{ and } (R^{\mathcal{J}})^{\bullet} d_1 Y\},$$

and put

$$\mathcal{X} = \bigcup \{\mathcal{X}_{d_1} \mid d_1 \in X \text{ and } d_1 \in (\geq i\ R)^{\mathcal{J}}\}.$$

Then $\mathcal{X}$ is a non-empty i-cloud such that X_1 is R-below $\mathcal{X}$ and $\mathcal{X} Z_i Y_2$, as required.

Next we turn to clause 7. Assume $X_1 Z_0 d_2$ and $R^{\mathcal{I}} d_2 e_2$; we need to find a 1-cloud $\mathcal{X}$ such that X_1 is R-below $\mathcal{X}$ and $(\bigcup \mathcal{X})\, Z_0 e_2$. For every concept C such that $e_2 \notin C^{\mathcal{I}}$, we can find e_1, $d_1 \in \Delta^{\mathcal{J}}$ with $d_1 \in X_1$ and $R^{\mathcal{J}} d_1 e_1$. Let $\mathcal{X}$ be the collection of all singletons $\{e_1\}$ obtained in this way; then X_1 is R-below $\mathcal{X}$ and $(\bigcup \mathcal{Z})\, Z_0 e_2$, as required.

This proves Claim A.9. Using a by now familiar argument the proof of Theorem 4.15 can now be completed. $\dashv$

Theorem 4.18 *Let $\alpha(x)$ be a first-order formula. Then $\alpha(x)$ is equivalent to an $\mathcal{FLR}^{-}$-concept iff it is preserved under $\mathcal{FLR}^{-}$-simulations.*

Proof. We first prove the left to right direction. We prove by induction on $\mathcal{FLR}^{-}$-concepts that if $X_1 Z_0 d_2$ and $X_1 \subseteq D^{\mathcal{I}}$, then $d_2 \in D^{\mathcal{J}}$. We only treat the quantificational cases. First, assume that $X_1 Z_0 d_2$ and $X_1 \subseteq (\exists (R_1 \sqcap \cdots \sqcap R_n).\top)^{\mathcal{I}}$, where all R_i are atomic role names. For each $d_1 \in X_1$ select $e_2 \in \Delta^{\mathcal{I}}$ with $(R_1 \sqcap \cdots \sqcap R_n)^{\mathcal{I}} d_2 e_2$, and collect these e_2's together in a set Y_1. Let $\mathcal{R}$ be a collection of (atomic) role names such that $R_1, \ldots, R_n \in \mathcal{R}$ and such that $\mathcal{R}$ is meet closed for X_1 and Y_1. By clause 4(a) of Definition 4.17 there exists $E_2 \subseteq \Delta^{\mathcal{I}}$ with $(X_1, Y_1) Z_1 (d_2, E_2)$. By clause 3(a) there exists $e_2 \in E_2$ such that $(\bigsqcap \mathcal{R})^{\mathcal{J}} d_2 e_2$. Hence, $d_2 \in (\exists (\bigsqcap \mathcal{R}).\top)^{\mathcal{J}}$, and therefore $d_2 \in (\exists (R_1 \sqcap \cdots \sqcap R_n).\top)^{\mathcal{J}}$, as required. Next, to prove preservation of $\forall R.C$, assume that $X_1 Z_0 d_2$ and $d_2 \notin (\forall R.C)^{\mathcal{J}}$. Let e_2 be such that $R^{\mathcal{J}} d_2 e_2$ and $e_2 \notin C^{\mathcal{J}}$. Then, by clause 4(b), there exists Y_1 with $(X_1, Y_1) Z_1 (d_2, e_2)$. By clause 3(b), $X_1 (R^{\mathcal{I}})_{\downarrow} Y_1$, and clause 5 gives $Y_1 Z_0 e_2$. Together with $e_2 \notin C^{\mathcal{J}}$ and the induction hypothesis this implies $X_1 \not\subseteq (\forall R.C)^{\mathcal{I}}$, and we're done.

Next, to prove the right to left direction, we assume that $\alpha(x)$ is preserved under $\mathcal{FLR}^{-}$-simulations, and proceed to prove that $Con(\alpha) \models \alpha$, where $Con(\alpha)$ is the set of $\mathcal{FLR}^{-}$-consequences of $\alpha(x)$. So, we assume that $\mathcal{I} \models Con(\alpha)[w]$, and we need to show that $\mathcal{I} \models \alpha(x)[w]$. Let $\Gamma = \{\neg C \mid C$

is in $\mathcal{FLR}^-$ and $w \notin C^{\mathcal{I}}\}$. As in Claim A.2 we find interpretations $\mathcal{I}_C$ and objects v_C such that $v_C \in \alpha(x)^{\mathcal{I}_C} \cap (\neg C)^{\mathcal{I}_C}$, and we form the disjoint union $\mathcal{J}$ of the interpretations $\mathcal{I}_C$. We leave it to the reader to check that there is an $\mathcal{FLR}^-$-simulation linking $\{v_C\}$ in $\mathcal{J}$ (or $\mathcal{I}_C$) to v_C in $\mathcal{I}_C$ (or $\mathcal{J}$). It follows that $v_C \in \alpha(x)^{\mathcal{J}} \setminus C^{\mathcal{J}}$.

We also leave it to the reader to establish an analog of Claim A.3. Next, take ω-saturated elementary extensions $\mathcal{J}^*$ and $\mathcal{I}^*$ of $\mathcal{J}$ and $\mathcal{I}$, respectively. Define the following relations Z_0, Z_1, and Z_2:

$$\begin{aligned}
Z_0 &:= \{(X_1, d_2) \mid \text{ for all } D,\ X_1 \subseteq D^{\mathcal{J}^*} \text{ implies } d_2 \in D^{\mathcal{I}^*}\}.\\
Z_1 &:= \{((X_1, Y_1), (d_2, E_2)) \mid \text{ for some } R,\ X_1 R^{\uparrow} Y_1, \text{ and for every}\\
&\qquad \text{meet closed collection of atomic concepts } \mathcal{R} \text{ for } X_1 \text{ and } Y_1\\
&\qquad \text{there exists } e_2 \in E_2 \text{ with } (\textstyle\bigsqcap \mathcal{R})^{\mathcal{I}^*} d_2 e_2\}.\\
Z_2 &:= \{((X_1, Y_1), (d_2, e_2)) \mid Y_1 Z_0 e_2 \text{ and for all concepts } R,\\
&\qquad R^{\mathcal{I}^*} d_2 e_2 \text{ implies } X_1 (R^{\mathcal{J}^*})_{\downarrow} Y_1\}.
\end{aligned}$$

Claim A.10 *The tuple $Z = (Z_0, Z_1, Z_2)$ is an $\mathcal{FLR}^-$-simulation.*

To prove the claim, observe first that conditions 1, 2, 3, and 5 of Definition 4.17 are trivially satisfied. As to condition 4(a), assume that $X_1 Z_0 d_2$ and $X_1 (R^{\mathcal{J}^*})^{\uparrow} Y_1$. Let $\mathcal{R}$ be any meet closed collection of atomic role names for X_1 and Y_1, and consider the set

$$\Sigma(d_2, y) := \{R d_2 y \mid R \in \mathcal{R}\}.$$

$\Sigma(d_2, y)$ is finitely satisfiable in $\mathcal{I}^*$. For, consider $R_1 d_2 y, \ldots, R_n d_2 y \in \Sigma(d_2, y)$. As $\mathcal{R}$ is meet closed for X_1 and Y_1, it follows that $X_1((R_1 \sqcap \cdots \sqcap R_n)^{\mathcal{J}^*})^{\uparrow} Y_1$, and hence $X_1 \subseteq (\exists (R_1 \sqcap \cdots \sqcap R_n).\top)^{\mathcal{J}^*}$. Since $X_1 Z_0 d_2$ it follows that $d_2 \in (\exists (R_1 \sqcap \cdots \sqcap R_n).\top)^{\mathcal{I}^*}$, hence there exists e_2 with $(R_1 \sqcap \cdots \sqcap R_n)^{\mathcal{I}^*} d_2 e_2$. Now, using the fact that $\mathcal{I}^*$ is ω-saturated, it follows that all of $\Sigma(d_2, y)$ is satisfiable in $\mathcal{I}^*$, say by e_2. Clearly, for this e_2 we have $(\bigsqcap \mathcal{R})^{\mathcal{I}^*} d_2 e_2$.

Repeat the above argument for every collection of atomic role names that is meet maximal for X_1 and Y_1, and collect the satisfying objects e_2 together in a set E_2. This proves clause 4(a).

As to clause 4(b), assume $X_1 Z_0 d_2$ and $R^{\mathcal{I}^*} d_2 e_2$. We need a $Y_1 \subseteq \Delta^{\mathcal{J}^*}$ with $(X_1, Y_1) Z_2 (d_2, e_2)$. Choose C with $e_2 \notin C^{\mathcal{I}^*}$, and define

$$\Sigma(x, y) := \{\neg C\} \cup \{Rxy \mid R^{\mathcal{I}^*} d_2 e_2\}.$$

We claim that $\Sigma(x, y)$ is finitely satisfiable in $\mathcal{J}^*$ in such a way that x takes its value in X_1. To see this, take $R_1(x, y), \ldots, R_n(x, y) \in \Sigma(x, y)$. Then $d_2 \notin (\forall (R_1 \sqcap \cdots \sqcap R_n).C)^{\mathcal{I}^*}$, and hence $X_1 \not\subseteq (\forall (R_1 \sqcap \cdots \sqcap R_n).C)^{\mathcal{J}^*}$, as $X_1 Z_0 d_2$. It follows that there exists $d_1 \in X_1$ and $e_1 \in \Delta^{\mathcal{J}^*}$ with $(R_1 \sqcap \cdots \sqcap R_n)^{\mathcal{J}^*} d_1 e_1$

and $e_1 \notin C^{\mathcal{J}^*}$. By ω-saturation, *all* of $\Sigma(x,y)$ is satisfiable in $\mathcal{J}^*$ in such a way that x takes its value in X_1. This yields an object e_1 such that for some $d_1 \in X_1$

$$(d_1, e_1) \in \bigcap\{R^{\mathcal{J}^*} \mid R^{\mathcal{I}^*} d_2 e_2\} \text{ and } e_1 \notin C^{\mathcal{J}^*}.$$

Repeating this argument for every C such that $e_2 \notin C^{\mathcal{I}^*}$, we obtain a set Y_1 as desired.

Using a by now familiar argument, we can use the existence of an $\mathcal{FLR}^-$-simulation as the main step in showing that $\mathcal{I} \models \alpha(x)[w]$. $\dashv$